

## ON SOME COPSON-TYPE INTEGRAL INEQUALITY

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ABSTRACT. In this paper, we give some new Copson-type integral inequality with a sharp constant.

### 1. Introduction

In 1976, E.T. Copson proved the following inequalities [5, Theorem1, Theorem2] and [5, Theorem3, Theorem4].

Let  $f$ ,  $\phi$  non-negative measurable functions on  $[0, \infty)$ ,

$$\Phi(x) = \int_0^x \phi(t)dt, \quad F(x) = \begin{cases} \int_0^x f(t)\phi(t)dt & \text{for } c > 1, \\ \int_x^\infty f(t)\phi(t)dt & \text{for } c < 1, \end{cases}$$

then for  $k \geq 1$ ,

$$(1) \quad \int_0^b F^k(x)\Phi^{-c}(x)\phi(x)dx \leq \left(\frac{k}{|c-1|}\right)^k \int_0^b f^k(x)\Phi^{k-c}(x)\phi(x)dx,$$

for  $0 < k \leq 1$ ,

$$(2) \quad \int_0^b F^k(x)\Phi^{-c}(x)\phi(x)dx \geq \left(\frac{k}{|c-1|}\right)^k \int_0^b f^k(x)\Phi^{k-c}(x)\phi(x)dx.$$

Inequalities (1) and (2) can be easily rewritten in the following forms

$$F(x) = \begin{cases} \int_0^x f(t)\phi(t)dt & \text{for } \alpha < k-1, \\ \int_x^\infty f(t)\phi(t)dt & \text{for } \alpha > k-1, \end{cases}$$

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then for  $k \geq 1$ ,

$$(3) \quad \int_0^b F^k(x) \Phi^{\alpha-k}(x) \phi(x) dx \leq \left( \frac{k}{|k-1-\alpha|} \right)^k \int_0^b f^k(x) \Phi^\alpha(x) \phi(x) dx,$$

for  $0 < k \leq 1$ ,

$$(4) \quad \int_0^b F^k(x) \Phi^{\alpha-k}(x) \phi(x) dx \geq \left( \frac{k}{|k-1-\alpha|} \right)^k \int_0^b f^k(x) \Phi^\alpha(x) \phi(x) dx.$$

Copson's type inequalities have been studied by many authors during the twentieth century and has motivated some important lines of study which are currently active. Several papers have been appeared in the literature which deals with the simple proofs.

We give the reverse Hölder's inequality [1] which will be used frequently in the proof.

Let  $\Omega \subset \mathbb{R}^n$  be a measurable set and  $p < 0$ , we suppose that  $f, g$  are measurable on  $\Omega$ . If  $f \in L_p(\Omega)$  and  $g \in L_{p'}(\Omega)$  ( $p'$  is the conjugate parameter), then

$$(5) \quad \int_{\Omega} |f \cdot g| dt \geq \|f\|_{L_p} \|g\|_{L_{p'}}.$$

The objective of this paper is to obtain new types of the Copson-type integral inequality which will be useful in applications by using some elementary methods of analysis with negative parameter  $k < 0$ . We adopt the usual convention  $\frac{0}{0} = 0$ .

## 2. Main Results

Let  $0 < b \leq +\infty$ . Throughout the paper, we will assume that the functions are non-negative integrable and the integrals throughout are assumed to exist and are finite (i.e, convergent).

LEMMA 2.1. *Let  $k < 0$ ,  $c < 1$  and  $f, \phi > 0$  be non-negative measurable functions on  $[0, +\infty)$ . Let*

$$\Phi(x) = \int_0^x \phi(t) dt, \quad F(x) = \int_0^x f(t) \phi(t) dt,$$

the inequality

$$(6) \quad \int_0^b F^k(x) \Phi^{-c}(x) \phi(x) dx \leq \left( \frac{k}{c-1} \right)^k \int_0^b f^k(x) \Phi^{k-c}(x) \phi(x) dx$$

hold if the right-hand side is finite.

*Proof.* Let

$$V(x) = F^k(x) \Phi^{1-c}(x),$$

we have

$$\frac{dV}{dx}(x) = kF^{k-1}(x)f(x)\phi(x)\Phi^{1-c}(x) + (1-c)F^k(x)\Phi^{-c}(x)\phi(x),$$

then

$$\begin{aligned} (1 - c) \int_0^b F^k(x) \Phi^{-c}(x) \phi(x) dx &= -k \int_0^b F^{k-1}(x) f(x) \Phi^{1-c}(x) \phi(x) dx + F^k(b) \Phi^{1-c}(b) \\ &\geq -k \int_0^b F^{k-1}(x) f(x) \Phi^{1-c}(x) \phi(x) dx. \end{aligned}$$

Using the reverse Hölder inequality for  $\frac{1}{k} + \frac{1}{k'} = 1$ , we deduce that

$$\begin{aligned} \int_0^b F^k(x) \Phi^{-c}(x) \phi(x) dx &\geq \frac{k}{c-1} \int_0^b F^{k-1}(x) f \Phi^{1-c}(x) \phi(x) dx \\ &= \frac{k}{c-1} \int_0^b \left( f(x) \Phi^{\frac{k-c}{k}}(x) \phi^{\frac{1}{k}}(x) \right) \times \left( F^{k-1}(x) \Phi^{\frac{-c}{k'}}(x) \phi^{\frac{1}{k'}}(x) \right) dx \\ &\geq \frac{k}{c-1} \left( \int_0^b f^k(x) \Phi^{k-c}(x) \phi(x) dx \right)^{\frac{1}{k}} \\ &\quad \times \left( \int_0^b F^k(x) \Phi^{-c}(x) \phi(x) dx \right)^{\frac{1}{k'}}, \end{aligned}$$

so

$$\left( \int_0^b F^k(x) \Phi^{-c}(x) \phi(x) dx \right)^{\frac{1}{k}} \geq \frac{k}{c-1} \left( \int_0^b f^k(x) \Phi^{k-c}(x) \phi(x) dx \right)^{\frac{1}{k}}.$$

Thus for  $k < 0$ , we get

$$\int_0^b F^k(x) \Phi^{-c}(x) \phi(x) dx \leq \left( \frac{k}{c-1} \right)^k \int_0^b f^k(x) \Phi^{k-c}(x) \phi(x) dx.$$

□

REMARK 2.2. The constant factor  $\left(\frac{k}{c-1}\right)^k$  is the best possible.

*Proof.* Putting  $\phi(x) = 1$  and for  $0 < \theta < 1 - c$ , we take

$$f_\theta(x) = \begin{cases} x^{\frac{c-1+\theta}{k}-1}, & x \in (0, 1] \\ 0, & x \in (1, b) \end{cases},$$

using the assumption on  $f_\theta$  in  $L_1$  the left-hand side of (6), we have

$$\begin{aligned} L_1 &= \int_0^b x^{-c} \left( \int_0^x f_\theta(t) dt \right)^k dx \\ &= \int_0^1 x^{-c} \left( \int_0^x t^{\frac{c-1+\theta}{k}-1} dt \right)^k dx \\ &= \left( \frac{k}{c-1+\theta} \right)^k \int_0^1 x^{\theta-1} dx \\ &= \left( \frac{k}{c-1+\theta} \right)^k \frac{1}{\theta}. \end{aligned}$$

Let  $R_1$  the right-hand side of (6), we get

$$\begin{aligned} R_1 &= \left(\frac{k}{c-1}\right)^k \int_0^b x^{-c} (x f_\theta(x))^k dx \\ &= \left(\frac{k}{c-1}\right)^k \int_0^1 x^{\theta-1} dx \\ &= \left(\frac{k}{c-1}\right)^k \frac{1}{\theta}. \end{aligned}$$

For  $\theta \rightarrow 0$ , we get the constant factor  $\left(\frac{k}{c-1}\right)^k$  is the best possible in (6).  $\square$

The following Corollary give us some particular cases.

**COROLLARY 2.3.** (i) For  $c = k$ , one has

$$(7) \quad \int_0^b F^k(x) \Phi^{-k}(x) \phi(x) dx \leq \left(\frac{k}{k-1}\right)^k \int_0^b f^k(x) \phi(x) dx.$$

(ii) For  $c = 0$ , one has

$$(8) \quad \int_0^b F^k(x) \phi(x) dx \leq (-k)^k \int_0^b f^k(x) \Phi^k(x) \phi(x) dx.$$

**LEMMA 2.4.** Let  $k < 0$ ,  $c > 1$  and  $f, \phi > 0$  be non-negative measurable functions on  $[0, +\infty)$ . Let

$$\Phi(x) = \int_0^x \phi(t) dt, \quad F(x) = \int_x^\infty f(t) \phi(t) dt,$$

the inequality

$$(9) \quad \int_0^b F^k(x) \Phi^{-c}(x) \phi(x) dx \leq \left(\frac{k}{1-c}\right)^k \int_0^b f^k(x) \Phi^{k-c}(x) \phi(x) dx$$

hold if the right-hand side is finite.

*Proof.* Let

$$U(x) = F^k(x) \Phi^{1-c}(x),$$

then

$$\frac{dU}{dx}(x) = -k F^{k-1}(x) f(x) \phi(x) \Phi^{1-c}(x) + (1-c) F^k(x) \Phi^{-c}(x) \phi(x) < 0,$$

we get

$$U(b) - U(0) < 0,$$

therefore

$$\begin{aligned} (1-c) \int_0^b F^k \Phi^{-c} \phi dx &= k \int_0^b F^{k-1} f \Phi^{1-c} \phi dx + U(b) - U(0) \\ &\leq k \int_0^b F^{k-1} f \Phi^{1-c} \phi dx. \end{aligned}$$

Using the reverse Hölder inequality for  $\frac{1}{k} + \frac{1}{k'} = 1$ , we deduce that

$$\begin{aligned} \int_0^b F^k \Phi^{-c} \phi \, dx &\geq \frac{k}{1-c} \int_0^b F^{k-1} f \Phi^{1-c} \phi \, dx \\ &\geq \frac{k}{1-c} \left( \int_0^b f^k \Phi^{k-c} \phi \, dx \right)^{\frac{1}{k}} \left( \int_0^b F^k \Phi^{-c} \phi \, dx \right)^{\frac{1}{k'}}, \end{aligned}$$

then

$$\left( \int_0^b F^k \Phi^{-c} \phi \, dx \right)^{\frac{1}{k}} \geq \frac{k}{1-c} \left( \int_0^b f^k \Phi^{k-c} \phi \, dx \right)^{\frac{1}{k}},$$

hence for  $k < 0$ , we get

$$\int_0^b F^k \Phi^{-c} \phi \, dx \leq \left( \frac{k}{1-c} \right)^k \int_0^b f^k \Phi^{k-c} \phi \, dx.$$

□

REMARK 2.5. The constant factor  $\left(\frac{k}{1-c}\right)^k$  is the best possible.

*Proof.* Putting  $\phi(x) = 1$  and for  $0 < \theta < c - 1$ , we take

$$f_\theta(x) = \begin{cases} x^{\frac{c-1-\theta}{k}-1}, & x \in [1, b) \\ 0, & x \in (0, 1) \end{cases}$$

using the assumption on  $f_\theta$  in  $L_2$  the left-hand side of (9), we get

$$\begin{aligned} L_2 &= \int_0^b x^{-c} \left( \int_x^\infty f_\theta(t) dt \right)^k dx \\ &= \int_1^b x^{-c} \left( \int_x^\infty t^{\frac{c-1-\theta}{k}-1} dt \right)^k dx \\ &= \left( \frac{k}{1-c+\theta} \right)^k \int_1^b x^{-\theta-1} dx. \end{aligned}$$

Let  $R_2$  the integral in right-hand side of (9), we have

$$\begin{aligned} R_2 &= \int_0^b x^{-c} (x f_\theta(x))^k dx \\ &= \int_1^b x^{-\theta-1} dx, \end{aligned}$$

for  $\theta \rightarrow 0$ , we get the constant factor  $\left(\frac{p}{1-c}\right)^p$  is the best possible in (9). □

THEOREM 2.6. Let  $k < 0$  and  $f, \phi > 0$  be non-negative measurable functions on  $[0, +\infty)$ . Let

$$\Phi(x) = \int_0^x \phi(t) dt, \quad F(x) = \begin{cases} \int_0^x f(t) \phi(t) dt & \text{for } c < 1, \\ \int_x^\infty f(t) \phi(t) dt & \text{for } c > 1, \end{cases}$$

then the inequality

$$(10) \quad \int_0^b F^k(x) \Phi^{-c}(x) \phi(x) dx \leq \left( \frac{k}{|c-1|} \right)^k \int_0^b f^k(x) \Phi^{k-c}(x) \phi(x) dx$$

hold if the right-hand side is finite.

*Proof.* By the inequalities (6) and (9), we have the inequality (10).  $\square$

The Theorem (2.6) can be easily rewritten in the following form.

**THEOREM 2.7.** Let  $k < 0$  and  $f, \phi > 0$  be non-negative measurable functions on  $[0, +\infty)$ . Let

$$\Phi(x) = \int_0^x \phi(t) dt, \quad F(x) = \begin{cases} \int_0^x f(t) \phi(t) dt & \text{for } \alpha > k - 1, \\ \int_x^\infty f(t) \phi(t) dt & \text{for } \alpha < k - 1, \end{cases}$$

then the inequality

$$(11) \quad \int_0^b F^k(x) \Phi^{\alpha-k}(x) \phi(x) dx \leq \left( \frac{k}{|k-1-\alpha|} \right)^k \int_0^b f^k(x) \Phi^\alpha(x) \phi(x) dx$$

hold if the right-hand side is finite.

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