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ON D-COMPACT TOPOLOGICAL SPACES

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Abstract. The aim of this work is to introduce for the first time the concept of D-set. This is done by defining a special type of cover called a D-cover. we present some results to study the properties of D-compact spaces and their relations with other topological spaces. Several examples are discussed to illustrate and support our main results. Our results extend and generalized many will known results in the literature.

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1. Introduction

In general topology, open sets play very important roles in defining a new type of sets and some important topological characteristics about these new concepts.

In 1982, Tong [17] introduced the notion of difference set (in shortly $D-$sets) by using open sets and used this notion to define and ivestigate a new separation axioms called D_i $(i = 0, 1, 2)$ spaces. Later, in 1997, Caldas [3] used semi open sets to define the concepts of $s - D_i$ $(i = 0, 1, 2)$. The implications of these new separation axioms among themselves and with the well known axioms D_i $(i = 0, 1, 2)$ are obtained. In 2001, Jafari, [8] used $p-$open sets to give the definitions of $p-D-$sets and use it to introduce the concepts of $p-D_i$ $(i = 0, 1, 2)$ spaces. The relation between these separation axioms are obtained. While in 2003, the topologists Caldas et. al. [4] defined $\alpha - D_i$ $(i = 0, 1, 2)$ spaces by using the notion of $\alpha-$open sets. In 2008, Ekici and Jafari [5] introduced the notions of $D-$sets, $DS-$sets, $D-$continuity and $DS-$continuity to obtain decompositions of continuous functions, $A-$continuous and $AB-$continuous functions. Also, properties of the classes of $D-$sets and $DS-$sets are discussed.

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Later, in 2009, Keskin and Noiri [10] introduced the concepts of $bD-$sets as a difference of two $b-$open sets and used it to obtain some weak separation axioms. They also, introduced the implications of these new separation axioms among themselves and with the well known axioms that we, mentioned earlier. In 2010, Balasubramanian [1] generalized the previously known postulates separation through the definition $g-D_i$ $(i=0,1,2)$ spaces . In 2011, Ilango et. al. [2] used $gpr-$sets to define $gpr-D_i$ $(i=0,1,2)$ spaces. In 2012, Sreeja and Janaki [16] introduced and investigated some weak separation axioms by using the notions of $\pi gb-$closed sets. They introduced a new generalized axiom called $\pi gb-$separation axioms and they incorporated $\pi gb-D_i$ spaces to characterize their fundamental properties. Gnanachandra and Thangavelu [7] used $pgpr-$open sets to introduce the notion of $pgpr-D-$sets as the difference of two $pgpr-$open sets and investigated their basic properties. They defined some weak separation axioms and studied some of their basic properties namely $pgpr-D_i$ $(i=0,1,2)$ spaces. The implications of these axioms among themselves and with the known axioms namely D_i , $p-D_i$, $s-D_i$, $\alpha-D_i$, $g-D_i$ and $gpr-D_i$ $(i=0,1,2)$ are discussed. Mustafa and Qoqazeh [12] introduced and investigated some weak separation axioms by using the notion of supra open sets. They studied the relationships between these new separation axioms and their relationships with some other properties. In 2017, Padma et. al. [14] introduced and investigated Q^*D-sets by using the notion of Q^*-open sets to obtain some new weak separation axiom namely Q^*D_i $(i=0,1,2)$ spaces. The relationships between existing $D-$sets namely D , $p-D$, $s-D$, $\alpha-D$, $g-D$ and $gpr-$sets and a new type of $D-$sets in topological spaces are explained. In 2018, Qoqazeh et. al. [15] introduced the concepts of pairwise D-metacompact spaces and studied their properties and their relations with other topological spaces. Vithya [18] introduced the notions and concepts of $b^\sharp D-$sets as a difference of two $b^\sharp-$open sets. Some weak separation axioms namely $b^\sharp-D_i$ $(i=0,1,2)$, $b^\sharp-R_0$, $b^\sharp-R_1$ are introduced and studied. Some lower separation axioms are characterized by using these separation axioms. In 2019, Sabiha and Abdul-Hady [11] introduced and characterized new types of soft sets in soft bitopological spaces, namely, soft $(1\ ,\ 2)^*$ $-$omega difference sets (briefly soft $(1\ ,\ 2)^*-\tilde{D}_\omega$-sets) and weak forms of soft $(1\ ,\ 2)^*$ $-$omega difference sets. Also, they used these soft sets to study a new types of soft separation axioms, namely, soft $(1\ ,\ 2)^*-\omega-\tilde{D}_j-$spaces, soft $(1\ ,\ 2)^*-\alpha-\omega-\tilde{D}_j-$spaces, soft $(1\ ,\ 2)^*-pre-\omega-\tilde{D}_j-$spaces, soft $(1\ ,\ 2)^*-b-\omega-\tilde{D}_j-$spaces, soft $(1\ ,\ 2)^*-\beta-\omega-\tilde{D}_j-$spaces, for $j=0,1,2$. Jardo [9] introduced the notions and concepts of $iD-$sets which are depend on $i-$open sets. He discussed the relationship between this sets and other types of sets. Many separation axioms are characterized by using this type of sets namely, $iD-(i=0\ ,1\ ,2)-$spaces.

In this work,first we introduce the notions of $D-$compact spaces by defining a special type of covers called a $D-$cover. Many new theories and illustrative examples is discussed.

Second, the main relationship between $D-$compact spaces and the compact spaces and many other topological spaces namely, $D-$lindlöf,$D-$countably compact spaces is presented.

Finally, some of the characteristics of the Cartesian product process between $D-$compact spaces and other spaces will be studied under extra conditions.

2. Preliminaries

Through out the paper, by (X, τ) and (Y, σ) or (X and Y) we always mean topological spaces on which no separation axioms are assumed, unless otherwise mentioned. The letters $\tau-$closure, $\tau-$interior of a set A will be denoted by $CL\,(A)$, $Int\,(A)$, respectively. The product of τ_1 and τ_2 will be denoted by $\tau_1 \times \tau_2$. Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ denote the set of all real numbers, integer numbers, natural numbers, and rational numbers, respectively. Let τ_{dis}, τ_{ind}, τ_u, τ_s, τ_{coc}, τ_{cof}, τ_l, τ_r denote to the discrete, the indiscrete usual, Sorgenfrey line, cocountable, cofinite, left-ray, and right-ray topologies, respectively. Also, ω_0 and ω_1 stand to the cardinal numbers of $\mathbb{Z}$ and $\mathbb{R}$, respectively.

3. D-Compact Spaces

In this section, we will introduce the concept of D-compactness in topological spaces, and provide some of their properties, and relate it to other spaces.

The following definitions will be used in the sequel.

Definition 3.1 (Tong [17])**.** A subset A of a topological space (X, τ) is called a $D-$set if there are two open sets U and V such that $U \neq X$ and $A = U - V$. In this case we say that A is a $D-$set generated by U and V.

Observe that every open set U different from X is a $D-$set if $A = U$ and $V = \phi$.

Remark 3.1. If X is the only open set in (X, τ), then τ is the indiscrete topology. In this case X called weakly $D-$set.

Remark 3.2. The converse of above definition need not be true as we see in the following example.

Example 3.2. (i) Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. Then $D = \{a, b\} - \{a\} = \{b\}$ is a $D-$set but not an open set.

(ii) In the topological space $(\mathbb{R}, \tau_{cof})$, $\{\{x\} : x \in \mathbb{R}\}$ is a collection of $D-$sets but not an open sets.

Definition 3.3. A cover $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$of a topological space (X, τ) is said to be $D-$cover if each D_α is a $D-$set for all $\alpha \in \Delta$.

It is clear that every open cover is a $D-$cover, but the converse needs not be true. In the topological space $(\mathbb{R}, \tau_{cof})$, $\{\{x\} : x \in \mathbb{R}\}$ is a $D-$cover that is not open cover.

Definition 3.4. A topological space (X, τ) is called $D-$compact if every $D-$cover of the space (X, τ) has a finite subcover.

Example 3.5. (i) Let $X = \mathbb{R}$ and $\tau = \{\phi, \mathbb{R}n, \{1\}, \mathbb{R} - \{1\}\}$. Then $(\mathbb{R}, \tau)$ a $D-$compact space.
(ii) Let $X = n\mathbb{R}$ and $\tau = (\mathbb{R}, \tau_u)$. Then $(\mathbb{R}, \tau)$ is not a $D-$compact space. Since $\forall\ n \in \mathbb{N}$ Given $U_n = (-n\ ,\ n)$ then the open cover $\tilde{U} = \{U_n : n \in \mathbb{N}\}$ is also a $D-$cover which has no finite subcover.

Theorem 3.6. *If X is a finite set , then (X, τ) is a $D-$compact space for any topology τ on X.*

Proof. Let $X = \{x_1\ ,\ x_2\ ,\ ...\ ,\ x_n\}$ be a finite set. Let $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$ be a $D-$cover of X. Now $\forall\ x_i \in X$, choose $D_i \in \tilde{D}$ such that $x_i \in D_i$. So $D^* = \{D_1\ ,\ D_2\ ,\ ...\ ,\ D_n\}$is a finite subcover of $\tilde{D}$ for X. So X is a $D-$compact. □

Remark 3.3. The intersection of any two $D-$sets is a $D-$set.

Proof. Let $D_1 = U_1 - V_1$ and $D_2 = U_2 - V_2$ be any two $D-$sets , then $D_1 \cap D_2 = (U_1 \cap U_2) - (V_1 \cup V_2)$ is a $D-$set. □

The following corollary is easy to prove using the method of mathematical induction:

Corollary 3.7. *In any topological space (X, τ) the finite intersection of $D-$sets is a $D-$set.*

Remark 3.4. In general the union of any collection of $D-$sets may not be a $D-$set.

Example 3.8. (i) Let $X = \{a, b\}$ and $\tau = \{\phi, X, \{a\}, \{b\}\}$. Then $D_1 = \{a\} - \phi$ and $D_1 = \{b\} - \phi$ are two $D-$sets. But $D_1 \cup D_2 = \{a\ , b\} = X$ is not a $D-$set. Since $X \neq U - V$ where U and V are two open sets and $X \neq U$.
(ii) Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{b\ , c\}\}$. Then $D_1 = \{a\} = \{a, c\} - \{b\ , c\}$ and $D_2 = \{b\} = \{b\ , c\} - \{c\}$ are two $D-$sets. But $D_1 \cup D_2 = \{a\ , b\}$is not a $D-$set. Since $\{a\ , b\} \neq U - V$ where U and V are two open sets and $X \neq U$.

Let us recall known definition and some important conclusions from it which will be used in the sequel :

Definition 3.9 (Sreeja and Jananki [12])**.** A space (X, τ) is said to be locally indiscrete if every open set is clopen.

It is clear that any discrete space (X, τ) is locally indiscrete. But the converse needs not be true as we will see in the following example.

Example 3.10. Let $X = \{1\ ,\ 2\ ,\ 3\}$ and $\tau = \{X\ ,\ \phi\ ,\ \{1\}\ , \{2, 3\}\}$. Then (X, τ) is locally indiscrete space but not discrete. Because $\{x\} \notin \tau$ when $x \in \{2, 3\}$.

Corollary 3.11. *In a locally indiscrete space every $D-$set is clopen.*

Proof. Let $D = U - V$, Where U and V be two open sets. Then they are clopen sets, so D is clopen since it is the difference of two clopen sets. □

Corollary 3.12. *In a locally indiscrete space the union of any collection of $D-$sets is a $D-$sets.*

Theorem 3.13. *Every $D-$compact space is compact.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of (X, τ). Then $\tilde{U}$ is a $D-$cover, so it has a finite subcover. Hence the result. □

The following example shows that the converse of the above theorem is not true in general.

Example 3.14. The topological space $(\mathbb{R}, \tau_{cof})$ is compact but not a $D-$compact . We know that any set of the form $\mathbb{R} - \{x\}, x \in \mathbb{R}$ is an open set in a topological space $(\mathbb{R}, \tau_{cof})$. Now let $U = \mathbb{R} - \{y\}$ and $V = \mathbb{R} - \{x\}$. Then $D = U - V = \{x\}$ is a $D-$set which is not open. So $\tilde{D} = \{\{x\} : x \in \mathbb{R}\}$ is a $D-$cover of $(\mathbb{R}, \tau_{cof})$ which has no finite subcover. If $\tilde{D} = \{\{x\} : x \in \mathbb{R}\}$ has a finite subcover $\{\{x_1\} , \{x_2\} , ... , \{x_n\}\}$, we have $\mathbb{R} \subseteq \bigcup_{i=1}^{n} \{x_i\}$, that means $\mathbb{R}$ is a finite set. Which is a contradiction.

The following example shows that the contrapositiv of the above theorem is true.

Example 3.15. The topological space $(\mathbb{R}, \tau_{l.r})$ is not compact , so it is not a $D-$compact .

The following theorem shows that the converse of the above theorem can be true under extra conditions.

Theorem 3.16. *Every locally indiscrete compact topological space (X, τ) is $D-$compact.*

Proof. Let $\tilde{D}$ be a $D-$cover of (X, τ). Then $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$, where D_α is a clopen set for each $\alpha \in \Delta$. So $\tilde{D}$ is an open cover of (X, τ). Since (X, τ) is compact , $\tilde{D}$ has a finite subcover. Hence the result. □

Example 3.17. (i) It is clear that the locally indiscrete topological space $(\mathbb{R}, \tau_{ind})$ is $D-$compact, since it is compact.
(ii) Let $X = \mathbb{R}$ and $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \{2\}, \{2\}\}$. Then (X, τ) is locally indiscrete compact topological space.Hence it is $D-$compact.

Theorem 3.18. *Let (X, τ) be a topological space and $A \subseteq X$, if D_α is a $D-$set in X , then $D_\alpha \cap A$ is a $D-$set in (A, τ_A) , where τ_A is the induced topology on A.*

Proof. Let D_α is a $D-$set in X , then $D_\alpha = U - V$. where U and V are two open sets in X and $U \neq X$. Now $D_\alpha \cap A = (U - V) \cap A = (U \cap A) - (V \cap A) = U_\alpha - V_\alpha$ is a $D-$set in (A, τ_A). Since $U_\alpha \in \tau_A$ and $V_\alpha \in \tau_A$. □

Theorem 3.19. *Let (X, τ) be a topological space and $A \subseteq X$, then (A, τ_A) is $D-$compact if and only if every cover of A by $D-$sets in X has a finite subcover.*

Proof. $\Longrightarrow$) Let (A, τ_A) be a $D-$compact space and $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$ be a $D-$cover of A by $D-$sets in X . Now $\forall\ \alpha \in \Delta$, let $D^*_\alpha = D_\alpha \cap A$. Then D^*_α is a $D-$sets in A. So $\tilde{D}^* = \{D^*_\alpha : \alpha \in \Delta\}$ is a $D-$cover of A by $D-$sets in A. Since (A, τ_A) is $D-$compact, $\tilde{D}^*$ has a finite subcover $\{D^*_{\alpha 1}\ ,\ D^*_{\alpha 2}\ ,\ ...\ ,\ D^*_{\alpha n}\ \}$ for A. Hence the family $\{D_{\alpha 1}\ ,\ D_{\alpha 2}\ ,\ ...\ ,\ D_{\alpha n}\ \}$ is a finite subcover of $\tilde{D}$ in X for A ,where $D^*_{\alpha i} = D_{\alpha i} \cap A$.Hence the result.

$\Longleftarrow$) Assume that any $D-$cover of A by $D-$sets in X has a finite subcover. Let $\tilde{A} = \{A_\alpha : \alpha \in \Delta\}$ be a $D-$cover of A by $D-$sets in A. So $\forall\ \alpha \in \Delta$, $\exists$ a $D-$sets D_α in X such that $A_\alpha = D_\alpha \cap A$. Now $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$ be a $D-$cover of A by $D-$sets in X. By assumption $\tilde{D}$ has a finite subcover $\{D_{\alpha 1}\ ,\ D_{\alpha 2}\ ,\ ...\ ,\ D_{\alpha n}\ \}$. Hence $\{A_{\alpha 1}\ ,\ A_{\alpha 2}\ ,\ ...\ ,\ A_{\alpha n}\ \}$ is a finite subcover of $\tilde{A}$ for A because $A_{\alpha i} \subseteq D_{\alpha i}$, $\forall\ \alpha \in \Delta$, $i = 1\ ,2\ ,\ ...\ ,\ n$. Hence the result. □

Since every open cover is a $D-$cover let use the following corollary:

Corollary 3.20. *If (A, τ_A) is a $D-$compact space then every open cover of A by open sets in X has a finite subcover.*

Corollary 3.21. *If every $D-$cover of A by $D-$sets in X has a finite subcover , then (A, τ_A) is compact .*

Proof. Since every $D-$compact space is compact , this corollary is a direct result of the second trend from the previous theory. □

Theorem 3.22. *Let (X, τ) be a topological space and β be a base for X. If (X, τ) is $D-$compact , then every $D-$cover generated by elements in β has a finite subcover.*

Proof. Assume that is a $D-$compact space and $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$ be a $D-$cover of X by $D-$sets generated by elements in β . So $\tilde{D}$ is also a $D-$cover of X generated by elements in τ,so it has a finite subcover. Hence the result. □

Corollary 3.23. *Let (X, τ) be a topological space and β be a base for X. If (X, τ) is $D-$compact , then every open cover generated by elements in β has a finite subcover.*

Proof. In the same way as in the previous theory , this corollary is easy to prove. □

Theorem 3.24. *Let (X, τ_1) ,(X, τ_2) be two topological spaces. If (X, τ_2) is a $D-$compact and $\tau_1 \subseteq \tau_2$, then (X, τ_1) is a $D-$compact.*

Proof. Let $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$ be a $D-$cover of (X, τ_1). Then $\tilde{D}$ is also a $D-$cover of (X, τ_2) , since $\tau_1 \subseteq \tau_2$. So it has a finite subcover. Hence the result. □

Remark 3.5. In a topological space (X, τ), the least upper bound topology of τ is the smallest topology defined on X that contains τ.

The following corollary can be proved easily.

Corollary 3.25. *The topological space (X, τ) is $D-$compact , if (X, τ) is $D-$compact, where τ is the least upper bound topology of τ.*

Theorem 3.26. *Every closed subspace of a $D-$compact space is $D-$compact.*

Proof. Let X be a $D-$compact space and A be a closed subset X . Let $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$ be a $D-$cover of A by $D-$sets in X , then $\tilde{D} \cup \{X - A\}$is a $D-$cover of X. So it has a finite subcover $\tilde{D}^*$ because X is $D-$compact. Now $\tilde{D}^* - \{X - A\}$ is a finite subcover of $\tilde{D}$ for A. Hence the result. □

Theorem 3.27. *Every closed subspace of a $D-$compact space is compact.*

Proof. Let X be a $D-$compact space and A be a closed subset X . Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of A by sets open in X , then $\tilde{U} \cup \{X - A\}$is an open cover of X. So it has a finite subcover $\tilde{U}^*$ because X is a $D-$compact. Now $\tilde{U}^* - \{X - A\}$ is a finite subcover of $\tilde{U}$ for A. Hence the result. □

The following definitions can be found in [17].

Definition 3.28. (i) A space (X, τ) is said to be D_0 if for any two distinct points x and y in X , there exists a $D-$set D_{xy} in X such that $x \in D_{xy}$ and $y \notin D_{xy}$ or $y \in D_{xy}$ and $x \notin D_{xy}$.
(ii) A space (X, τ) is said to be D_1 if for any two distinct points x and y in X , there exists a $D-$sets G and H in X such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.
(iii) A space (X, τ) is said to be D_2 if for any two distinct points x and y in X , there exists disjoint $D-$sets G and H in X such that $x \in G$ and $y \in H$.

Theorem 3.29. *Let K be a $D-$compact subset of a locally indiscrete D_2-space (X, τ) , then $\forall x \notin K$ there exists two $D-$sets D_x and D_y such that $x \in D_x$, $K \subseteq D_y$ and $D_x \cap D_y = \phi$.*

Proof. Let $x \in X - K$, then it is clear that $\forall\ y \in K$ we have $x \neq y$. Since X is a D_2-space , there exists two $D-$sets D_{1y} and D_{2y} such that $x \in D_{1y}$, $y \in D_{2y}$ and $D_{1y} \cap D_{2y} = \phi$. Now , let $\tilde{D} = \{D_{2y} : y \in K\}$, then $\tilde{D}$ is a $D-$cover of K. Since K is $D-$compact; $\tilde{D}$ has a finite subcover $\tilde{D}^* = \{D_{2y_1}\ ,\ D_{2y_2},\ ...\ ,\ D_{2y_n}\}$ .Now Let $D_y = \bigcup_{i=1}^{n} D_{2yi}$ and $D_x = \bigcap_{i=1}^{n} D_{1yi}$, then D_x and D_y are two $D-$sets such that $x \in D_x$, $K \subseteq D_y$ and $D_x \cap D_y = \phi$. □

The following theorem can be found in [13].

Theorem 3.30. *Let K be a compact subset of a T_2-space (X,τ) , then $\forall x \notin K$ there exists two open sets U_x and V_x such that $x \in U_x$, $K \subseteq V_x$ and $U_x \cap V_x = \phi$.*

Through this study, we will present to you strong and important results in $D-$compact spaces that deepen the concept of the above theory. So we give some theorems that illustrate the relation between compactness and $D-$compactness in T_2 and D_2 spaces.

Theorem 3.31. *Let K be a $D-$compact subset of a T_2-space (X,τ) , then $\forall x \notin K$ there exists two open sets U_x and V_x such that $x \in U_x$, $K \subseteq V_x$ and $U_x \cap V_x = \phi$.*

Proof. Let $x \in X - K$, then it is clear that $\forall\ y \in K$ we have $x \neq y$. Since X is a T_2-space , there exists two open sets U_y and V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$. Now , let $\tilde{V} = \{V_y : y \in K\}$, then $\tilde{V}$ is an open cover of K , hence it is a $D-$cover of K.. Since K is $D-$compact ; $\tilde{V}$ has a finite subcover $\tilde{V}^* = \{V_{y_1}\ ,\ V_{y_2},\ ...\ ,\ V_{y_n}\}$.Now Let $V_x = \bigcup\limits_{i=1}^{n} V_{yi}$ and $U_x = \bigcap\limits_{i=1}^{n} U_{yi}$, then U_x and V_x are two open sets such that $x \in U_x$, $K \subseteq V_x$ and $U_x \cap V_x = \phi$. To explain the last fact ; if $U_x \cap V_x \neq \phi$ then $U_x \cap V_{yk} \neq \phi$ for some $k \in \{1, 2, ..., n\}$. So $U_{yk} \cap V_{yk} \neq \phi$ since $U_x \subseteq U_{yk}$. Which is a contradiction. □

Theorem 3.32. *Any $D-$compact subset of a T_2-space (X,τ) is closed.*

Proof. Let X be a T_2-space and K be a $D-$compact subset of X. Let $x \in X - K$, then by a previous theorem there exists two open sets U_x and V_x such that $x \in U_x$, $K \subseteq V_x$ and $U_x \cap V_x = \phi$.Now $x \in U_x \subseteq X - V_x \subseteq X - K$. So $X - K$ is open , hence the result. □

Theorem 3.33. *Any $D-$compact subset of a locally indiscrete D_2-space (X,τ) is closed.*

Proof. Let X be a locally indiscrete D_2-space and K be a $D-$compact subset of X. Let $x \in X - K$, then by a previous theorem there exists two $D-$sets D_x and D_y such that $x \in D_x$, $K \subseteq D_y$ and $D_x \cap D_y = \phi$. Now $x \in D_x \subseteq X - D_y \subseteq X - K$. Since D_x is clopen then $X - K$ is open , hence the result. □

As we mentioned at the beginning of this section, we will now present to you some new definitions and general concepts related to them and their relationship to $D-$compact spaces without delving into the study of these concepts.

Definition 3.34. (i) A topological space (X,τ) is called $D-$lindlöf if every $D-$cover of the space (X,τ) has a countable subcover.

(ii) A topological space (X,τ) is called $D-$countably compact if every countable $D-$cover of the space (X,τ) has a finite subcover.

The following theories are very easy to prove.

Theorem 3.35. *(i) Every $D-$compact space (X,τ) is $D-$lindlöf.*
(ii) Every $D-$compact space (X,τ) is $D-$countably compact.

Theorem 3.36. *If X is a countable set , then (X,τ) is a $D-$lindlöf space for any topology τ on X.*

Theorem 3.37. *Every $D-$lindlöf $D-$countably compact space (X,τ) is $D-$compact.*

Theorem 3.38. *Every closed subspace of a $D-$lindlöf (resp. $D-$countably compact) space is $D-$lindlöf (resp. $D-$countably compact).*

Theorem 3.39. *The continuous image of a $D-$countably compact space is $D-$countably compact.*

4. Product of D-compact topological Spaces

In this section, we will consider all topological spaces are T_2 unless otherwise indicated.

Theorem 4.1. *The continuous image of a $D-$compact space is $D-$compact.*

Proof. Let $f : X \longrightarrow Y$ be a continuous on-to function and X is a $D-$compact space. Let $\tilde{D}_y = \{D_\alpha : \alpha \in \Delta\}$ be a $D-$cover of Y. Now $\tilde{D}_x = \left\{f^{-1}(D_\alpha) : \alpha \in \Delta\right\}$ is a $D-$cover of X. Since $\bigcup\limits_{\alpha\in\Delta} f^{-1}(D_\alpha) = f^{-1}\left(\bigcup\limits_{\alpha\in\Delta} D_\alpha\right) = f^{-1}(Y) = X$ and X is $D-$compact, $\tilde{D}_x$ has a finite subcover $\left\{f^{-1}(D_{\alpha 1})\ ,\ f^{-1}(D_{\alpha 2}),\ ...\ ,\ f^{-1}(D_{\alpha n})\right\}$. So $\{D_{\alpha 1}\ ,\ D_{\alpha 2},\ ...\ ,\ D_{\alpha n}\}$is a finite subcover of $\tilde{D}_y$ for Y. Hence the result. □

Definition 4.2. Let X and Y be two topological spaces. A function $f : X \longrightarrow Y$ is said to be $D-$irresolute function if the inverse image of any $D-$set in Y is an open set in X.

Theorem 4.3. *The $D-$irresoluteness image of a compact space is $D-$compact.*

Proof. Let $f : X \longrightarrow Y$ be a $D-$irresoluteness on-to function and X is a compact space. Let $\tilde{D}_y = \{D_\alpha : \alpha \in \Delta\}$ be a $D-$cover of Y. Since $f : X \longrightarrow Y$ is $D-$irresolute, then $\tilde{D}_x = \left\{f^{-1}(D_\alpha) : \alpha \in \Delta\right\}$ is an open cover of X because $\bigcup\limits_{\alpha\in\Delta} f^{-1}(D_\alpha) = f^{-1}\left(\bigcup\limits_{\alpha\in\Delta} D_\alpha\right) = f^{-1}(Y) = X$. Since X is compact, $\tilde{D}_x$ has a finite subcover $\left\{f^{-1}(D_{\alpha 1})\ ,\ f^{-1}(D_{\alpha 2}),\ ...\ ,\ f^{-1}(D_{\alpha n})\right\}$. So $\{D_{\alpha 1}\ ,\ D_{\alpha 2},\ ...\ ,\ D_{\alpha n}\}$is a finite subcover of $\tilde{D}_y$ for Y. Hence the result. □

Theorem 4.4. *Let $f : X \longrightarrow Y$ be a perfect function. If X is locally indiscrete space, then X is $D-$compact space if Y is so.*

Proof. Let $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$ be any $D-$cover of X. Now, since f is perfect, for every $y \in Y$ we have $f^{-1}(y)$ is compact subset of X. So there exist finite subsets Δ_y of Δ such that $f^{-1}(y) \subseteq \bigcup\limits_{\alpha\in\Delta_y} D_\alpha$. Now; $\bigcup\limits_{\alpha\in\Delta_y} D_\alpha$ is an open subset of X. Hence $O_y = Y - f\left(X - \bigcup_{\alpha\in\Delta_y} D_\alpha\right)$ is an open subset of Y containing y and $f^{-1}(O_y) \subseteq \bigcup_{\alpha\in\Delta_y} D_\alpha$. So, $\tilde{O} = \{O_y : y \in Y\}$ is an open cover of Y. Since Y is $D-$compact, $\tilde{O}$ has a finite subcover $\tilde{O}^* = \{O_{y_i}\}_{i=1}^{n}$ i.e. $Y = \bigcup_{i=1}^{n} O_{y_i}$. Thus, $X = f^{-1}(Y) = f^{-1}\left(\bigcup_{i=1}^{n} O_{y_i}\right) = \bigcup_{i=1}^{n} f^{-1}(O_{y_i})$ .Since $f^{-1}(O_{y_i})$ is subset of a union of a finite number of members of $\tilde{D}$. Hence X is covered by a finite members of $\tilde{D}$. Hence the result. □

Theorem 4.5. *Let $f : X \longrightarrow Y$ be a perfect function. Then X is compact space if Y is $D-$compact.*

Proof. In the same way as in the previous theory, this theory is easy to prove. □

For completeness, we recall the two well known definition and theorem in general topology which play an important role here after.

Definition 4.6 (Engleking [6])**.** Let (X, τ) and (Y, σ) be two topological spaces. Then the Cartesian product of (X, τ) and (Y, σ) is the topological space $(X \times Y, \tau \times \sigma)$.

Theorem 4.7 (Engleking [6])**.** *Let (X, τ) and (Y, σ) be two T_2 topological spaces and (X, τ) is compact , then the projection function $P_y : X \times Y \longrightarrow Y$ is a perfect function.*

Theorem 4.8. *Let (X, τ) and (Y, σ) be two Hausdorff locally indiscrete topological spaces such that X is compact and Y is a $D-$compact spaces. Then $X \times Y$ is a $D-$compact.*

Proof. We know that the projection function $P_y : X \times Y \longrightarrow Y$ is a perfect function. Since Y is $D-$compact , we conclude that $X \times Y$ is a $D-$compact. □

Corollary 4.9. *The product of a compact Hausdorff topological space and a locally indiscrete $D-$compact topological space is compact.*

Theorem 4.10. *Let (X, τ) and (Y, σ) be two T_2 topological spaces and (X, τ) is $D-$compact , then the projection function $P_y : X \times Y \longrightarrow Y$ is a perfect function.*

Proof. We know that the projection function $P_y : X \times Y \longrightarrow Y$ is a continuous function. Since $X \times \{y\} \simeq X$ and X is $D-$compact , we conclude that $X \times \{y\}$ is a $D-$compact. So for each $y \in Y$ we have $P_y^{-1}(y) = X \times \{y\}$ is $D-$compact , hence it is compact. Finally we show that P_y is closed. Let $y \in Y$ and $P_y^{-1}(y) = X \times \{y\} \subseteq U$ where U is an open subset in $X \times Y$.Now for each $x \in X$ there exists a basic open sets V_{yx} and U_x such that $x \in U_x$, $y \in V_{yx}$. Now , $\tilde{U} = \{U_x : x \in X\}$ is an open cover of X. Since X is a $D-$compact space

, $\tilde{U}$ has a finite subcover say $\{U_{xi}\}_{i=1}^{n}$. Let $O_y = \bigcap_{i=1}^{n} V_{yx_i}$, then O_y is an open set containing y and $P_y^{-1}(O_y) = X \times O_y \subseteq U$. Hence the result. □

Theorem 4.11. *Let (X,τ) and (Y,σ) be two $D-$compact T_2 locally indiscrete topological spaces then the product $X \times Y$ is $D-$compact.*

Proof. The projection function $P_y : X \times Y \longrightarrow Y$ is a perfect function. So by the previous theorem we obtain that $X \times Y$ is $D-$compact. □

Theorem 4.12. *The product space $\prod_{i=1}^{n} X_i$ is $D-$compact if and only if X_i is locally indiscrete $D-$compact. for all $i = 1, 2, ..., n$.*

Proof. $\Longrightarrow$) The projection function $P_k : \prod_{i=1}^{n} X_i \longrightarrow X_k$ is continuous onto function. Since $\prod_{i=1}^{n} X_i$ is $D-$compact then X_k is $D-$compact for all $i = 1, 2, ..., n$.

$\Longleftarrow$) Using the method of mathematical induction of proof it is easy to prove this direction of theory. □

Theorem 4.13. *Every continuous function from a $D-$compact space X on to a T_2 space Y is closed.*

Proof. Let X be a $D-$compact space,Y be a T_2 space and C be a closed subset of X. Then C is $D-$compact. So $f(C)$ is a $D-$compact subset in a T_2 space Y. Hence $f(C)$ is closed. □

Theorem 4.14. *Let $f : X \longrightarrow Y$ be a continuous function of a $D-$compact space X onto a T_2 space Y then $f(CL(A)) = CL(f(A))$.*

Proof. $\subseteq$) Since f is continuous function we have $f(CL(A)) \subseteq CL(f(A))$.

$\supseteq$) Since$CL(A)$ is a closed subset of a $D-$compact space X then $CL(A)$ is $D-$compact. Since f is continuous we have $f(CL(A))$ is $D-$compact subset in a T_2 space Y.Hence $f(CL(A))$ is closed. Now $f(A) \subseteq f(CL(A)) \Longrightarrow CL(f(A)) \subseteq f(CL(A))$. Hence the result. □

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