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RESIDUAL FINITENESS AND ABELIAN SUBGROUP SEPARABILITY OF SOME HIGH DIMENSIONAL GRAPH MANIFOLDS

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Abstract. We generalize 3-manifolds supporting non-positively curved metric to construct manifolds which have the following properties : (1) They are not locally CAT(0). (2) Their fundamental groups are residually finite. (3) They have subgroup separability for some abelian subgroups.

1. Introduction

A subgroup H of a group G is *separable* if, given an element $g \in G \backslash H$, there exists a subgroup K of finite index in G such that $H \subset K$ and $g \notin K$. A group G is *residually finite* if the trivial subgroup is separable. Equivalently, G is residually finite if, given a nontrivial element g, there exists a finite group F and a homomorphism $\phi : G \to F$ such that $\phi(g)$ is nontrivial. A group G is *subgroup separable*, or *LERF (locally extended residually finite)* if every finitely generated subgroup of G is separable.

Residual finiteness and subgroup separability are interesting for a number of reasons. For example, if finitely generated residually finite group has the solvable word problem. In other words, there exists an algorithm to decide whether a given element represents the identity or not. Also a finitely generated residually finite group G is Hopfian, i.e., every epimorphism $G \to G$ is an isomorphism. If a finitely presented group is subgroup separable, then it has the solvable generalized word problem. See [8] for details. One can also find the connection of subgroup separability to geometric topology in [12].

Related to low dimensional topology, the following fundamental groups of manifolds are known to be residually finite and subgroup separable.: free groups ([5]), surface groups([12]), Seifert manifold groups([12]), and hyperbolic 3-manifold groups([1], [15]). Nevertheless, 3-manifold groups are much more complicated. Every Haken 3-manifold group is residually finite. But nontrivial graph manifold groups and fibered 3-manifold groups with some conditions are known to be non-LERF. See [13]. Finally, related to this paper, every Haken 3-manifold group is abelian subgroup separable, i.e., every (finitely generated) abelian subgroup is separable. See [6].

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In [4], Frigerio, Lafont and Sisto generalized 3-manifolds supporting nonpositively curved metric to introduce the notion of high dimensional graph manifolds (graph n-manifolds, in short). They established various rigidity results, including topological, smooth and quasi-isometric rigidities, inspired by corresponding results in the theory of nonpositively curved spaces and groups. They also studied many interesting algebraic and algorithmic properties of the fundamental groups of graph n-manifolds, mainly by analyzing the canonical graph of groups structure. Finally, they presented examples showing that their construction of the manifolds is more general than nonpositively curved 3-manifolds. A geodesic metric space X is CAT(0) if every geodesic triangle in X is thinner than the comparison triangle in the Euclidean plane having the same side lengths. By definition, every nonpositively curved manifold is locally CAT(0). See [3] for broad understanding about CAT(0) spaces and groups. Contrast to 3-manifolds theory, Frigerio, Lafont and Sisto proved that

Theorem 1.1. [4, Corollary 11.11] *In each dimension $n \geq 4$, there exist infinitely many closed irreducible graph n-manifolds not supporting any locally* CAT(0) *metric.*

The main goal of this paper is that the fundamental groups of some of the manifolds appeared in the theorem above are residually finite and have subgroup separability for some abelian subgroups. One can find the precise construction of graph n-manifolds not supporting locally CAT(0) metric in [4, Chapter 11]. More specifically, we analyze the canonical graph of groups structure of the fundamental group of a graph n-manifold to prove the following.

Theorem 1.2. *In each dimension $n \geq 4$, there exist infinitely many closed n-manifolds M, which generalize* 3*-manifolds supporting nonpositively curved metric, with following properties:*

1. *M is not locally* CAT(0).
2. *$\pi_1(M)$ is residually finite.*
3. *$\pi_1(M)$ has subgroup separability for some abelian subgroups.*

2. Construction

We briefly recall the notion of a graph of groups and introduce the construction of the manifolds for the main theorem. Then we prove the fundamental groups are residually finite and have subgroup separability for some abelian subgroups in the following sections. The author refers to readers Serre's book [14] for the details about a graph of groups.

A graph of groups $(\mathcal{G}, X)$ consists of a graph X, a vertex group G_v for each vertex v in X, an edge group G_e for each edge e in X, and a monomorphism $i_e : G_e \to G_v$ for the edge e with the initial vertex v. We also require that $G_e = G_{\overline{e}}$, where $\overline{e}$ is the edge e with the opposite direction. Once a base point $*$ in X is fixed, one defines the fundamental group $\pi_1(\mathcal{G}, X, *)$ of a graph of groups based at $*$ and it turns out that this definition is independent of the choice of a base point so that we just denote by $\pi_1(\mathcal{G}, X)$ (or $\pi_1(\mathcal{G})$, if the underlying graph is obvious). We denote $i_e(G_e)$ in G_v by G_e^- and $i_{\overline{e}}(G_e)$ by G_e^+.

Given a graph of groups $(\mathcal{G}, X)$, there exists a tree T, called the *Bass-Serre tree*, on which $\pi_1(\mathcal{G})$ acts by isometries. An element is called *elliptic* if it fixes a vertex in T and *hyperbolic*, otherwise. If $h \in \pi_1(\mathcal{G})$ acts hyperbolically on T, there exists an infinite geodesic γ in T on which h acts by nontrivial translation. Finally, for some $k > 0$, we say $\pi_1(\mathcal{G})$ is *k-acylindrical* if every element which pointwise fixes any path in T of length $\geq k$ is automatically trivial.

For $n \geq 3$, let N be a complete finite volume hyperbolic n-manifold with toric cusps. It is well known that each cusp supports a canonical smooth foliation by closed tori and defines a diffeomorphism between the cusp and $Y^{n-1} \times [0, \infty)$, where $Y^{n-1} = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$ is the standard torus. Note that we save the letter "T" for later use. Truncate the cusps of N to set $\overline{N} = N \setminus \bigcup (Y^{n-1} \times (4, \infty))$. Denote the cusps of $\overline{N}$ by $Y_1, \cdots, Y_k$.

THEOREM 2.1. [4, Corollary 11.5] *For the inclusion $i : \partial\overline{N} \to \overline{N}$, the induced map $i_* : H_1(\partial\overline{N}) \to H_1(\overline{N})$ is not injective. In other words, for each $i = 1, \cdots, k$, there exists $b_i \in H_1(Y_i)$ such that $0 \neq b_1 + \cdots + b_k \in H_1(Y_1) \oplus \cdots \oplus H_1(Y_k) = H_1(\partial\overline{N})$, but $i_*(b_1 + \cdots + b_k) = 0$ in $H_1(\overline{N})$.*

Let $V = \overline{N} \times S^1$, where S^1 is the standard circle. Note that, for each $i = 1, \cdots, k$ and the natural projection $p : V \to \overline{N}$, $p^{-1}(Y_i)$ is n-torus and denote by T_i. We obtain the closed manifold D^*V of dimension $n+1$ by gluing two pieces, say V^- and V^+, of V in the following manner. For each $i = 1, \cdots, k$, denote Y_i and T_i in V^- (V^+, respectively) by Y_i^- and T_i^- (Y_i^+ and T_i^+, respectively). In the theorem above,

- if $b_i \neq 0$ in $H_1(Y_i)$, glue T_i^+ and T_i^- via the affine diffeomorphism $\psi_i : T_i^+ \to T_i^-$ inducing $(\psi_i)_* : H_1(Y_i^+) \oplus H_1(S^1) \to H_1(Y_i^-) \oplus H_1(S^1)$ such that $(\psi_i)_*(v, 0) = (v, 0)$ for all $v \in H_1(Y_i^+)$ and $(\psi_i)_*(0, \lambda) = (b_i, \lambda)$, where λ is the positive generator of $H_1(S^1)$.
- if $b_i = 0$ in $H_1(Y_i)$, glue T_i^+ and T_i^- via the affine diffeomorphism $\psi_i : T_i^+ \to T_i^-$ inducing $(\psi_i)_* : H_1(Y_i^+) \oplus H_1(S^1) \to H_1(Y_i^-) \oplus H_1(S^1)$ such that $(\psi_i)_*(v, 0) = (v, 0)$ for all $v \in H_1(Y_i^+)$ and $(\psi_i)_*(0, \lambda) = (w_i, \lambda)$, where λ is the positive generator of $H_1(S^1)$ and w_i is any nonzero element in $H_1(Y_i^-)$.

For each case, $(\psi_i)_*$ can be written in the matrix form as follows :

$$\left(\begin{array}{c|c} \text{Id.} & b_i \\ \hline O & 1 \end{array}\right) \qquad \left(\begin{array}{c|c} \text{Id.} & w_i \\ \hline O & 1 \end{array}\right)$$

,where (Id.)-matrix is of size $(n-1) \times (n-1)$ and b_i and w_i are written as $(n-1) \times 1$ matrices.

In general, the fundamental group of a graph n-manifold has the canonical structure of the fundamental group of a graph of groups induced by the decomposition of the manifold into"pieces". We briefly explain the graph of groups structure only for $\pi_1(D^*V)$. See [4, Section 2.3] for the general case.

The decomposition of D^*V into two pieces $V^\pm$ induces on $\pi_1(D^*V)$ the fundamental group of a graph of groups. More precisely, let $(\mathcal{G}, X)$ be the graph of groups defined as follows : (1) the underlying graph X has two vertices and k- (undirected) edges. (2) the vertex groups are $G_{v^\pm} = \pi_1(V^\pm)$ and the edge groups are free abelian groups of rank n. (3) the homomorphism of an edge group into the adjacent vertex group is induced by the inclusion of $T_i^\pm$ into $V^\pm$. Note that, for each edge e with the initial vertex v^-, $i_e \circ i_{\bar{e}}^{-1} = (\psi_i)_*$. A basic property of the Bass-Serre theory implies that $\pi_1(D^*V)$ is isomorphic to $\pi_1(\mathcal{G}, X)$.

Theorem 2.2. [4, Corollary 11.11] *The closed manifold D^*V constructed as above has the following properties.*

1. *D^*V is not locally* CAT(0).
2. *$\pi_1(D^*V)$ is 3-acylindrical.*

Remark 2.3. In the construction, the requirement that w_i is nonzero is necessary for $\pi_1(D^*V)$ to be acylindrical and the acylindrical action of $\pi_1(D^*V)$ on the Bass-Serre tree will be crucial when we classify abelian subgroups of $\pi_1(D^*V)$.

We close the section with the following simple lemma.

Lemma 2.4. *Let $\phi : \mathbb{Z}^{n-1} \times \mathbb{Z} \to \mathbb{Z}^{n-1} \times \mathbb{Z}$ be the element in $\mathrm{GL}_n(\mathbb{Z})$ defined as follows : $\phi(v, 0) = (v, 0)$, $\forall v \in \mathbb{Z}^{n-1}$ and $\phi(0, 1) = (\alpha, 1)$ for some nonzero $\alpha \in \mathbb{Z}^{n-1}$. Let H be a subgroup of finite index in $\mathbb{Z}^{n-1}$. Then there exist infinitely many integers k such that $\phi(H \times k\mathbb{Z}) = H \times k\mathbb{Z}$.*

Proof. Since H is a subgroup of finite index in $\mathbb{Z}^{n-1}$, there exists a positive integer k such that $k\mathbb{Z}^{n-1} \subset H$. For $(h, l) \in H \times k\mathbb{Z}$, $\phi(h, l) = (h+l\alpha, l) \in H \times k\mathbb{Z}$. Therefore, $\phi(H \times k\mathbb{Z}) \subset H \times k\mathbb{Z}$. For the surjectivity, given $(h, l) \in H \times k\mathbb{Z}$, it can be easily checked that $\phi(h - l\alpha, l) = (h, l)$. In fact, any multiple of k has the same property. Therefore, there exist infinitely many integers k such that $\phi(H \times k\mathbb{Z}) = H \times k\mathbb{Z}$. □

3. Residual Finiteness

In [7], Hempel analyzed the graph of groups structure induced by JSJ decomposition to prove that every Haken 3-manifold group is residually finite. The main ingredient of the proof is the existence of the compatible collection of normal subgroups of finite index, each obtained from pieces appeared in JSJ decomposition of the manifold. Following Hempel's argument, we prove that $\pi_1(D^*V)$ is residually finite.

Definition 3.1. For a graph of groups $(\mathcal{G}, X)$, the collection of subgroups

$$\{H_v \leq G_v : v \in V(X)\},\ \{H_e \leq G_e : e \in E(X)\}$$

is called a *compatible collection of subgroups* if

$$i_e(G_e) \cap H_{i(e)} = i_e(H_e)$$

, where $V(X)(E(X)$, respectively) is the set of vertices (edges, respectively) in X and $i_e : G_e \to G_{i(e)}$ is the edge homomorphism from an edge group G_e to the initial vertex group $G_{i(e)}$.

For a graph of groups induced by the decomposition of a graph n-manifold M into "pieces", if we are given a compatible collection of normal subgroups of finite index, then it is an elementary fact from covering space theory that we can find a finite cover $\overline{M}$ of M with the property that $\pi_1(\overline{M})$ is the fundamental group of a graph of groups such that vertex groups and edge groups are the fundamental groups of elements in the collection. For the precise argument, see [7, Section 2], for example. Furthermore,

Theorem 3.2. [7, Theorem 3.1] *For a graph of groups $(\mathcal{G}, X)$, let $g \in G_{v_0} - \{1\}$ be a nontrivial element in some vertex group G_{v_0}. Suppose that there exists a compatible collection $\{H_v\}, \{H_e\}$ of subgroups each normal and of finite index in the* *corresponding* G_v, G_e *with* $g \notin H_{v_0}$. *Suppose further that if for some fixed* e_0 *with* $i(e_0) = v_0, g \notin i_{e_0}(G_{e_0})$, *we can choose the collection so that* $g \notin H_{v_0} \cdot i_{e_0}(G_{e_0})$. *Then there exists a subgroup* K *of finite index in* $\pi_1(\mathcal{G}, X)$ *such that* $g \notin K$.

THEOREM 3.3. $\pi_1(D^*V)$ *is residually finite.*

Proof. Let $g \in \pi_1(D^*V)$ be a nontrivial element. There are two possibilities for g depending how it acts on the Bass-Serre tree T.

1. Suppose that g is hyperbolic. In other words, g does not fix a vertex in T.
   Consider the natural projection $p : \pi_1(D^*V)(\simeq \pi_1(\mathcal{G}, X)) \to \pi_1(X)$. Since g is assumed to be hyperbolic, $p(g)$ is nontrivial. Note that $\pi_1(X)$ is free and the free group is residually finite. Therefore, there exists a normal subgroup H of finite index in $\pi_1(X)$ such that $p(g) \notin H$. It follows that $p^{-1}(H)$ is a subgroup of finite index in $\pi_1(D^*V)$ not containing g.
2. Suppose that g is elliptic. In other words, g fixes a vertex in T.
   By a basic property of the Bass-Serre theory, g is conjugate to an element in a vertex group. Without loss of generality, we assume $g \in G_{v^-}(= \pi_1(V^-))$. Note that $\pi_1(V^-) \simeq \pi_1(\overline{N}) \times \mathbb{Z}$. Write $g = (g_1, g_2) \in \pi_1(\overline{N}) \times \mathbb{Z}$. If g_1 is the identity in $\pi_1(\overline{N})$, choose any integer $l > 1$ such that $g_2 \notin l\mathbb{Z}$. Take
   $$H_{v^+} = H_{v^-} = \pi_1(\overline{N}) \times l\mathbb{Z}$$
   and, for each edge e,
   $$H_e = \mathbb{Z}^{n-1} \times l\mathbb{Z}.$$
   It can be easily verified that this choice of subgroups is the compatible collection of normal subgroups of finite index. By Theorem 3.2, there exists a subgroup K of finite index in $\pi_1(D^*V)$ not containing g.
   Suppose that g_1 is not the identity. By Malcev's lemma, $\pi_1(\overline{N})$, as a linear group, is residually finite. (See [9].) Therefore, there exists a normal subgroup H of finite index in $\pi_1(\overline{N})$ such that $g_1 \notin H$. For each $i = 1, \cdots, k$, by Lemma 2.4, there exist infinitely many integers l_i such that $\psi_i((H \cap \pi_1(Y_i)) \times l_i\mathbb{Z}) = (H \cap \pi_1(Y_i)) \times l_i\mathbb{Z}$. Let l be the product of such l_i's and take
   $$H_{v^+} = H_{v^-} = H \times l\mathbb{Z}$$
   and for each edge e.
   $$H_e = (H \cap \mathbb{Z}^{n-1}) \times l\mathbb{Z}.$$
   By construction, this choice of subgroups is the compatible collection of normal subgroups of finite index. By Theorem 3.2, there exists a subgroup K of finite index in $\pi_1(\mathcal{G})$ not containing g.

□

4. Abelian Subgroup Separability

For $n \geq 4$, it has been proved in [13] that the fundamental group of noncompact arithmetic hyperbolic n-manifold is non-LERF. Since subgroup separability inherits into a subgroup, it is unlikely that $\pi_1(D^*V)$ is subgroup separable. As for abelian subgroup separability, Hamilton proved in [6] that, for Haken 3-manifolds and closed hyperbolic n-orbifolds, its fundamental group is abelian subgroup separable. In the case of finite volume hyperbolic manifolds, she also applied the same idea in [6] to prove that every infinite cyclic subgroup generated by a loxodromic element is separable. But it is unknown whether an abelian subgroup contained in a cusp subgroup is separable or not. We firstly classify abelian subgroups in $\pi_1(D^*V)$ and prove that an abelian subgroup H of $\pi_1(D^*V)$ is separable unless H is contained in $\pi_1(T_i)$.

Lemma 4.1. *Let $(\mathcal{G}, X)$ be a graph of groups and H be a nontrivial abelian subgroup in $\pi_1(\mathcal{G})$. Suppose $\pi_1(\mathcal{G})$-action on the Bass-Serre tree T is 3-acylindrical. Then H is either infinite cyclic generated by a hyperbolic element or H is conjugate to a subgroup of a vertex group.*

Proof. Note that, by the Bass-Serre theory, if H fixes a vertex in T, then H is conjugate to a subgroup of a vertex group. Suppose that H does not fix a vertex in T. At first, we prove that H has a hyperbolic element. If H consists only in elliptic elements, then, by [2, Proposition 3.7], there exists an infinite path in T, say with vertices $v_0, \cdots, v_n, \cdots$ such that $H_{v_i} \subset H_{v_{i+1}}$ for every $i \geq 0$ and $H = \cup H_{v_i}$. But $\pi_1(\mathcal{G})$-action on T is assumed 3-acylindrical. Therefore, H must be trivial, which is a contradiction. By [10, Section 2], there are two possibilities for H.

1. There exists an infinite geodesic γ in T such that H is a subgroup of the stabilizers of γ. Since H leaves γ invariant, we obtain the following short exact sequence:
$$0 \to H_\gamma \to H \to \mathrm{Isom}_H(\gamma) \to 0,$$
where H_γ is the subgroup of H which fixes γ pointwise and $\mathrm{Isom}_H(\gamma)$ is the image of H in the group of isometries of γ. Note that $\pi_1(\mathcal{G})$-action on T is 3-acylindrical. Therefore, H_γ is trivial and H is isomorphic to $\mathrm{Isom}_H(\gamma)$. On the other hand, $\mathrm{Isom}_H(\gamma)$ is a subgroup of the group of simplicial automorphisms of $\mathbb{R}$, which is either $1, \mathbb{Z}_2, \mathbb{Z}$ or D_∞(infinite dihedral). It follow that H is infinite cyclic (generated by a hyperbolic element).
2. H is a subgroup of $\mathrm{Stab}(\mathcal{E})$, where $\mathcal{E}$ is an end of T and $\mathrm{Stab}(\mathcal{E})$ is the stabilizers of $\mathcal{E}$. In this case, one can invoke the basic properties of the relative translation length to define a homomorphism $\alpha : \mathrm{Stab}(\mathcal{E}) \to \mathbb{Z}$ and obtain the following short exact sequence: (See [10, Lemma 4].)
$$0 \to H_0 \to H \to \mathbb{Z} \to 0,$$
where $H_0 = H \cap \ker(\alpha)$, which consists in elliptic elements. By the discussion above, H_0 is trivial, and therefore H is infinite cyclic (generated by a hyperbolic element).

□

Lemma 4.2. *Let B be an abelian subgroup in $\pi_1(\overline{N}) \times \mathbb{Z}$ and $\overline{B}$ the image of B under the natural projection $\rho : \pi_1(\overline{N}) \times \mathbb{Z} \to \pi_1(\overline{N})$. Suppose that $\overline{B}$ is nontrivial. Then $\overline{B}$ is either infinite cyclic generated by a loxodromic element or contained in $\pi_1(Y_i)$ for some i.*

Proof. Note that $\pi_1(\overline{N})$ acts by isometries on the hyperbolic space $\mathbb{H}^{n-1}$. Since every nonelementary discrete group of isometries of $\mathbb{H}^{n-1}$ has a non-abelian free group (See [11, Exercise 12.2.15]), $\overline{B}$ must be elementary. Furthermore, $\overline{B}$, as a subgroup of discrete torsion free subgroup of $\mathrm{Isom}(\mathbb{H}^{n-1})$, is torsion free, i.e., it does not contain any elliptic element. Therefore, $\overline{B}$ is contained in either infinite cyclic subgroup of $\pi_1(\overline{N})$ or $\pi_1(Y_i)$ for some i. □

Let H be a nontrivial abelian subgroup of $\pi_1(D^*V)$ and A be the maximal abelian subgroup containing H. Then, by Lemma 4.1 and Lemma 4.2, there are three types for A and H.

1. A does not fix a vertex in the Bass-Serre tree T. In this case, A is infinite cyclic and H is a subgroup of A.
2. A is conjugate to a subgroup of a vertex group. If we assume that A is contained in a vertex group, then, by Lemma 4.2 and the maximality of A, A is the product of an infinite cyclic group generated by a loxodromic element in $\pi_1(\overline{N})$ and $\mathbb{Z}$, and H is an abelian subgroup of A.
3. If we assume that A is contained in a vertex group, then, by Lemma 4.2 and the maximality of A, A is isomorphic to $\pi_1(T_i)$ for some i and H is an abelian subgroup of A.

THEOREM 4.3. *An abelian subgroup in $\pi_1(D^*V)$ is separable, unless it is of type* 3.

Proof. Let H be an abelian subgroup in $\pi_1(D^*V)$ and $g \notin H$ be an element in $\pi_1(D^*V)$. Since we proved that $\pi_1(D^*V)$ is residually finite, let us assume that H is nontrivial. Suppose that A is the maximal abelian subgroup containing H. By [6, Proposition 1], A is separable and therefore if $g \notin A$, then there exists a subgroup K of finite index such that $H \subset A \subset K$ and $g \notin K$. It suffices to consider the case that $g \in A \backslash H$.

Suppose that A does not fix a vertex in the Bass-Serre tree T. By Lemma 4.1, A is infinite cyclic generated by an element h which acts on T hyperbolically. Let $H = \langle h^m \rangle$ and $g = h^a$ with $m \nmid a$. Consider the natural projection $p : \pi_1(D^*V) \to \pi_1(X)$. Note that h is hyperbolic so that $p(h)$ is nontrivial. Since $\pi_1(X)$ is free and the free group is residually finite, there exist a finite group F and a homomorphism $\beta : \pi_1(X) \to F$ such that $\beta(p(h)^m)$ is not the identity. By precomposing with p, it follows that there exist a finite group F and a homomorphism $\phi : \pi_1(D^*V) \to F$ such that $\phi(h^m)$ is not the identity. Note that m divides the order of $\phi(h)$ in F. Let $K = \phi^{-1}\langle \phi(h^m) \rangle$. Then K is a subgroup of finite index in $\pi_1(D^*V)$, K contains H and, by construction, $g = h^a \notin K$.

Suppose that A fixes a vertex in T. Then A is conjugate to a subgroup of a vertex group. Without loss of generality, assume that A is contained in G_{v^-}. By the argument above (and we only consider type 2), A is a free abelian group of rank 2, which is the product of infinite cyclic generated by a loxodromic element and $\mathbb{Z}(= \pi_1(S^1))$, and H, as a subgroup of free abelian groups, has rank ≤ 2.

1. Suppose that H is of rank 1. Let $A = \langle h, \lambda \rangle \leq \pi_1(\overline{N}) \times \pi_1(S^1)(= G_{v^-})$, where h is a generator for an infinite cyclic group generated by a loxodromic element in $\pi_1(\overline{N})$ and λ is the positive generator for $\pi_1(S^1)$, $H = \langle h^a \lambda^b \rangle$ and $g = h^x \lambda^y$. Note that $a \neq 0$. (If $a = 0$, then H is a subgroup of the form $\{1\} \times H' \leq \pi_1(\overline{N}) \times \pi_1(S^1)$ and this is of type 3). Depending on whether a divides x and/or b divides y, we have several cases. Most of the cases are very straight forward so that the argument in Section 3 is applicable.

   Suppose that a does not divide x. Let $\{h_1, \cdots, h_{a-1}\}$ be a set of nontrivial coset representatives of $\langle h \rangle / \langle h^a \rangle$. By [6] , every abelian subgroup in $\pi_1(\overline{N})$ generated by a loxodromic element is separable. Therefore, there exists a subgroup K of finite index in $\pi_1(\overline{N})$ such that $\langle h^a \rangle \subset K$, but $K \cap \{h_1, \cdots, h_{a-1}\} = \emptyset$.

Then $K \cap \langle h \rangle = \langle h^a \rangle$. By using the method in Theorem 3.3, we obtain a subgroup $\overline{K}$ of finite index in $\pi_1(D^*V)$ with the property that $\overline{K} \cap G_{v^-} = K \times l\mathbb{Z}$ for some l. $\overline{K}$ contains a normal subgroup $\overline{K_0}$ of finite index in $\pi_1(D^*V)$ and it is obvious that the subgroup $H\overline{K_0}$ is of finite index in $\pi_1(D^*V)$ and contains H. In order to complete the case, it remains to show that $g \notin H\overline{K_0}$. For this, it suffices to prove that $g \notin H\overline{K}$, and moreover, by a basic property from the Bass-Serre theory, this is equivalent to proving that $g \notin H((K \times l\mathbb{Z}) \cap A)$. By construction, since $H = \langle h^a \lambda^b \rangle, K \cap \langle h \rangle = \langle h^a \rangle$ and $a \nmid x$, it follows that $g \notin H((K \times l\mathbb{Z}) \cap A)$, as required.

Suppose that x is divisible by a. Note that if $b = y = 0$, then $g \in H$.

- Suppose that $b = 0$ and $y \neq 0$. Choose any integer s_0 such that $\lambda^y \notin s_0\mathbb{Z} \leq \pi_1(S^1)$ and, as in Theorem 3.3, take

$$H_{v^+} = H_{v^-} = \pi_1(\overline{N}) \times s_0\mathbb{Z}$$

  and, for each edge e,

$$H_e = \mathbb{Z}^{n-1} \times s_0\mathbb{Z}.$$

  It can be easily verified that this choice of subgroups is the compatible collection of normal subgroups of finite index. By Theorem 3.2, there exists a subgroup K of finite index in $\pi_1(D^*V)$ such that $H \subset K$, but $g \notin K$.
- Suppose that $b \neq 0$ and $b \nmid y$. Replace $-b$, if $b < 0$ and take

$$H_{v^+} = H_{v^-} = \pi_1(\overline{N}) \times b\mathbb{Z}$$

  and, for each edge e,

$$H_e = \mathbb{Z}^{n-1} \times b\mathbb{Z}.$$

  It can be easily verified that this choice of subgroups is the compatible collection of normal subgroups of finite index. By Theorem 3.2, there exists a subgroup K of finite index in $\pi_1(D^*V)$ such that $H \subset K$, but $g \notin K$.

The remaining nontrivial case is when $b|y$ (and $a|x$). Since $a|x$ and $b|y$, $g = h^x\lambda^y$ can be written $g = h^{a_1}(h^a\lambda^b)^u$ for some integer $a_1 \neq 0$ and u. Since $\pi_1(\overline{N})$ is residually finite, there exists a normal subgroup K of finite index in $\pi_1(\overline{N})$ such that $h^{a_1} \notin K$. Suppose that $[\pi_1(\overline{N}) : K] = d$. Then, for any integer c, $h^{a_1dc+a_1} \notin K$. Recall that, in Theorem 3.3, we chose the integer l in order for the choice of subgroups to be compatible. In fact, it can be easily verified that any multiple of l has the same property. At this time, choose any integer l' such that l' is divisible by dba_1 and the following choice of subgroups is the collection of normal subgroups of finite index :

$$H_{v^+} = H_{v^-} = K \times l'\mathbb{Z}$$

and for each edge e.

$$H_e = (K \cap \mathbb{Z}^{n-1}) \times l'\mathbb{Z}.$$

As in Theorem 3.3, we can construct a subgroup $\overline{K}$ of finite index in $\pi_1(D^*V)$ with the property that $\overline{K} \cap G_{v^-} = K \times l'\mathbb{Z}$. Take the normal closure $\overline{K_0}$ of $\overline{K}$. Then $H\overline{K_0}$ is a subgroup of finite index in $\pi_1(D^*V)$ and it contains H. It remains to show that $g \notin H\overline{K_0}$. Furthermore, in order to do this, it suffices to show that $h^{a_1} \notin H((K \times l'\mathbb{Z}) \cap A)$. The generic element in $H((K \times l'\mathbb{Z}) \cap A)$ can be written as, for some integers m_1, m_2 and m_3,

$$(h^a \lambda^b)^{m_1}(h^{m_2}\lambda^{l'm_3}) = h^{am_1+m_2}\lambda^{bm_1+l'm_3}.$$

Therefore, in order for $h^{a_1} \in H((K \times l'\mathbb{Z}) \cap A)$, there exist m_1, m_2 and m_3 satisfying the following equations.:

$$am_1 + m_2 = a_1,\ bm_1 + l'm_3 = 0$$

By the choice of l', m_1 must be divisible by da_1, and therefore m_2 must be of the form $a_1dc' + a_1$ for some integer c'. But $h^{a_1dc'+a_1} \notin K$, by construction. Therefore, $h^{a_1} \notin H((K \times l'\mathbb{Z}) \cap A)$

2. Suppose that H is of rank 2. Write $g \in A\backslash H$ as $g = (g_1, g_2) \in \pi_1(\overline{N}) \times \pi_1(S^1)$. Let $p_1 : \pi_1(\overline{N}) \times \pi_1(S^1) \to \pi_1(\overline{N})$ and $p_2 : \pi_1(\overline{N}) \times \pi_1(S^1) \to \pi_1(S^1)$ be the natural projections. Since $g \notin H$, either $g_1 \notin p_1(H)$ or $g_2 \notin p_2(H)$.

Suppose that $g_1 \notin p_1(H)$. By [6], there exists a subgroup K of finite index in $\pi_1(\overline{N})$ such that $K \cap p_1(A) = p_1(H)$, but $g_1 \notin K$. By using the construction in Theorem 3.3, we obtain a subgroup $\overline{K}$ of finite index in $\pi_1(D^*V)$ with the property that $\overline{K} \cap G_{v^-} = K \times l\mathbb{Z}$ for some l. $\overline{K}$ contains a normal subgroup, say $\overline{K_0}$, of finite index. It is obvious that $H\overline{K_0}$ is of finite index in $\pi_1(D^*V)$ and contains H. Furthermore, by construction, $g \notin H\overline{K_0}$.

Suppose that $g_2 \notin p_2(H)$. Note that $p_2(H)$ is an infinite cyclic subgroup in $\pi_1(S^1)$, say $s\mathbb{Z}$. Take, as in Theorem 3.3,

$$H_{v^+} = H_{v^-} = \pi_1(\overline{N}) \times s\mathbb{Z}$$

and, for each edge e,

$$H_e = \mathbb{Z}^{n-1} \times s\mathbb{Z}.$$

It can be easily verified that this choice of subgroups is the compatible collection of normal subgroups of finite index. By Theorem 3.2, there exists a subgroup K of finite index in $\pi_1(D^*V)$. Furthermore, by construction, K contains H, but $g \notin K$.

□

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