Korean J. Math. **29** (2021), No. 3, pp. 613–620
http://dx.doi.org/10.11568/kjm.2021.29.3.613

WEAKLY CONVERGENT SEQUENCES IN FUZZY NORMED SPACES

KYUGEUN CHO

ABSTRACT. In this paper, we introduce the definition of weakly convergent sequence in fuzzy normed spaces. We investigate relationship between convergent sequence and weakly convergent sequence in fuzzy normed spaces. We also study the dual spaces of a standard fuzzy normed space and 01-fuzzy normed space.

1. Introduction and preliminaries

Various definitions of fuzzy normed spaces have been investigated by several authors (see [1, 2, 6, 7]). In this paper, we take definitions of fuzzy normed spaces introduced by C. Alegre and S. Romaguera [1].

DEFINITION 1.1. If X is a real vector space, a fuzzy norm on X is a pair $(N, \wedge)$ such that $\wedge$ is defined by $a \wedge b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and N is a fuzzy set in $X \times [0, \infty)$ satisfying the following conditions for every $x, y \in X$, and $t, s \geq 0$:

(FN1) $N(x, 0) = 0$,
(FN2) $N(x, t) = 1$ for all $t > 0$ if and only if $x = 0$,
(FN3) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ for every $c \in \mathbb{R} \setminus \{0\}$,
(FN4) $N(x + y, t + s) \geq N(x, t) \wedge N(y, s)$,
(FN5) $\lim_{t\to\infty} N(x, t) = 1$,
(FN6) $N(x, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

The triple $(X, N, \wedge)$ is called a fuzzy normed space. If condition (FN5) is omitted we say that $(N, \wedge)$ is a weak fuzzy norm on X and the triple $(X, N, \wedge)$ will be called a weak fuzzy normed space.

Let $(X, N, \wedge)$ be a fuzzy normed space. A sequence $\{x_n\}$ is said to be convergent to x if, for all $\epsilon \in (0, 1)$ and $t > 0$, there exists $n_{\epsilon,t} \in \mathbb{N}$, depending on ϵ and t, such that $N(x_n - x, t) > 1 - \epsilon$, for $n \geq n_{\epsilon,t}$ [2]. A sequence $\{x_n\}$ is said to be s-convergent if, for all $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$ such that $N(x_n - x, \frac{1}{n}) > 1 - \epsilon$, for $n \geq n_\epsilon$ [4]. A sequence $\{x_n\}$ is said to be st-convergent if, for all $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$ such that $N(x_n - x, t) > 1 - \epsilon$, for $n \geq n_\epsilon$ and for all $t > 0$ [4]. In [4], the authors proved the following strict implications in fuzzy normed spaces.

$$st - convergence \Rightarrow s - convergence \Rightarrow convergence$$

Received February 28, 2021. Revised September 20, 2021. Accepted September 21, 2021.
2010 Mathematics Subject Classification: 54A40, 46A22.
Key words and phrases: fuzzy normed spaces, convergent sequences, weakly convergent sequences.

A sequence (x_n) in a fuzzy normed space (X, N) is said to be s_p-convergent to x_0, for $p \in \mathbb{N}$ if $\lim_{n\to\infty} N\left(x_n - x_0, \frac{1}{n^p}\right) = 1$ [5]. A sequence (x_n) in (X, N) is said to be s_∞-convergent to x_0 if (x_n) is s_p-convergent to x_0, for all $p \in \mathbb{N}$ [5]. In [5], the authors proved the following strict implications in fuzzy normed space.

$$st - conv. \Rightarrow s_\infty - conv. \Rightarrow \cdots \Rightarrow s_{k+1} - conv. \Rightarrow s_k - conv. \cdots \Rightarrow s - conv.$$

In Section 2, we introduce the definition of weakly convergent sequence in fuzzy normed spaces. We investigate relationship between convergent sequence and weakly convergent sequence in fuzzy normed spaces. We also study the dual spaces of a standard fuzzy normed space and 01-fuzzy normed space.

2. Convergent sequence and weakly convergent sequence in fuzzy normed spaces

The followings are well-known examples of fuzzy normed spaces [1].

Example 2.1. Let $(X, \|\cdot\|)$ be a normed space.

(a) Let $N_s : X \times [0, \infty) \to [0, 1]$ given by $N_s(x, 0) = 0$ for all $x \in X$ and let

$$N_s(x, t) = \frac{t}{t + \|x\|},$$

for all $x \in X$ and $t > 0$. Then $(N_s, \wedge)$ is a fuzzy norm on X. This fuzzy norm is called the standard fuzzy norm induced by $\|\cdot\|$.

(b) Let $N_{01} : X \times [0, \infty) \to [0, 1]$ given by $N_{01}(x, t) = 0$ if $t \le \|x\|$ and $N_{01}(x, t) = 1$ if $t > \|x\|$. Then $(N_{01}, \wedge)$ is a fuzzy norm on X. This fuzzy norm will be called the 01-fuzzy norm induced by $\|\cdot\|$.

For a fuzzy normed space $(X, N, \wedge)$, the open ball $B_N(x, r, t)$ is defined by

$$B_N(x, r, t) = \{y \in X : N(y - x, t) > 1 - r\}.$$

Let τ_N be a topology on X which has as a base the collection $\{B_N(x, r, t) : x \in X, 0 < r < 1, t > 0\}$. Then τ_N is metrizable and the countable collection of balls $\{B_N(x, 1/n, 1/n) : x \in X, n = 2, 3, \cdots\}$ forms a fundamental system of neighborhoods of x, for all $x \in X$ [1]. Let $(X, N, \wedge)$ be a fuzzy normed space and let $(N_s, \wedge)$ be the standard fuzzy norm on $\mathbb{R}$. Denote by X^* the set of all continuous linear mappings from (X, τ_N) to $(\mathbb{R}, \tau_{N_s})$. We now define a weakly convergent sequence.

Definition 2.2. A sequence $\{x_n\}$ in a fuzzy normed space $(X, N, \wedge)$ is said to be weakly convergent to $x \in X$ if $\lim_{n\to\infty} N_s(f(x_n) - f(x), t) = 1$, for all $f \in X^*$ and $t > 0$.

The followings are found in [1].

Proposition 2.3. *Let $(X, N, \wedge)$ be a fuzzy normed space and let $\alpha \in (0, 1)$. Then the following hold.*

(a) *The function $\|\cdot\| : X \to [0, \infty)$ given by*

$$\|x\|_\alpha = \inf\{t > 0 : N(x, t) \ge \alpha\}$$

is a seminorm on X.

(b) *The family $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ is separating.*

PROPOSITION 2.4. *Let $(X, N, \wedge)$ be a fuzzy normed space and let $\{\|\cdot\|_\alpha : \alpha \in (0,1)\}$ be the α-seminorms corresponding to the fuzzy norm $(N, \wedge)$. Then $f \in X^*$ if and only if there exist $\alpha \in (0,1)$ and $M > 0$ such that $|f(x)| \leq M\|x\|_\alpha$ for every $x \in X$.*

The following theorem is a modification of [8, Theorem 14.1].

THEOREM 2.5. *(Hahn-Banach theorem) Let $(X, N, \wedge)$ be a fuzzy normed space, Y a subspace of X and p a real mapping on X such that for all $x, y \in X$ and $t > 0$,*

$$p(x+y) \leq p(x) + p(y), \quad p(tx) = tp(x).$$

Let f be a linear functional on Y such that

$$f(x) \leq p(x) \quad \text{for all } x \in Y.$$

Then there exists a linear functional $\tilde{f}$ on X such that

$$\tilde{f}(x) = f(x) \quad \text{for all } x \in Y$$

and

$$\tilde{f}(x) \leq p(x) \quad \text{for all } x \in X.$$

Proof. We note that $g : Z \to \mathbb{R}$ is a linear extension of $f : Y \to \mathbb{R}$ if it is linear, Z and Y are subspaces of a vector space X with $Y \subset Z \subset X$ and $g(x) = f(x)$ for all $x \in Y$.
Let $\mathcal{A} = \{(Z, g) \mid\ g : Z \to \mathbb{R}$ is a linear extension of $f,\ g(x) \leq p(x)$ on $Z\}$. Let us endow $\mathcal{A}$ with the partial order relation $\preceq$ defined by

$$(Z_1, g_1) \preceq (Z_2, g_2) \text{ if } Z_1 \subset Z_2 \text{ and } g_1(x) = g_2(x) \text{ for all } \ x \in Z_1.$$

Let $\mathcal{C}$ be a totally ordered subset of $\mathcal{A}$. Then an upper bound (Z_0, g_0) of $\mathcal{C}$ is defined by

$$Z_0 = \cup_{(Z,g)\in\mathcal{C}} Z \quad \text{ and } \ g_0(x) = g(x) \text{ for all } x \in Z,\ (Z, g) \in \mathcal{C}.$$

By Zorn's lemma, there exists a maximal element (Z, g) of $\mathcal{A}$. In order that Z has our desired properties, it suffices that $Z = X$. We will prove it by contradiction, assuming that there exists $x_0 \in X \setminus Z$.
Let $X_1 = \{x + tx_0 : x \in Z,\ t \in \mathbb{R}\}$. Define $f_1 : X_1 \to \mathbb{R}$ by

$$f_1(x + tx_0) = g(x) + t\alpha,\ x \in Z,\ t \in \mathbb{R},$$

where $\alpha = \inf_{x\in Z} \left(p(x + x_0) - g(x)\right)$.

This α is finite because $p(x + x_0) - g(x) \geq p(x) - p(-x_0) - g(x) \geq -p(-x_0)$, for all $x \in Z$. Let us show that $f_1(x) \leq p(x)$ for all $x \in X_1$. For $t > 0$ and $x \in Z$,

$$\begin{aligned} f_1(x + tx_0) = g(x) + t\alpha \ &\leq\ g(x) + t\left(p\left(\frac{x}{t} + x_0\right) - g\left(\frac{x}{t}\right)\right) \\ &=\ p(x + tx_0) \end{aligned}$$

Let us now bound $f_1(x - tx_0)$ for $t > 0$. For all $x, y \in Z$,

$$g(x) + g(y) = g(x + y) \leq p(x + y) \leq p(x + x_0) + p(y - x_0)$$

and so for all $y \in Z$,

$$-\alpha = \sup_{x\in Z} \left(g(x) - p(x + x_0)\right) \leq p(y - x_0) - g(y)$$

For $z \in Z$, we get

$$\begin{aligned} f_1(z - tx_0) = g(z) - t\alpha &\leq g(z) + t\left(p\left(\frac{z}{t} - x_0\right) - g\left(\frac{z}{t}\right)\right) \\ &= p(z - tx_0) \end{aligned}$$

Finally, for $t = 0$, we have $f_1(x) = g(x) \leq p(x)$ for all $x \in Z$. Thus, we have proved that $f_1(x) \leq p(x)$ for all $x \in X_1$. Then $(Z, g) \prec (X_1, f_1)$. This contradicts the maximality of (Z, g). Thus, $Z = X$ and g has our desired properties.

□

Using the Hahn-Banach theorem, we can get the following theorem.

THEOREM 2.6. *Let $(X, N, \wedge)$ be a fuzzy normed space and $x_0 \in X$. For all $\alpha \in (0, 1)$, there exists $f \in X^*$ such that $f(x_0) = \|x_0\|_\alpha$ and $|f(x)| \leq \|x\|_\alpha$ for all $x \in X$*

Proof. Let $Y = \{tx_0 : t \in \mathbb{R}\}$ and $\alpha \in (0, 1)$. Define $f_0 : Y \to \mathbb{R}$ by $f_0(tx_0) = t\|x_0\|_\alpha$, for all $t \in \mathbb{R}$. Then $f_0(x) \leq \|x\|_\alpha$ for all $x \in Y$. By the Hahn-Banach theorem, there exists a linear extension f of f_0 such that $f(x_0) = f_0(x_0) = \|x_0\|_\alpha$ and $f(x) \leq \|x\|_\alpha$ for all $x \in X$.

If $f(x) \geq 0$, $|f(x)| \leq \|x\|_\alpha$ and otherwise, $|f(x)| = f(-x) \leq \|-x\|_\alpha = \|x\|_\alpha$. Thus, $|f(x)| \leq \|x\|_\alpha$ for all $x \in X$. By Proposition 2.4, $f \in X^*$. □

COROLLARY 2.7. *Let $(X, N, \wedge)$ be a fuzzy normed space and $x \in X$. If $f(x) = 0$ for all $f \in X^*$, then $x = 0$.*

Proof. Let $\alpha \in (0, 1)$. Then there exists $f_\alpha \in X^*$ such that $f_\alpha(x) = \|x\|_\alpha$ by Theorem 2.6. Since $f_\alpha(x) = 0$, $\|x\|_\alpha = 0$ for all $\alpha \in (0, 1)$. This implies that $x = 0$, by (b) of Proposition 2.3. □

We now prove the weakly convergent limit of a sequence in a fuzzy normed space $(X, N, \wedge)$ is unique.

PROPOSITION 2.8. *Let $(X, N, \wedge)$ be a fuzzy normed space, $\{x_n\}$ a sequence in X and $x, y \in X$. If $\{x_n\}$ is weakly convergent to x and $\{x_n\}$ is weakly convergent to y, then $x = y$.*

Proof. Let $f \in X^*$ and $t > 0$. Then

$$\lim_{n\to\infty} N_s\left(f(x_n) - f(x), \frac{t}{2}\right) = \lim_{n\to\infty} N_s\left(f(x_n) - f(y), \frac{t}{2}\right) = 1$$

For all $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that

$$N_s\left(f(x_{n_k}) - f(x), \frac{t}{2}\right) > 1 - \frac{1}{k}$$

and

$$N_s\left(f(x_{n_k}) - f(y), \frac{t}{2}\right) > 1 - \frac{1}{k}.$$

This implies that

$$\begin{aligned} N_s(f(x) - f(y), t) &\geq N_s\left(f(x) - f(x_{n_k}), \frac{t}{2}\right) \wedge N_s\left(f(y) - f(x_{n_k}), \frac{t}{2}\right) \\ &> 1 - \frac{1}{k} \end{aligned}$$

This means that $N_s(f(x) - f(y), t) = 1$, for all $t > 0$. Thus $f(x) = f(y)$ for all $f \in X^*$. By Corollary 2.7, we have $x = y$. □

We introduce another definition of fuzzy norm introduced by T. Bag and S. K. Samanta [2].

Definition 2.9. Let X be a linear space over a field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Let N be a fuzzy subset of $X \times \mathbb{R}$. Then N is called a BS-fuzzy norm on X if for all $x, y \in X$ and $c \in \mathbb{F}$,

(N1) for all $t \le 0$, $N(x, t) = 0$,
(N2) (for all $t > 0$, $N(x, t) = 1$) if and only if $x = 0$,
(N3) for all $t > 0$ and $c \ne 0$, $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$,
(N4) for all $t, s \in \mathbb{R}$, $N(x + y, s + t) \ge \min\{N(x, t), N(y, s)\}$,
(N5) $N(x, \cdot)$ is non-decreasing function of $\mathbb{R}$ such that $\lim_{t\to\infty} N(x, t) = 1$.
The pair (X, N) will be referred to as a BS-fuzzy normed linear space.
(N6) If $N(x, t) > 0$ for all $t > 0$, then $x = 0$.
For a BS-fuzzy normed linear space (X, N) with (N6), define a function $\|\cdot\|_\alpha$ on X by

$$\|x\|_\alpha = \wedge\{t : N(x, t) \ge \alpha\}$$

for $\alpha \in (0, 1)$. Then $\|\cdot\|_\alpha$ is a norm on X [2]. We call these norms as α-norms on X corresponding to the BS-fuzzy norm N on X.
(N7) For $x \ne 0$, $N(x, \cdot)$ is a continuous function of $\mathbb{R}$ and strictly increasing on $\{t : 0 < N(x, t) < 1\}$.

The following proposition is found in [4].

Proposition 2.10. *Let $\{x_n\}$ be a sequence in a BS-fuzzy normed space (X, N) satisfying (N6) and let $x \in X$. Then $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$ if and only if $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$ for all $\alpha \in (0, 1)$, where $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of α-norms on X corresponding to the BS-fuzzy norm N on X.*

Similarly, we get the following lemma in case of fuzzy normed spaces introduced by C. Alegre and S. Romaguera [1]. We note that $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of seminorms on X corresponding to the fuzzy norm N on X, by (a) of Proposition 2.3.

Lemma 2.11. *Let $\{x_n\}$ be a sequence in a fuzzy normed space $(X, N, \wedge)$ and let $x \in X$. $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$ if and only if $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$ for all $\alpha \in (0, 1)$, where $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of seminorms on X corresponding to the fuzzy norm N on X.*

Remark 2.12. The limit of any convergent sequence in $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ is unique, by (b) of Proposition 2.3.

We now show that convergence implies weak convergence in fuzzy normed spaces.

Theorem 2.13. *Let $(X, N, \wedge)$ be a fuzzy normed space, $\{x_n\}$ a sequence in X and $x \in X$. If $\{x_n\}$ is convergent to $x \in X$ then $\{x_n\}$ is weakly convergent to $x \in X$.*

Proof. Suppose that $\{x_n\}$ is convergent to $x \in X$. Then for all $\alpha \in (0,1)$, $\|x_n - x\|_\alpha \to 0$ as $n \to \infty$, by Lemma 2.11. For $f \in X^*$,

$$N_s\left(f(x_n) - f(x), t\right) = \frac{t}{t + |f(x_n - x)|} \geq \frac{t}{t + M\|x_n - x\|_\alpha},$$

for some $\alpha \in (0,1)$ and $M > 0$, by Proposition 2.4. Since $\|x_n - x\|_\alpha \to 0$ as $n \to \infty$, $N_s\left(f(x_n) - f(x), t\right) \to 1$ as $n \to \infty$. This completes the proof. □

The following theorem is found in [1].

THEOREM 2.14. *Let $\{\|\cdot\|_\alpha : \alpha \in (0,1)\}$ be an ascending family of separating extended seminorms on a real linear space X, and let $\|\cdot\|_0$ be given by $\|x\|_0 = 0$, for all $x \in X$. Then, the pair $(N, \wedge)$ is a weak fuzzy norm on X, where $N : X \times [0,\infty) \to [0,1]$ is given by $N(x,0) = 0$, for all $x \in X$, and*

$$N(x,t) = \sup\left\{\alpha \in [0,1) : \|x\|_\alpha < t\right\}$$

for all $x \in X$ and $y > 0$. $(N, \wedge)$ will be called the weak fuzzy norm induced by the seminorms $\{\|\cdot\|_\alpha : \alpha \in (0,1)\}$.

We now consider the converse of Theorem 2.13.

THEOREM 2.15. *Let $(X, \|\cdot\|)$ be a normed space and $(X', \|\cdot\|)$ a dual space of X. Let $(X, N_{01}, \wedge)$ be a 01-fuzzy normed space induced by $\|\cdot\|$ and X^*_{01} the set of all continuous linear mapping form $(X, \tau_{N_{01}})$ to $(\mathbb{R}, \tau_{N_s})$. Then $X^*_{01} = X'$ and $(X^*_{01}, N^*, \wedge)$ is a fuzzy normed space, where*

$$N^*(f,t) = \begin{cases} 1, & \|f\| < t \\ 0, & \|f\| \geq t, \quad \textit{for } t > 0. \end{cases}$$

Proof. For each $f \in X^*_{01}$, define $\|f\|^*_0 = 0$ and

$$\|f\|^*_\alpha = \sup\left\{|f(x)| : \|x\|_{1-\alpha} \leq 1\right\},$$

whenever $\alpha \in (0,1)$. Then $\{\|\cdot\|^*_\alpha : \alpha \in (0,1)\}$ is an ascending family of extended norms on X^*, by Proposition 18 of [1]. Since

$$\|x\|_{1-\alpha} = \inf\{t > 0 : N_{01}(x,t) \geq 1 - \alpha\} = \|x\|,$$

$\|f\|^*_\alpha = \|f\|$ for all $\alpha \in (0,1)$. This implies that $X^*_{01} = X'$. Define $N^* : X^*_{01} \to [0,1]$ by $N^*(f,0) = 0$ and

$$N^*(f,t) = \sup\{\alpha \in [0,1) : \|f\|^*_\alpha < t\}, \quad \text{for all } t > 0.$$

Then $(N^*, \wedge)$ is a weak fuzzy norm on X^*_{01}, by Theorem 2.14. Since $\|f\|^*_\alpha = \|f\|$ for all $\alpha \in (0,1)$,

$$N^*(f,t) = \begin{cases} 1, & \|f\| < t \\ 0, & \|f\| \geq t, \quad \text{for } t > 0. \end{cases}$$

Since for each $f \in X^*$, $\lim_{t\to\infty} N^*(f,t) = 1$, $(X^*_{01}, N^*, \wedge)$ is a fuzzy normed space. □

We now show that weak convergence in fuzzy normed spaces does not imply convergence, in general.

Example 2.16. Consider a fuzzy normed space $(l_2, N_{01}, \wedge)$, where $l_2 = \{x = (x_n) : x_n \in \mathbb{R}, \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ is a normed space with norm $\|x\|_2 = (\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}}$. We note that $(l_2)_{01}^* = l_2' = l_2$. For all $x = (x_n) \in l_2$, $< x, e_n >= x_n \to 0$ as $n \to \infty$, where $e_n = (0, \cdots, 0, 1, 0, \cdots)$ is a unit vector in l_2. Since

$$\begin{aligned} N_s(< x, e_n >, t) &= \frac{t}{t + < x, e_n >} \\ &= \frac{t}{t + x_n} \to 1 \text{ as } n \to \infty, \end{aligned}$$

$\{e_n\}$ is weakly convergent to 0 in $(l_2, N_{01}, \wedge)$. But for all $t > 0$,

$$\begin{aligned} N_{01}(e_n, t) &= \begin{cases} 1, & \|e_n\|_2 < t \\ 0, & \|e_n\|_2 \geq t \end{cases} \\ &= \begin{cases} 1, & t > 1 \\ 0, & t \leq 1 \end{cases}. \end{aligned}$$

This implies that $N_{01}(e_n, t) \nrightarrow 1$ for $t \leq 1$. Thus, $\{e_n\}$ is not convergent to 0 in $(l_2, N_{01}, \wedge)$.

In similar way with Theorem 2.15, we finally study the dual space of $(X, N_s, \wedge)$, where $(X, \|\cdot\|)$ is a normed space.

Theorem 2.17. *Let $(X, \|\cdot\|)$ be a normed space and $(X', \|\cdot\|)$ a dual space of X. Let $(X, N_s, \wedge)$ be a standard fuzzy normed space induced by $\|\cdot\|$ and X_s^* the set of all continuous linear mapping form (X, τ_{N_s}) to $(\mathbb{R}, \tau_{N_s})$. Then $X_s^* = X'$ and $(X_s^*, N^*, \wedge)$ is a fuzzy normed space, where*

$$N^*(f, t) = \frac{t}{t + \|f\|}, \quad \text{for } t > 0.$$

Proof. For each $f \in X_s^*$, define $\|f\|_0^* = 0$ and

$$\|f\|_\alpha^* = \sup\{|f(x)| : \|x\|_{1-\alpha} \leq 1\},$$

whenever $\alpha \in (0, 1)$. Then $\{\|\cdot\|_\alpha^* : \alpha \in (0, 1)\}$ is an ascending family of extended norms on X^*, by Proposition 18 of [1]. Since

$$\begin{aligned} \|x\|_{1-\alpha} &= \inf\{t > 0 : N_s(x, t) \geq 1 - \alpha\} \\ &= \inf\left\{t > 0 : \frac{t}{t + \|x\|} \geq 1 - \alpha\right\} = \frac{1 - \alpha}{\alpha}\|x\|, \end{aligned}$$

$\|f\|_\alpha^* = \sup\left\{|f(x)| : \|x\| \leq \frac{\alpha}{1-\alpha}\right\} = \frac{\alpha}{1-\alpha}\|f\|$ for all $\alpha \in (0, 1)$. This implies that $X_s^* = X'$. Define $N^* : X_s^* \to [0, 1]$ by $N^*(f, 0) = 0$ and

$$N^*(f, t) = \sup\{\alpha \in [0, 1) : \|f\|_\alpha^* < t\}, \quad \text{for all } t > 0.$$

Then $(N^*, \wedge)$ is a weak fuzzy norm on X_s^*, by Theorem 2.14. Since $\|f\|_\alpha^* = \frac{\alpha}{1-\alpha}\|f\|$ for all $\alpha \in (0, 1)$,

$$N^*(f, t) = \frac{t}{t + \|f\|}, \quad \text{for } t > 0.$$

Since for each $f \in X_s^* = X'$, $\lim_{t\to\infty} N^*(f, t) = 1$, $(X_s^*, N^*, \wedge)$ is a fuzzy normed space. □

Acknowledgment. The author is grateful to the referee for very valuable comments and suggestions.

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Kyugeun Cho
Bangmok College of General Education, Myong Ji University
Yong-In 17058, Korea
E-mail: kgjo@mju.ac.kr