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ORDERED FUZZY FILTERS OF HEYTING ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT. In this paper the concept of ordered fuzzy filters is introduced in Heyting almost distributive lattices and the properties of these ordered fuzzy filters are studied. We characterized and proved a set of theorems of ordered fuzzy filters. Some topological properties of prime ordered fuzzy filters are also studied.

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1. Introduction

Swamy and Rao in [9] introduced the notion of an almost aistributive lattices(ADLs). An ADL $(L, \wedge, \vee, 0)$ satisfies all the axioms of distributive lattice, except possibly the commutativity of operations $\wedge$ and $\vee$. It is known that, in any ADL the commutativity of $\vee$ is equivalent to that of $\wedge$ and also to the right distributivity of $\vee$ over $\wedge$. In [4], Rao, Berhanu and Mani introduced the concept of Heyting almost distributive lattice as a generalization of Heyting algebra in the class of ADLs. In [8], Rao and Rao introduced the notions of ordered filters of Heyting almost distributive lattices. Basic properties of prime ordere filters of Heyting ADLs are also studied by Rao and Rao in [7]. Again, Rao and Rao in [6] studied some topological properties of Heyting ADLs. On the other hand, fuzzy set theory was introduced by Zadeh [13]. Later in 1971, Zadeh [14], defined a fuzzy ordering as a generalization of the concept of ordering. Next, fuzzy groups were studied by Rosenfield [5]. Many scholars have used this idea to different mathematical branches such as semi-group, ring, semi-ring, near-ring, lattice etc. For instance Yuan and Wu [12] introduced the notion of fuzzy sublattice and fuzzy ideals of lattice.

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More recently, Swamy, Raji and Teshale [10] introduced fuzzy ideals of ADLs. In adition to this Alaba and Alemayehu [1] studied e-fuzzy filters of MS-algebras also in [2] Addis studied fuzzy prime spectrum of C-algebras. Again, Swamy, Raji and Teshale [11] studied L-fuzzy filters of almost distributive lattices.

In this paper, the notion of ordered fuzzy filters is introduced in Heyting almost distributive lattices(HADLs). Some necessary and sufficient conditions are derived for a nonempty set of HADL to become an ordered fuzzy filter. Finally the topological properties of prime ordered fuzzy filters of Heyting almost distributive lattices are studied.

2. Preliminaries

In this section, we recall basic definitions and results which will be used in this article.

Definition 2.1. [9] An algebra $L = (L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following conditions for all a, b and $c \in L$:

(1) $0 \wedge a = 0$,
(2) $a \vee 0 = a$,
(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
(4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
(5) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,
(6) $(a \vee b) \wedge b = b$.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Definition 2.2. [9] If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$,
(2) $a \vee b = b \Leftrightarrow a \wedge b = a$,
(3) $\wedge$ is associative in L,
(4) $a \wedge b \wedge c = b \wedge a \wedge c$,
(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$,
(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
(8) $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ and $a \vee (b \wedge a) = a$,
(9) $a \leq a \vee b$ and $a \wedge b \leq b$,
(10) $a \wedge a = a$ and $a \vee a = a$,
(11) $0 \vee a = a$ and $a \wedge 0 = 0$,
(12) If $a \leq c, b \leq c$ then $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L, m \leq a \Rightarrow m = a$.

Theorem 2.3. [9] *Let L be an ADL and $m \in L$. Then the following are equivalent:*

(1) *m is maximal with respect to $\leq$,*
(2) *$m \vee a = m$, for all $a \in L$,*
(3) *$m \wedge a = a$, for all $a \in L$,*
(4) *$a \vee m$ is maximal, for all $a \in L$.*

Definition 2.4. [4] Let $(L, \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element m. Suppose $\rightarrow$ is abinary operation on L satisfying the following conditions:

(1) $a \rightarrow a = m$,
(2) $(a \rightarrow b) \wedge b = b$,
(3) $a \wedge (a \rightarrow b) = a \wedge b \wedge m$,
(4) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$,
(5) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$ for all $a, b, c \in L$. Then $(L, \vee, \wedge, 0, m)$ is called a Heyting ADL.

Definition 2.5. [4] Let $(L, \vee, \wedge, 0, m)$ and $(L^{'}, \vee, \wedge, 0^{'}, m^{'})$ be two HADLs. Then the mapping $f : L \rightarrow L^{'}$ is called a homomorphism of L into $L^{'}$ if for any $x, y \in L$, the following conditions hold:

(1) $f(x \wedge y) = f(x) \wedge f(y)$,
(2) $f(x \vee y) = f(x) \vee f(y)$,
(3) $f(x \rightarrow y) = f(x) \rightarrow f(y)$,
(4) $f(0) = 0^{'}$.

Lemma 2.6. [4] *Let m be a maximal element in a Heyting ADL L.Then for any $a, b, c \in L$, the following conditions hold:*

(1) *$b \wedge m \leq (a \rightarrow b) \wedge m$,*
(2) *$a \rightarrow (a \wedge c) = a \rightarrow c$,*
(3) *$a \wedge b \wedge m = a \wedge c \wedge m \Leftrightarrow (a \rightarrow b) \wedge m = (a \rightarrow c) \wedge m$,*
(4) *$a \wedge m \leq b \wedge m \Leftrightarrow (a \rightarrow b) \wedge m = m$,*
(5) *$a \wedge c \wedge m \leq b \wedge m \Leftrightarrow c \wedge m \leq (a \rightarrow b) \wedge m$,*
(6) *$a \wedge m \leq ((a \rightarrow b) \rightarrow b) \wedge m$,*
(7) *$a \wedge m \leq (b \rightarrow c) \wedge m \Leftrightarrow b \wedge m \leq (a \rightarrow c) \wedge m$,*
(8) *$(a \rightarrow (b \rightarrow c)) \wedge m = ((a \wedge b) \rightarrow c) \wedge m$,*
(9) *$((a \wedge b) \rightarrow c) \wedge m = ((b \wedge a) \rightarrow c) \wedge m$,*
(10) *$(a \rightarrow (b \rightarrow c)) \wedge m = (b \rightarrow (a \rightarrow c)) \wedge m$.*

Definition 2.7. [8] Let L be a HADL with a maximal element m. A nonempty subset F of L is called an ordered filter if it satisfies the following conditions for all $x, y \in L$:

(1) $x, y \in F$ implies $x \wedge y \in F$,
(2) $x \in F$ and $x \wedge m \leq y \wedge m$ implies $y \in F$.

Remember that, for any set S a function $\mu : S \longrightarrow ([0,1], \wedge, \vee)$ is called a fuzzy subset of S, where $[0,1]$ is a unit interval, $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$ for all $\alpha, \beta \in [0,1]$.

Definition 2.8. [11] Let λ be a fuzzy subset of an ADL L. For any $\alpha \in [0,1]$, we denote the level subset λ_α, i.e

$$\lambda_\alpha = \{x \in L : \alpha \leq \lambda(x)\}.$$

Swamy, Raji and Teshale [11] $\mu : L \longrightarrow L^{'}$, where L is an ADL and $L^{'}$ is a complete lattice satisfing infinite meet distiributive law. Now in our cases take $L^{'}$ as $[0,1]$.

λ is said to be a fuzzy filter of an ADL L if λ_α is a filter of L for all $\alpha \in L$.

Theorem 2.9. [11]

Let λ be a fuzzy subset of an ADL L. Then the following are equivalent to each other.

(1) *λ is a fuzzy filter of L,*
(2) *$\lambda(m) = 1$ for all maximal element m and $\lambda(x \wedge y) = \lambda(x) \wedge \lambda(y)$, for all $x, y \in L$,*
(3) *$\lambda(m) = 1$ for all maximal element m and $\lambda(x \vee y) \geq \lambda(x) \vee \lambda(y)$ and $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$, for all $x, y \in L$.*

We define the binary operations " + " and "." on all fuzzy subsets of an ADL L as: $(\mu + \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \vee b = x\}$ and $(\mu.\theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \wedge b = x\}$.

The intersection of fuzzy filters of L $(FF(L))$ is a fuzzy filter. However the union of fuzzy filters may not be fuzzy filter. The least upper bound of a fuzzy filters μ and θ of L is denoted as $\mu \vee \theta = \cap\{\sigma \in FF(L) : \mu \cup \theta \subseteq \sigma\}$. If μ and θ are fuzzy filters of L, then $\mu.\theta = \mu \vee \theta$ and $\mu + \theta = \mu \cap \theta$

In the next sections L stands for a Heyting ADL unless otherwise mentioned.

3. Ordered fuzzy filters of HADLs

In this section, the concept of ordered fuzzy filters is introduced in HADLs and these fuzzy filters are then characterized. A necessary and sufficient condition is derived for any nonempty set to become an ordered fuzzy filter. The homomorphic images of ordered fuzzy filters are studied.

Definition 3.1. Let L be a HADL with a maximal element m. A nonempty fuzzy subset μ of L is called an ordered fuzzy filter if it satisfies the following conditions for all $x, y \in L$:

(1) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$,
(2) $x \wedge m \leq y \wedge m$ implies $\mu(y) \geq \mu(x)$.

Theorem 3.2. *Let L be a HADL with a maximal element m. A nonempty fuzzy subset μ of L is an ordered fuzzy filter iff the level subset μ_α, $\alpha \in [0,1)$ is an ordered filter of L.*

Proof. Suppose μ be an ordered fuzzy filter of L. Let $x, y \in \mu_\alpha$. Then $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y) \geq \alpha$. This implies $x \wedge y \in \mu_\alpha$. Let $x \wedge m \leq y \wedge m$ and $x \in \mu_\alpha$. Then $\mu(y) \geq \mu(x) \geq \alpha$. This implies $y \in \mu_\alpha$.
Conversely, suppose that μ_α is an ordered filter of L. Let $\mu(x) \wedge \mu(y) = \alpha$. Then $\mu(x) \geq \alpha$ and $\mu(y) \geq \alpha$. This implies $x, y \in \mu_\alpha$ and so $x \wedge y \in \mu_\alpha$. Hence $\mu(x \wedge y) \geq \alpha = \mu(x) \wedge \mu(y)$. Let $x \wedge m \leq y \wedge m$ and $\mu(x) = \alpha$. Then $x \in \mu_\alpha$. This implies $y \in \mu_\alpha$. Hence $\mu(y) \geq \mu(x)$. □

Corollary 3.3. *A nonempty subset F of L is an ordered filter if and only if χ_F is an ordered fuzzy filter.*

Corollary 3.4. *Let μ be an ordered fuzzy filter and m be any maximal element of an HADL L. Then $\mu(m) = 1$.*

Proof. Let $\mu(x) = \alpha$ for any $x \in L$ and $\alpha \in [0,1]$. Since $x \wedge m \leq m = m \wedge m$, by Defination 3.1(2) $\mu(m) \geq \mu(x) = \alpha$, $\alpha \in [0,1]$, we have get $\mu(m) = 1$. □

The following Lemma can help us to understand the relation between a fuzzy filter and an ordered fuzzy filter of L.

Lemma 3.5. *Let m be a maximal element of L. Then every fuzzy filter of L is an ordered fuzzy filter.*

Proof. Suppoe that μ is a fuzzy filter of L, $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$ for any $x, y \in L$, and by Corollarly 3.4 , $\mu(m) = 1$. Let $x \wedge m \leq y \wedge m$ and $\mu(x) = \alpha, \alpha \in [0,1]$. Then $\mu(x \wedge m) \geq \mu(x) \wedge \mu(m) = \mu(x) \wedge 1 = \mu(x)$.
Also $y \wedge x \wedge m = y \wedge x \wedge m \wedge m = y \wedge m \wedge x \wedge m = x \wedge m$. Since μ is a fuzzy filters of L,

$$\begin{aligned}
\mu(y) &= \mu(y \vee (y \wedge x \wedge m)) \\
&\geq \mu(y) \vee \mu(y \wedge x \wedge m) \\
&\geq \mu(y \wedge x \wedge m) \\
&= \mu(x \wedge m) \\
&\geq \mu(x)
\end{aligned}$$

This implies every fuzzy filter of L is an ordered fuzzy filter. □

Example 3.6. The set $M_\circ$ of all maximal elements of L. $\chi_{M_\circ}$ is an ordered fuzzy filter.

Theorem 3.7. *Let m be a maximal element of L and μ a non-empty fuzzy subset of L. Then μ is an ordered fuzzy filter of L if and only if it satisfies the following properties.*

(1) $\mu(m) = 1$,

(2) $\mu(y) \geq \mu(x \to y) \wedge \mu(x)$ *for all* $x, y \in L$.

Proof. Suppose that μ is an ordered fuzzy filter of L. Then clearly $\mu(m) = 1$. Since $x \wedge y \wedge m = x \wedge (x \mapsto y)$ and $x \wedge y \wedge m \wedge m = x \wedge y \wedge m \leq y \wedge m$, $\mu(y) \geq \mu(x \wedge y \wedge m) = \mu(x \wedge (x \mapsto y)) \geq \mu(x) \wedge \mu(x \mapsto y)$.
Conversely, suppose (1) and (2) holds true. Since L is an HADL. Then for any $x, y \in L$, $y \mapsto (x \mapsto (x \wedge y)) = m$. Now by condition (2)

$$\mu(x \wedge y) \geq \mu(x) \wedge \mu(x \mapsto (x \wedge y))$$

and

$$\mu(x \mapsto (x \wedge y)) \geq \mu(y) \wedge \mu(y \mapsto (x \mapsto (x \wedge y))) = \mu(y) \wedge \mu(m) = \mu(y).$$

Thus by these two equations we have get

$$\mu(x \wedge y) \geq \mu(x) \wedge \mu(y).$$

Let $x \wedge m \leq y \wedge m$. Then $x \to y = m$. Now by condition (2), $\mu(y) \geq \mu(x \to y) \wedge \mu(x) = \mu(m) \wedge \mu(x) = \mu(x)$

Hence μ is an ordered fuzzy filter of L. □

Theorem 3.8. *Let m be a maximal element of L and μ be a non-empty fuzzy subset of L. Then μ is an ordered fuzzy filter if and only if $x \wedge m \leq (y \mapsto z) \wedge m$ implies $\mu(z) \geq \mu(x) \wedge \mu(y)$ for all $x, y, z \in L$.*

Proof. Suppose that μ is an ordered fuzzy filter of L. Let m be a maximal element of L and $x \wedge m \leq (y \mapsto z) \wedge m$. Then $x \mapsto (y \mapsto z) = m$. By Theorem 3.7

$$\mu(z) \geq \mu(y) \wedge \mu(y \mapsto z)$$

and

$$\mu(y \mapsto z) \geq \mu(x) \wedge \mu(x \mapsto (y \mapsto z)) = \mu(x) \wedge \mu(m) = \mu(x)$$

Thus we get

$$\mu(z) \geq \mu(y) \wedge \mu(y \mapsto z) \geq \mu(y) \wedge \mu(x) \wedge \mu(x \mapsto (y \mapsto z)) \geq \mu(y) \wedge \mu(x)$$

Conversely, suppose that $x \wedge m \leq (y \mapsto z) \wedge m$ implies $\mu(z) \geq \mu(x) \wedge \mu(y)$ for all $x, y, z \in L$. Let for any $x \in L$, $\mu(x) = \alpha,\ \alpha \in [0, 1]$ and $x \wedge m \leq m = m \wedge m$. This implies by the given conditions $\mu(m) \geq \mu(x) = \alpha,\ \forall \alpha \in [0, 1]$. Thus we have $\mu(m) = 1$. For any $x, y \in L$, $x \wedge m \leq ((x \mapsto y) \mapsto y) \wedge m$. This implies by the given conditions $\mu(y) \geq \mu(x) \wedge \mu(x \mapsto y)$. Hence μ is an ordered fuzzy filter. □

Corollary 3.9. *Let m be a maximal element of an HADL L and μ a nonempty fuzzy subset of L. Then μ is an ordered fuzzy filter if and only if $x \mapsto (y \mapsto z) = m$ implies $\mu(z) \geq \mu(x) \wedge \mu(y)$for all $x, y, z \in L$.*

Lemma 3.10. *Let m be an ordered fuzzy filter of L. Then $\mu(x \mapsto (y \mapsto z)) \geq \mu((x \mapsto y) \mapsto z)$ for all $x, y, z \in L$.*

Theorem 3.11. *Let μ be an ordered fuzzy filter of a HADL L with a maximal element m. Then $\mu(x \mapsto y) \geq \mu(a) \wedge \mu((x \mapsto a) \mapsto y)$ for all $x, y \in L$.*

Theorem 3.12. *Let* $(L, \vee, \wedge, \mapsto, 0, m)$ *and* $(L^{'}, \vee, \wedge, \mapsto, 0^{'}, m^{'})$ *be two HADLs and* $f : L \mapsto L^{'}$ *an onto mapping such that* $f(x \mapsto y) = f(x) \mapsto f(y)$ *for all* $x, y \in L$. *Then the following properties hold.*

(1) *If* μ *is an ordered fuzzy filter of* L, *then* $f(\mu)$ *is an ordered fuzzy filter of* $L^{'}$,

(2) *If* λ *is an ordered fuzzy filter of* $L^{'}$, *then* $f^{-1}(\lambda)$ *is an ordered fuzzy filter of* L.

Definition 3.13. Let m be a maximal element of L. For any ordered fuzzy filter μ of L and $a \in L$, the fuzzy subset μ^a is defined as follows:

$$\mu^a(x) = \mu((a \mapsto x) \wedge m), \forall x \in L$$

It is obvious that $\mu^0 = \chi_L$ and $\mu^m = \mu$.

Theorem 3.14. *Let* m *be a maximal element of* L. *If* μ *is an ordered fuzzy filter of* L *and* $a \in L$, *then* μ^a *is an ordered fuzzy filter of* L *containing* μ.

Proof. $\mu^a(m) = \mu((a \mapsto m) \wedge m) = \mu(m) = 1$

$$\begin{aligned} \mu^a(x) \wedge \mu^a(y) &= \mu((a \mapsto x) \wedge m) \wedge \mu((a \mapsto y) \wedge m) \\ &\leq \mu(((a \mapsto x) \wedge m) \wedge ((a \mapsto y) \wedge m))) \\ &= \mu((a \mapsto x) \wedge (a \mapsto y) \wedge m) \\ &= \mu((a \mapsto (x \wedge y) \wedge m) \\ &= \mu^a(x \wedge y) \end{aligned}$$

Let $x \wedge m \leq y \wedge m$. Then $a \mapsto (x \wedge m) \leq a \mapsto (y \wedge m)$

$$\begin{aligned} \mu^a(y) &= \mu((a \mapsto y) \wedge m) \\ &= \mu((a \mapsto y) \wedge (a \mapsto m)) \\ &= \mu(a \mapsto (y \wedge m)) \\ &\geq \mu(a \mapsto (x \wedge m)) \\ &= \mu((a \mapsto x) \wedge m)) \\ &= \mu^a(x) \end{aligned}$$

This implies μ^a is an ordered fuzzy filter of L.

$$\begin{aligned} \mu^a(x) &= \mu((a \mapsto x) \wedge m) \\ &\geq \mu(a \mapsto x) \wedge \mu(m) \\ &= \mu(a \mapsto x) \\ &\geq \mu(x) \text{ as } x \wedge m \leq (a \mapsto x) \wedge m \end{aligned}$$

This implies $\mu \subseteq \mu^a$. □

Lemma 3.15. *Let* μ, λ *be two ordered filters of* L. *Then for any* $a, b \in L$, *we have the following properties:*

(1) $a \leq b$ *implies* $\lambda^b \subseteq \lambda^a$,
(2) $\mu \subseteq \lambda$ *implies* $\mu^a \subseteq \lambda^a$,
(3) $\mu^a \cap \lambda^a = (\mu \cap \lambda)^a$,
(4) $\lambda^{a\vee b} = \lambda^a \cap \lambda^b = \lambda^{b\vee a}$,
(5) $\lambda^{a\wedge b} = \lambda^{b\wedge a}$.
(6) $\lambda^{a\wedge b} = (\lambda^a)^b = (\lambda^b)^a$,
(7) $(\lambda^a)^a = \lambda^a$,
(8) $\mu(a) = 1$ *if and only if* $\mu^a = \mu$

Proof. (1) Let $a \leq b$ for any $a, b \in L$. Then $(b \to x) \wedge m \leq (a \to x)$ for any $x \in L$. Now, $\lambda^b(x) = \lambda((b \to x) \wedge m) \leq \lambda((a \to x) \wedge m) = \lambda^a(x)$. Hence $\lambda^b \subseteq \lambda^a$.
(2) and (3) is straightforward.

$$\begin{aligned}(4)\ (\lambda^a \cap \lambda^b)(x) &= \lambda^a(x) \wedge \lambda^b(x)\\ &= \lambda((a \to x) \wedge m) \wedge \lambda((b \to x) \wedge m)\\ &= \lambda((a \to x) \wedge m \wedge (b \to x) \wedge m)\\ &= \lambda((a \to x) \wedge (b \to x) \wedge m)\\ &= \lambda(((a \vee b) \to x) \wedge m)\\ &= \lambda^{a\vee b}(x).\end{aligned}$$

Since $(\lambda^a \cap \lambda^b) = (\lambda^b \cap \lambda^a)$, $\lambda^{a\vee b} = \lambda^a \cap \lambda^b = \lambda^{b\vee a}$.

$$\begin{aligned}(5)\ \lambda^{a\wedge b}(x) &= \lambda(((a \wedge b) \to x) \wedge m)\\ &= \lambda(((b \wedge a) \to x) \wedge m) = \lambda^{b\wedge a}(x)\end{aligned}$$

$$\begin{aligned}(6)\ \lambda^{a\wedge b}(x) &= \lambda(((a \wedge b) \to x) \wedge m)\\ &= \lambda((a \to (b \to x)) \wedge m)\\ &= \lambda^a((b \to x) \wedge m)\\ &= (\lambda^a)^b.\end{aligned}$$

By (5) $\lambda^{a\wedge b} = (\lambda^b)^a$. Hence $\lambda^{a\wedge b} = (\lambda^a)^b = (\lambda^b)^a$

By (6), (7) is clear.

(8) Suppose that $\mu(a) = 1$.

$$\begin{aligned}\mu^a(x) &= \mu((a \to x) \wedge m)\\ &= \mu((a \to x) \wedge m) \wedge 1\\ &= \mu((a \to x) \wedge m) \wedge \mu(a)\\ &\leq \mu(a \wedge (a \to x) \wedge m)\\ &= \mu(a \wedge x \wedge m)\\ &= \mu(x), \quad \text{since } a \wedge x \wedge m \wedge m \leq a \wedge x \wedge m \leq x \wedge m\end{aligned}$$

This implies $\mu^a \subseteq \mu$. Clearly $\mu \subseteq \mu^a$. Hence $\mu^a = \mu$.
Conversely, suppose that $\mu \subseteq \mu^a$. Now $\mu(a) = \mu^a(a) = \mu((a \to a) \wedge m) = 1$. □

For any ordered fuzzy filter μ of a HADL L, $\overline{\mu}$ denotes the set of all ordered filters of the form $\mu^a,\ a \in L$. Then we have the following result.

Theorem 3.16. *Let μ be an ordered fuzzy filter of a HADL L. Then $\overline{\mu}$ forms a distributive lattice.*

Proof. For any $\mu^a, \mu^b \in \overline{\mu}$, by (4) $\mu^{a\vee b}$ is the infimum for μ^a and μ^b. Next, we prove that $\mu^{a\wedge b}$ is the supremum of μ^a and μ^b. By (1) $\mu^a,\ \mu^b \subseteq \mu^{a\wedge b}$. Let μ^c be any upper bound for both μ^a and μ^b.

$$\begin{aligned}
\mu^{a\wedge b}(x) &= \mu((a\wedge b)\to x)\wedge m) \\
&= \mu((a\to(b\to x))\wedge m) \\
&= \mu^a((b\to x)\wedge m) \\
&\leq \mu^c((b\to x)\wedge m) \\
&= \mu(((c\to(a\to x))\wedge m) \\
&= \mu(((b\to(c\to x))\wedge m) \\
&= \mu^b((c\to x)\wedge m) \\
&\leq \mu^c((c\to x)\wedge m) \\
&= \mu^c(x)
\end{aligned}$$

$\mu^{a\wedge b} = \sup\{\mu^a, \mu^b\}$. It is denoted by $\mu^{a\wedge b} = \mu^a \sqcup \mu^b$. Thus $(\overline{\mu}, \sqcup, \cap)$ is lattice. Finally it can be easily verified that $(\overline{\mu}, \sqcup, \cap)$ is a distributive lattice. □

Let μ be a nonempty fuzzy subset of L. Then the smallest ordered fuzzy filter containing μ is called the ordered fuzzy filter generated by μ and denoted by $<\mu>$. Then the following theorem explains about the description of $<\mu>$.

Theorem 3.17. *Let m be a maximal element of L. For any non-empty fuzzy subset μ of L, we have*

$$<\mu>(x) = \sup\{\mu(a_1)\wedge\mu(a_2)\wedge\ldots\wedge\mu(a_n) : ((a_1\wedge a_2\wedge\ldots\wedge a_n)\mapsto x)\wedge m = m\}.$$

Proof. Clearly $<\mu>(m) = 1$ and $\mu \subseteq <\mu>$.
$<\mu>(x)\wedge <\mu>(x\to y) = \sup\{\mu(a_1)\wedge\mu(a_2)\wedge\ldots\wedge\mu(a_n) : ((a_1\wedge a_2\wedge\ldots\wedge a_n)\mapsto x)\wedge m = m\}\wedge\sup\{\mu(b_1)\wedge\mu(b_2)\wedge\ldots\wedge\mu(b_n) : ((b_1\wedge b_2\wedge\ldots\wedge b_n)\mapsto(x\to y))\wedge m = m\} = \sup\{\mu(a_1)\wedge\mu(a_2)\wedge\ldots\wedge\mu(a_n)\wedge\mu(b_1)\wedge\mu(b_2)\wedge\ldots\wedge\mu(b_n) : ((a_1\wedge a_2\wedge\ldots\wedge a_n)\mapsto x)\wedge m = m,\ ((b_1\wedge b_2\wedge\ldots\wedge b_n)\mapsto(x\to y))\wedge m = m\} \leq \sup\{\mu(a_1)\wedge\mu(a_2)\wedge\ldots\wedge\mu(a_n)\wedge\mu(b_1)\wedge\mu(b_2)\wedge\ldots\wedge\mu(b_n) : ([(a_1\wedge a_2\wedge\ldots\wedge a_n)\wedge(b_1\wedge b_2\wedge\ldots\wedge b_n)]\mapsto y)\wedge m = m\} = <\mu>(y)$. Thus $<\mu>$ is an ordered fuzzy filter of L containing μ. Let ν be any ordered fuzzy filter of L containing μ. This implies $<\mu>\subseteq<\nu> = \nu$. Hence $<\mu>$ is the smallest ordered fuzzy filter containing μ. □

Corollary 3.18. *Let m be a maximal element of L. If μ is an ordered fuzzy filter of L and $a \in L$, then μ^a is the smallest ordered fuzzy filter containing μ and $\mu^a(a) = 1$.*

4. The space of prime ordered fuzzy filters

For each $x \in L$ and $\alpha \in [0,1)$ remember from [3] that, the fuzzy subset x_α of L given by:

$$x_\alpha(z) = \begin{cases} \alpha & if\ x = z \\ 0 & if\ \text{ otherwise} \end{cases}$$

for all $z \in L$ is called a fuzzy point of L. In this case x is called the support of x_α and α its value. For a fuzzy point x_α of L and a fuzzy subset μ of L we write $x_\alpha \in \mu(x_\alpha \subseteq \mu)$ to say that $\mu(x) \geq \alpha$.

Theorem 4.1. *Let $\alpha \in [0,1)$, μ be an ordered fuzzy filter and λ be a fuzzy ideal of L such that $\mu \cap \lambda \leq \alpha$. Then there exists a prime ordered fuzzy filter η of L such that $\mu \subseteq \eta$ and $\eta \cap \lambda \leq \alpha$.*

Proof. Put $\xi = \{\theta : \theta \text{ is an ordered fuzzy filter},\ \mu \subseteq \theta,\ \theta \cap \sigma \leq \alpha\}$. Clearly $\mu \in \xi$, $\xi \neq \emptyset$, $(\xi, \subseteq)$ is a poset. Let $Q = \{\mu_i : i \in \Omega\}$ be a chain in ξ. We prove that $\cup_{i\in\Omega}\mu_i \in \xi$. Clearly $(\cup_{i\in\Omega}\mu_i)(m) = 1$. For any $x, y \in L$,

$$\begin{aligned}(\cup_{i\in\Omega}\mu_i)(x) \wedge (\cup_{i\in\Omega}\mu_i)(y) &= \sup\{\mu_i(x) : i \in \Omega\} \wedge \sup\{\mu_j(y) : j \in \Omega\} \\ &= \sup\{\mu_i(x) \wedge \mu_j(y) : i, j \in \Omega\} \\ &\leq \sup\{(\mu_i \cup \mu_j)(x) \wedge (\mu_i \cup \mu_j)(y) : i, j \in \Omega\}\end{aligned}$$

Since Q is a chain, $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$. Without loss of generality, assume $\mu_j \subseteq \mu_i$. This implies $\mu_i \cup \mu_j = \mu_i$. This shows,

$$\begin{aligned}(\cup_{i\in\Omega}\mu_i)(x) \wedge (\cup_{i\in\Omega}\mu_i)(y) &\leq \sup\{\mu_i(x) \wedge \mu_i(y),\ i \in \Omega\} \\ &\leq \sup\{\mu_i(x \wedge y),\ i \in \Omega\} \\ &= (\cup_{i\in\Omega}\mu_i)(x \wedge y)\end{aligned}$$

Suppose that $x \wedge m \leq y \wedge m$. Now $(\cup_{i\in\Omega}\mu_i)(x) = \sup\{\mu_i(x) : i \in \Omega\} \leq \sup\{\mu_i(y) : i \in \Omega\} = (\cup_{i\in\Omega}\mu_i)(y)$. Hence $\cup_{i\in\Omega}\mu_i$ is an ordered fuzzy filter of L. Since $\mu_i \cap \sigma \leq \alpha$ for each $i \in \Omega$,

$$\begin{aligned}((\cup_{i\in\Omega}\mu_i) \cap \sigma)(x) &= (\cup_{i\in\Omega}\mu_i)(x) \wedge \sigma(x) \\ &= \sup\{\mu_i(x),\ i \in \Omega\} \wedge \sigma(x) \\ &= \sup\{\mu_i(x) \wedge \sigma(x),\ i \in \Omega\} \\ &= \sup\{(\mu_i \wedge \sigma)(x),\ i \in \Omega\} \leq \alpha\end{aligned}$$

Thus $(\cup_{i\in\Omega}\mu_i) \cap \sigma) \leq \alpha$. Hence $\cup_{i\in\Omega}\mu_i \in \xi$. By applying Zorn's Lemma, we get a maximal element, say ϕ, i.e, ϕ is an ordered fuzzy filter of L such that $\mu \subseteq \phi$ and $\phi \cap \sigma \leq \alpha$. Next, we show that ϕ is a prime ordered fuzzy filter of L. Assume that ϕ is not a prime ordered fuzzy filter.

For any $a, b \in L$, such that $\phi(a) \neq 1$ and $\phi(b) \neq 1$. This implies $\phi \subset \phi^a$ and $\phi \subset \phi^b$. Since ϕ is a maximal in ξ, we get $\phi^a, \phi^b \notin \xi$. This implies there exist

$x, y \in L$ such that $(\phi^a \cap \sigma)(x) > \alpha$ and $(\phi^b \cap \sigma)(y) > \alpha$. This implies $\phi^a(x) > \alpha$, $\sigma(x) > \alpha$, and $\phi^b(y) > \alpha$, $\sigma(y) > \alpha$. Hence $\sigma(x \vee y) > \alpha$ and

$$\begin{aligned}
(\phi^a \cap \phi^b)(x \vee y) &= \phi^{a\vee b}(x \vee y) \\
&= \phi(((a \vee b) \to (x \vee y)) \wedge m) \\
&= \phi(((a \to (x \vee y)) \wedge m) \wedge (b \to (x \vee y)) \wedge m) \\
&\geq \phi(((a \to (x \vee y)) \wedge m) \wedge \phi((b \to (x \vee y)) \wedge m) \\
&\geq \phi((a \to x) \wedge m \wedge (b \to y) \wedge m) \\
&= \phi^a(x) \wedge \phi^b(y) \\
&> \alpha
\end{aligned}$$

If $\phi^a \cap \phi^b \subseteq \phi$, then $\phi \cap \sigma > \alpha$, which is a contradiction. Therefore ϕ is a prime ordered fuzzy filter. ☐

Corollary 4.2. *Let μ be an ordered fuzzy filter and λ be a fuzzy ideal of L such that $\mu \cap \lambda = 0$. Then there exists a prime ordered fuzzy filter ϕ such that $\mu \subseteq \phi$ and $\phi \cap \lambda = 0$.*

Corollary 4.3. *Let $\alpha \in [0, 1)$, μ be an ordered fuzzy filter of L and $\mu(x) \leq \alpha$. Then there exists a prime ordered fuzzy filter θ of L such that $\mu \subseteq \theta$ and $\theta(x) \leq \alpha$.*

Proof. Put $\xi = \{\theta : \theta$ is an ordered fuzzy filter, $\mu \subseteq \theta$ and $\theta(x) \leq \alpha\}$. Clearly $\mu \in \xi$, $\xi \neq \emptyset$, and $(\xi, \subseteq)$ is a poset. Let $Q = \{\mu_i : i \in \Omega\}$ be a chain in ξ. We prove that $\cup_{i\in\Omega}\mu_i \in \xi$. By Theorem 4.1, $(\cup_{i\in\Omega}\mu_i)$ is an ordered fuzzy filte of L. Since $\mu_i \subseteq \theta$ for each $i \in \Omega$ and $\theta(x) \leq \alpha$.

$(\cup_{i\in\Omega}\mu_i)(x) = \sup\{\mu_i(x),\ i \in \Omega\} \leq \theta(x) \leq \alpha$.

Hence $\cup_{i\in\Omega}\mu_i \in \xi$. By applying Zorn's Lemma, we get a maximal element of ξ, say δ, i.e, ϕ is an ordered fuzzy filte such that $\mu \subseteq \phi$ and $\phi(x) \leq \alpha$. Next we show that ϕ is a prime ordered fuzzy filte of L. Assume that ϕ is not a prime ordered fuzzy filte. For any $a, b \in L$, such that $\phi(a) \neq 1$ and $\phi(b) \neq 1$. This implies $\phi \subset \phi^a$ and $\phi \subset \phi^b$. Since ϕ is a maximal in ξ, we get $\phi^a, \phi^b \notin \xi$. This implies $\phi^a(x) > \alpha$ and $\phi^b(x) > \alpha$. Now,

$$\begin{aligned}
(\phi^a \cap \phi^b)(x) &= \phi^{a\vee b}(x) \\
&= \phi(((a \vee b) \to x) \wedge m) \\
&= \phi(((a \to x) \wedge m) \wedge (b \to x) \wedge m) \\
&\geq \phi((a \to x) \wedge m) \wedge \phi((b \to y) \wedge m) \\
&= \phi^a(x) \wedge \phi^b(y) \\
&> \alpha
\end{aligned}$$

If $\phi^a \cap \phi^b \subseteq \phi$, then $\phi(x) > \alpha$, which is a contradiction. Therefore ϕ is a prime ordered fuzzy filter. ☐

Corollary 4.4. *For any ordered fuzzy filter μ of an L, we have $\mu = \cap\{\sigma : \sigma$ is a prime ordered fuzzy filter of L, $\mu \subseteq \sigma\}$.*

Corollary 4.5. *Then the intersection of all prime ordered fuzzy filters of L is equal to $\chi_{M_\circ}$.*

Let L be an HADL and X° denotes the set of all prime ordered fuzzy filters of L. For a fuzzy subset θ of L, define $H^\circ(\theta) = \{\mu \in X^\circ : \theta \subseteq \mu\}$, and $X^\circ(\theta) = \{\mu \in X^\circ : \theta \nsubseteq \mu\}$.

Theorem 4.6. *The collection $\mathcal{T} = \{X^\circ(\theta) : \theta$ is an ordered fuzzy filter of $L\}$ is a topology on X°.*

Proof. Consider the fuzzy subsets λ_1, λ_2 of L defined as : $\lambda_1(x) = 0$ and $\lambda_2(x) = 1$ for all $x \in L$. Clearly $[\lambda_1)$ and λ_2 are fuzzy filters of L. $[\lambda_1) \subseteq \mu$ for all $\mu \in X^\circ$, which is impossible. Thus $X^\circ([\lambda_1)) = \emptyset$. Since each $\mu \in X^\circ$ is non-constant, $\lambda_2 \nsubseteq \mu$ for all $\mu \in X^\circ$. Thus $X^\circ(\lambda_2) = X^\circ$. This implies $\emptyset, X^\circ \in \mathcal{T}$.

Also for any fuzzy filters λ_1 and λ_2 of L, we have $X^\circ(\lambda_1) \cap X^\circ(\lambda_2) = X^\circ(\lambda_1 \cap \lambda_2)$. This show that $\mathcal{T}$ is closed under finite intersections. Next, let $\{\lambda_i, i \in \Omega\}$ be any family of fuzzy filters of L. Now we prove that $\cup_{i\in\Omega} X^\circ(\lambda_i) = X^\circ([\cup_{i\in\Omega}\lambda_i))$. Let $\mu \in X^\circ([\cup_{i\in\Omega}\lambda_i))$, then $[\cup_{i\in\Omega}\lambda_i) \nsubseteq \mu$, which implies that $\lambda_i \nsubseteq \mu$ for some $i \in \Omega$. Otherwise if $\lambda_i \subseteq \mu$ for each $i \in \Omega$, it will be true that $[\cup_{i\in\Omega}\lambda_i) \subseteq \mu$. Thus $\mu \in \cup_{i\in\Omega} X^\circ(\lambda_i)$ whence $X^\circ([\cup_{i\in\Omega}\lambda_i)) \subseteq \cup_{i\in\Omega} X^\circ(\lambda_i)$. Clearly $\cup_{i\in\Omega} X^\circ(\lambda_i) \subseteq X^\circ([\cup_{i\in\Omega}\lambda_i))$. Hence $\cup_{i\in\Omega} X^\circ(\lambda_i) = X^\circ([\cup_{i\in\Omega}\lambda_i))$. Therefore, $\mathcal{T}$ is closed under arbitrary unions and hence, it is Topology on X°. □

Definition 4.7. The topological space $(X^\circ, \mathcal{T})$ is called the prime ordered fuzzy filter Spectrum of L and it is denoted by $F - Spac^\circ_F(L)$.

Lemma 4.8. *Let λ be a fuzzy subset of L. Then $X^\circ(\lambda) = X^\circ([\lambda))$.*

Proof. Since $\lambda \subseteq [\lambda)$, $X^\circ(\lambda) \subseteq X^\circ([\lambda))$. Let $\mu \in X^\circ([\lambda))$, then $\lambda \nsubseteq \mu$. Otherwise, if $\lambda \subseteq \mu$, then $[\lambda) \subseteq \mu$. Which is impossible. So that $\mu \in X^\circ(\lambda)$ and so $X^\circ(\lambda) = X^\circ([\lambda))$. □

Lemma 4.9. *For any fuzzy subsets λ and ν of L*

$$X^\circ(\lambda) = X^\circ(\nu) \Rightarrow [\lambda) = [\nu)$$

Proof. Prove that using a contradiction. Suppose if possible that $[\lambda) \neq [\nu)$ Then there exists $x \in L$ such that $[\lambda)(x) \geq [\nu)(x)$. Let say $[\nu)(x) = r$. Then by Corollary 4.3 there exists a prime ordered fuzzy filter θ of L such that $[\nu) \subseteq \theta$ and $\theta(x) = r < [\lambda)(x)$. So $\theta \in X^\circ(\lambda)$ and $\theta \notin X^\circ(\nu)$. Therefore $X^\circ(\lambda) \neq X^\circ(\nu)$ and this completes the proof. □

Lemma 4.10. *Let $x, y \in L$, and $\alpha \in (0, 1]$. Then*

(1) $\cup_{x\in L,\ \alpha\in(0,1]} X^\circ(x_\alpha) = X^\circ$,
(2) $X^\circ(x_\alpha) \cap X^\circ(y_\alpha) = X^\circ((x \vee y)_\alpha)$,
(3) $X^\circ(x_\alpha) \cup X^\circ(y_\alpha) = X^\circ((x \wedge y)_\alpha)$,
(4) $X^\circ(x_\alpha) = \emptyset \Leftrightarrow x \in M_\circ$,
(5) $X^\circ((x \vee m)_\alpha) = X^\circ((m \vee x)_\alpha) = X^\circ(x_\alpha)$

Theorem 4.11. *Let μ be an ordered filter of a HADL L. Then $x_\alpha \in \mu$ if and only if $X^\circ(x_\alpha) \subseteq X^\circ(\mu)$.*

Proof. Assume that $x_\alpha \in \mu$. Let $\theta \in X^\circ$ be such that $\theta \in X^\circ(x_\alpha)$. Then $x_\alpha \nsubseteq \theta$. Hence we get $\mu \nsubseteq \theta$. Thus it yields $\theta \in X^\circ(\mu)$. Therefore $X^\circ(x_\alpha) \subseteq X^\circ(\mu)$. Conversely, assume that $X^\circ(x_\alpha) \subseteq X^\circ(\mu)$. Suppose that $x_\alpha \notin \mu$. This implies $\mu(x) < \alpha$. Then by Corollary 4.3, there exists $\theta \in X^\circ$ such that $\mu \subseteq \theta$ and $\theta(x) < \alpha$. Hence $x_\alpha \nsubseteq \theta$ and so $\theta \in X^\circ(x_\alpha)$. Since $\theta \in X^\circ(\mu)$, it follows that $X^\circ(x_\alpha) \nsubseteq X^\circ(\mu)$, which is a contradiction. □

Theorem 4.12. *Let $\mathcal{B} = \{X^\circ(x_\alpha) : x \in L, \alpha \in (0, 1]\}$. Then $\mathcal{B}$ forms a base for some topology on τ.*

Proof. By Lemma 4.10, (1) am (2), it follows that $\mathcal{B}$ forms a base for some topology on τ. □

Theorem 4.13. *Let L be a HADL with maximal element m. Then we have the following:*

(1) *For any $x \in L$ and $\alpha \in (0, 1]$, $X^\circ(x_\alpha)$ is compact in X°.*
(2) *The space X° is a T_0-space,*
(3) *Let A be a compact open subset of X°. Then $A = X^\circ(x_\alpha)$ for some $x \in L$ and $\alpha \in (0, 1]$.*

Corollary 4.14. *X° is a compact space.*

For any fuzzy subset θ of L, $X^\circ(\theta) = \{\mu \in X^\circ : \mu \nsubseteq \theta\}$ is open set of X° and $H^\circ(\theta) = X^e - X^\circ(\theta)$ is a closed set of X°. Also every closed set in X° is the form of $H^\circ(\theta)$ for all fuzzy subset of L. Then we have the following:

Theorem 4.15. *The closure of any $A \subseteq X^\circ$ is given by $\overline{A} = H^\circ(\cap_{\mu\in A}\mu)$.*

Proof. Let $A \subseteq X^\circ$ and $\beta \in A$. Then $\cap_{\mu\in A}\mu \subseteq \beta$. Thus $\beta \in H^\circ(\beta) \subseteq H^\circ(\cap_{\mu\in A}\mu)$. Therefore, $H^\circ(\cap_{\mu\in A}\mu)$ is a closed set containing A. Let C be any closed set containing A in X° . Then $C = H^\circ(\theta)$ for some fuzzy subset θ of L. Since $A \subseteq C = H^\circ(\theta)$, we have $\theta \subseteq \mu$ for all $\mu \in A$. Hence $\theta \subseteq \cap_{\mu\in A}\mu$. Therefore, $H^\circ(\cap_{\mu\in A}\mu) \subseteq H^\circ(\theta) = C$. Hence $H^\circ(\cap_{\mu\in A}\mu)$ is the smallest closed set containing A. Therefore, $\overline{A} = H^\circ(\cap_{\mu\in A}\mu)$. □

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