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ON D-METACOMPACTNESS IN TOPOLOGICAL SPACES

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ABSTRACT. This paper intends to define the metacompact spaces and the D-metacompact spaces as well as study their properties along with their relations with some other topological spaces. Several theoretical results are stated and proved with meticulous care through generalizing many well known theorems concerning with metacompact spaces. These results are supported via handling some illustrative examples.

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1. Introduction

The notion of a metacompact in the topological space (X, τ) introduced by Steen and Seebach in [2]. Further intensive studies on such spaces was then carried out, see for examples [1, 4, 7]. In this paper, we will study the notions of metacompact and D-metacompact in topological spaces as well as we will derive some important theoretical results related to this topic. To this aim, we intend to introduce a sufficient survey of some basic concepts and main results in topological spaces needed for going forward. In particular, we will introduce the concept of metacompactness in topological spaces, and introduce some of their properties, and relate them to other spaces. We will also provide certain well known definitions that will be used in the sequel. This background will pave the way for studying the concept of locally metacompactness in the topological spaces, and proving several properties related to these spaces. In the same context, we will introduce the concept of D-metacompactness in the topological spaces, and introduce some of their properties, and relate it to other spaces. In addition, the notion of mappings as well as the product of two D-metacompactness spaces and also with other spaces will be introduced.

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However, we will recall some required notations to make this paper self-contained. For instance, the sets $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$, and $\mathbb{Z}$ will denote the sets of real numbers, rational numbers, natural numbers and integer numbers, respectively. Furthermore, the terms τ_u, τ_{dis}, τ_{cof} and τ_{coc} will denote the usual, discrete, co-finite and the co-countable topologies, respectively.

2. Basic Definitions

In this section, some basic concepts and definitions associated with the meta-compact spaces are introduced so that our upcoming results can be understood easily.

Definition 2.1. [3, 4] A topological spaces (X, τ) is called metacompact, if every open cover of the space (X, τ) has point a finite parallel refinement.

Definition 2.2. [4, 5] A topological space (X, τ) is called countably metacompact, if every countable open cover of the space (X, τ) has a point finite parallel refinement.

Definition 2.3. [4, 6] If (X, τ) is a topological space, then τ is said to be locally metacompact, if each point of X, has an open neighborhood whose closure is metacompact. For instance, the topological space $(\mathbb{R}, \tau_{dis})$ is locally metacompact.

Corollary 2.4. [7] *Every metacompact space is locally metacompact.*

Proof. To show τ is locally metacompact, we let $x \in X$ and U be any open neighborhood of x. Then, the clU is a closed proper subset of a metacompact space X, and consequently clU is metacompact. Thus, (X, τ) is locally metacompact. $\square$

3. D-Metacompactness in Topological Spaces

In this part, we will recall some necessary definitions related to the concept of D-metacompactness in topological spaces, and introduce some of their properties. This would lay the foundation to provide some theoretical results that stem from the subject at hand.

Definition 3.1. A subset A of a topological space (X, τ) is called a D-set if there are two open sets U and V such that $U \neq X$ and $A = U - V$. In this case, we say that A is a D-set generated by U and V. It should be noted that every open set U different from X is a D-set if $A = U$ and $V = \phi$.

Definition 3.2. A cover $\check{\mathrm{D}} = \{D_\alpha : \alpha \in \Delta\}$ of a topological space (X, τ) is said to be D-cover if each D_α is a D-set for all $\alpha \in \Delta$.

Definition 3.3. A D-cover $\tilde{U}$ of the topological space (X, τ) is called point finite if each $x \in X$ is contained in a finite number of D-member of $\tilde{U}$. It is clear that every open cover is a D-cover, but the converse needs not be true.

Definition 3.4. A topological space (X, τ) is called D-metacompact if every D-cover of the space (X, τ) has a point finite parallel refinement.

Theorem 3.5. *Every D-metacompact space (X, τ) is metacompact.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a open cover of (X, τ). Then, $\tilde{U}$ is a D-cover, and so it has a point finite parallel refinement, which yield the desired result. $\square$

Example 3.6. The topological space $(\mathbb{R}, \tau_u)$ is metacompact, since $(\mathbb{R}, \tau_u)$ is a D-metacompact.

Definition 3.7. A topological space (X, τ) is called locally indiscrete if every τ_i-open set is τ_i-clopen, where $i = 1, 2$.

In this regard, we will introduce some new results in view of the aforesaid definitions and basic facts. For attaining this purpose, we begin first by proposing the next theorem that aims to show that the converse of the above theorem can be true under extra conditions.

Theorem 3.8. *Every locally indiscrete metacompact topological space (X, τ) is D-metacompact.*

Proof. Let $\tilde{U}$ be a $D-$cover of the space (X, τ). Then, $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$, where $U_\alpha \in D_\tau$ for each $\alpha \in \Delta$. Since (X, τ) is locally indiscrete, U_α is clopen set, for each $\alpha \in \Delta$. Hence, $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ is an open cover, and so it has a point finite parallel refinement. This, immediately, gives the desired result. $\square$

In the same context, one might observe that the locally indiscrete topological space $(\mathbb{R}, \tau_{dis})$ is clearly D-metacompact since it is metacompact. Anyhow, the following results can be moreover derived using the aforesaid definitions.

Theorem 3.9. *The topological space (X, τ) is D-metacompact if and only if it is D-metacompact and metacompact.*

Proof. $\Rightarrow$) Assume that (X, τ) is D-metacompact. Let $\tilde{U}$ be a D-open cover of X. Then, $\tilde{U}$ is a D-open cover of the space (X, τ). Since (X, τ) is D-metacompact, $\tilde{U}$ has a point finite parallel refinement. Hence, (X, τ) is D-metacompact, and consequently (X, τ) is also metacompact.
$\Leftarrow$) Assume that (X, τ) is D-metacompact. Let $\tilde{U}$ be a D-cover of (X, τ). If $\tilde{U}$ is a $D-$cover, then the result follows. If $\tilde{U}$ is not D-cover, then it will be τ-D-cover of (X, τ). Since (X, τ) is D-metacompact, $\tilde{U}$ has a point finite parallel refinement. So, (X, τ) is D-metacompact. $\square$

Theorem 3.10. *The topological space (X, τ) is metacompact if and only if (X, τ) is D-metacompact, where τ is the least upper bound topology.*

Corollary 3.11. *The topological space (X, τ) is metacompact if and only if (X, τ) is metacompact, where τ is the least upper bound topology.*

Theorem 3.12. *If a topological space (X, τ) is D-metacompact, then any subspace of (X, τ) must be D-metacompact spaces.*

Corollary 3.13. *If a topological space (X, τ) is metacompact, then (X, τ) must be metacompact space.*

By recalling that a space has a hereditary property if every subspace of it has this property, we can established the next result.

Theorem 3.14. *If a topological space (X, τ) is hereditray D-metacompact, then it is D-metacompact.*

Proof. Let $\tilde{U}$ be a D-cover of (X, τ). Then, $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$, where U_α is a D-set for each $\alpha \in \Delta$. since $U = \cup\{U_\alpha : \alpha \in \Delta\}$ is D-metacompact, then it has a point finite parallel refinement, say $\{U^*_\alpha : \alpha \in \Delta\}$ in which $U = \cup_{\alpha\in\Delta} U^*_\alpha$. Hence, $\{U^*_\alpha : \alpha \in \Delta\}$ is a D-point finite parallel refinement of $\tilde{U}$, and this completes the proof. □

It is worth mentioning that if a topological space (X, τ) has a D-topological property, then it will be possessed certain topological properties. In particular, we have the following features:

- If a topological space (X, τ) is D-metacompact, then it is metacompact.
- If a topological space (X, τ) is D-Lindelöf, then it is Lindelöf.
- If a topological space (X, τ) is D-compact, then it is compact.

As a matter of fact, the above list leads us to recall some definitions and necessary fundamentals related to the Lindelöf and D-Lindlöf spaces. Such definitions are stated below for completeness.

Definition 3.15. A topological space (X, τ) is called D-Lindlöf if every D-Lindlöf cover of the space (X, τ) has a countable subcover.

Definition 3.16. A subset A of a topological space (X, τ) is called D-dense, if we have $D_x \cap A \neq \phi$, for all $x \in X$ and every D-set D_x containing x.

In view of the above definition, we can clearly observe that every D-dense set is dense. The converse of the above statement is not true. This is, actually, because we observe that the set of all irrational numbers $\Bbbk = \mathbb{R}/\mathbb{Q}^c$ is dense but not D-dense in the space $(\mathbb{R}, \tau_{cof})$, since $\{5\}$ is a D-set and $\Bbbk \cap \{5\} = \phi$.

Definition 3.17. A topological space (X, τ) is called D-separable, if it has a D-dense countable subset D.

As a consequence of the above definitions, we state and prove the next result, followed by stating two other results that can be yielded immediately.

Theorem 3.18. *A D-separable, D-metacompact (X, τ) is D-Lindelöf.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a D-cover of X. Assume that $\tilde{U}$ has no countable subcover. Let $\tilde{A} = \{A_\beta : \beta \in \Gamma\}$ be an uncountable point finite parallel refinement subcover of $\tilde{U}$. Let D be a countable D-dense subset of X. Then, $A_\beta \cap D \neq \phi$, for each $\beta \in \Gamma$. Thus, D is an uncountable set because $\tilde{A}$ is uncountable, which is a contradiction, and hence the result. □

Corollary 3.19. *A D-separable, D-metacompact space (X, τ) is Lindelöf.*

Corollary 3.20. *A separable, metacompact space (X, τ) is Lindelöf.*

In the following content, we will state further definitions and fundamentals for the purpose of deeply exploring more results. Some of these results, however, will be proved, while the others will be only stated.

Definition 3.21. A topological space (X, τ) is called countably D-metacompact if every countable D-cover of the space (X, τ) has a point finite parallel refinement.

Definition 3.22. A topological space (X, τ) is called D-countably compact if every countable D-cover of the space (X, τ) has a finite subcover.

Definition 3.23. A space (X, τ) is said to be D-paracompact if (X, τ) is a D-paracompact space.

Remark 3.1. It is clear that every D-paracompact space is D-metacompact.

Theorem 3.24. *Every D-Lindelöf countably D-metacompact space (X, τ) is a D-metacompact space.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a D-cover of X. Since X is D-Lindelöf, then $\tilde{U}$ has a countable subcover $\tilde{A} = \{A_{\alpha i}\}_{i=1}^{\infty}$. Since X is countably D-metacompact, then $\tilde{A}$ has a point finite parallel refinement $\hat{\mathrm{W}}$ of $\tilde{U}$. Hence, (X, τ) is indeed D-metacompact. $\square$

Corollary 3.25. *Every D-Lindelöf countably D-metacompact space (X, τ) is a metacompact space.*

Corollary 3.26. *Every Lindelöf countably D-metacompact space (X, τ) is a metacompact space.*

Corollary 3.27. *Every Lindelöf countably metacompact space (X, τ) is metacompact space.*

Remark 3.2. It is clear that the topological space $(\mathbb{Z}, \tau_{dis})$ is D-metacompact, since it is countably D-metacompact and D-Lindelöf.

Definition 3.28. A space (X, τ) is said to be D-metalindelöf if every D-open cover of (X, τ) has a point countable parallel refinement.

Theorem 3.29. *Every D-metaLindelöf countably D-metacompact space (X, τ) is a D-metacompact space.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a D-cover of X. Since X is D-metalindelöf, then $\tilde{U}$ has a point countable parallel refinement $\tilde{A} = \{A_{\alpha i}\}_{i=1}^{\infty}$, which is also a D-cover of (X, τ). Since X is countably D-metacompact, then $\tilde{A}$ has a point finite parallel refinement $\hat{\mathrm{W}}$ of $\tilde{U}$. Hence, (X, τ) is indeed D-metacompact. $\square$

Corollary 3.30. *Every D-metalindelöf, countably D-metacompact space (X, τ) is a metacompact space.*

Corollary 3.31. *Every metalindelöf, countably D-metacompact space (X, τ) is a metacopmpact space.*

Corollary 3.32. *Every metalindelöf, countably metacompact space (X, τ) is metacompact space.*

Theorem 3.33. *Every D-metalindelöf, countably D-metacompact space (X, τ) is a D-metacompact space*

Definition 3.34. A topological space (X, τ) is called D-compact if every D-cover of the space (X, τ) has a finite subcover.

Theorem 3.35. *Every D-compact space (X, τ) is a compact.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of (X, τ). Then, $\tilde{U}$ is a D-cover and so it has a finite subcover. Hence the result. $\square$

4. Product of D-metacompact topological spaces

In this section, we intend to establish several theoretical results that related to the notion of mappings and the product of two D-metacompactness spaces.

Theorem 4.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a perfect function. If X is locally indiscrete space, then X is a D-metacompact space if Y is so.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be any D-cover of X, where $\{U_\alpha : \alpha \in \Delta\}$ is the set of D-members of $\tilde{U}$. Now, since f is perfect, then $f^{-1}(y)$ is a compact subset of X, for every $y \in Y$. So, there exists a finite subset of Δ such that $f^{-1}(y) \subseteq \{\cup_{i=1}^{n} U_i : i \in \Delta\}$. Now, $O_y = Y - f(X - \cup_{i=1}^{n} U_i : i \in \Delta)$ is an open subset of Y and $f^{-1}(O_y) \subseteq \{\cup_{i=1}^{n} U_i : i \in \Delta\}$, where $y \in O_y$. Therefore, $\tilde{O} = \{\tilde{O} = \{O_y : y \in Y\}\}$ is an open cover of Y. Since Y is D-metacompact, then $\tilde{O}$ has a point finite parallel refinement such that $\tilde{O}^* = \tilde{O} = \{O_y^* : y \in Y\}$. Consequently, O_y^* is a D-open subset of X. Since f is perfect, the set $\{f^{-1}(O_y^*) : y \in Y\}\}$ is a point finite parallel refinement of X, and hence X is D-metacommpact. $\square$

Corollary 4.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a perfect function. Then X is metacompact space if Y is so.*

Lemma 4.3. *Every D-compact subset A of a topological space (X, τ) is compact.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of A. Then $\tilde{U}$ is a D-cover of A. Thus, it has a finite subcover, and hence A is compact. $\square$

Theorem 4.4. *Let (X, τ) and (Y, σ) be two topological spaces. If X is D-compact, then the projection function $P : X \times Y \to Y$ is closed.*

Proof. To show that the projection function $P : X \times Y \to Y$ is closed, we show that the projection function $P : (X \times Y, \tau \times \sigma) \to (Y, \sigma)$ is closed. To this aim, let $y \in Y$ and let U be an open set in $(X \times Y, \tau \times \sigma)$ such that $P^{-1}(\{y\}) \subseteq U$. So, based on Wallace Lemma, there exists an open set in Y, say V_y, such that $P^{-1}(\{y\}) = X \times \{y\} \subseteq X \times V_y \subseteq U$. This, however, implies $y \in V_y$ and $P^{-1}(V_y) = X \times V_y \subseteq U$. Therefore, $P : (X \times Y, \tau \times \sigma) \to (Y, \sigma)$ is closed function. □

Corollary 4.5. *Let (X, τ) and (Y, σ) be two topological spaces. If X is compact, then the projection function $P : X \times Y \to Y$ is closed.*

Theorem 4.6. *Let (X, τ) and (Y, σ) be two topological spaces such that X is a D-compact space and Y a D-metacompact space. Then $X \times Y$ is a D-metacompact space.*

Proof. In view of the fact that the projection function $P : X \times Y \to Y$ is continuous and $P^{-1}\{y\} = X \times \{y\} \simeq X$ is D-compact, for all $y \in Y$, then $P : X \times Y \to Y$ is a perfect function. Now, since Y is D-metacompact, then $X \times Y$ is also D-metacompact. □

Corollary 4.7. *The product of a compact topological space and a metacompact topological space is metacompact.*

Example 4.8. The topological space $(\mathbb{R}, \tau_{cof})$ is compact, so it is metacompact but not D-compact.

Theorem 4.9. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous, closed, onto function and Y is locally indiscrete space. Then, Y is D-metacompact, if X is so.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be any D-cover of Y, where $\{U_\alpha : \alpha \in \Delta\}$ is a D-member of $\tilde{U}$. Since f is continuous, onto function, the set $\tilde{U} = \{f^{-1}(U_\alpha)\alpha \in \Delta\}$ is an open cover of X. Since X is D-metacompact space, there exists a point finite open parallel refinement of $\tilde{U}$, say $\tilde{U}^* = \{f^{-1}(U^*_\alpha) : \alpha \in \Delta\}$. Therefore,$Y$ is D-metacompact. □

Corollary 4.10. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous, closed, onto function. Then Y is D-metacompact, if X is metacompact.*

Corollary 4.11. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous, closed, onto function. Then Y is metacompact, if X is so.*

5. Conclusion

In this paper, the metacompact spaces and the D-metacompact spaces as well as study their properties along with their relations with some other topological spaces have been studied and discussed to outline several theoretical results. These results are generalization of several well known theorems concerning with metacompact spaces.

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