J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2021.28.3.247 ISSN(Online) 2287-6081
Volume 28, Number 3 (August 2021), Pages 247–252

REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM WITH SPECIAL STRUCTURE TENSOR FIELD

Dong Ho Lim [a] and Hoonjoo Kim [b,*]

Abstract. Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. In this paper, we prove that if $(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0$ holds on M, then M is a Hopf hypersurface, where ϕ is the tangential projection of the complex structure of $M_n(c)$. We characterize such Hopf hypersurfaces of $M_n(c)$.

1. Introduction

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. It is well-known that a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n\mathbf{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper, we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The Reeb vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant [2] and that M is called a *Hopf hypersurface*.

Typical examples of Hopf hypersurfaces in $P_n\mathbf{C}$ are homogeneous ones, and these real hypersurfaces are given as orbits under the subgroup of the projective unitary group $PU(n+1)$. Takagi [9] completely classified all such hypersurfaces into six model spaces: A_1, A_2, B, C, D and E.

Received by the editors April 07, 2021. Accepted August 12, 2021.
2010 *Mathematics Subject Classification.* 53C15, 53B25.
Key words and phrases. real hypersurface, differential operator of structure tensor field, Hopf hypersurface, model spaces.
This paper was supported by the Sehan University Research Fund in 2021.
*Corresponding author.

On the other hand, real hypersurfaces in $H_n\mathbf{C}$ have been investigated by Berndt [1], Montiel and Romero [6] and so on. Berndt [1] catagorized all homogeneous Hopf hypersurfaces in $H_n\mathbf{C}$ as four model spaces which are said to be A_0, A_1, A_2 and B.

If a real hypersurface M is of A_1 or A_2 in $P_n\mathbf{C}$ or of A_0, A_1, A_2 in $H_n\mathbf{C}$, then M is said to be *type A* for simplicity.

The following theorem is a typical characterization of real hypersurfaces of type A due to Okumura [8] for $c > 0$ and Montiel and Romero [6] for $c < 0$.

Theorem A. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to one of the model spaces of type A.*

We define a structure tensor field $(\nabla_X \phi)Y$ on M in $M_n(c)$ by

$$(\nabla_X \phi)Y = \nabla_X(\phi Y) - \phi \nabla_X Y = \eta(Y)AX - g(AX, Y)\xi,$$

by using the tangential projection and parallelism of J.

Many geometricians have studied real hypersurfaces with the conditions of the structure tensor field and obtain some results on the classification of real hypersurfaces in complex space form $M_n(c)$ [4, 5, 6, 8, etc].

For the Codazzi type of structure tensor field, Lim and Kim [3] have proved the following theorem;

Theorem B. *There exists no real hypersurface of $M_n(c)$, $c \neq 0$, whose structure tensor field is Codazzi type.*

In this paper, we shall study a real hypersurface in a nonflat complex space form $M_n(c)$, with special conditions of structure tensor field, and give some characterizations of such a real hypersurface in $M_n(c)$.

All manifolds are assumed to be connected and of class C^∞ and real hypersurfaces supposed to be orientable.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$, we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\widetilde{g}$ of $M_n(c)$. Then the Gauss and

Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \widetilde{\nabla}_X N = -AX$$

respectively, where X and Y are any vector fields tangent to M, g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M, we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. And we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify the followings from the Gauss and Weingarten formulas:

$$\nabla_X \xi = \phi AX,$$
$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.1}$$

Let X, Y and Z be vector fields on M and R denote the Riemannian curvature tensor of M. As the ambient space has holomorphic sectional curvature c, the equations of Gauss and Codazzi are given by

$$\begin{aligned} R(X, Y)Z =& \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \\ (\nabla_X A)Y - (\nabla_Y A)X =& \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \end{aligned}$$

respectively.

Let Ω be an open subset of M defined by

$$\Omega = \{p \in M \mid A\xi - \alpha\xi \neq 0\}$$

where $\alpha = \eta(A\xi)$. We put

$$A\xi = \alpha\xi + \mu W, \tag{2.2}$$

where W be a unit vector field orthogonal to ξ and μ does not vanish on Ω.

3. Some Lemmas

In this section, we assume that Ω is not empty and recall some well known results in [7] which will be used to prove our results.

Lemma 3.1. *Let M be a Hopf hypersurface in a nonflat complex space form $M_n(c)$. If X is a unit vector field such that $AX = \lambda X$, Then*

$$(\lambda - \frac{\alpha}{2})A\phi X = \frac{1}{2}(\alpha\lambda + \frac{c}{2})\phi X. \tag{3.1}$$

Lemma 3.2. *The B type hypersurface in $H_n\boldsymbol{C}$ has three distinct principal curvatures, $\frac{1}{r}\coth u$, $\frac{1}{r}\tanh u$ of multiplicity $n-1$ and $\alpha = \frac{2}{r}\tanh 2u$ of multiplicity 1. On the other hand, in $P_n\boldsymbol{C}$, the type B hypersurface also has three distinct principal curvatures, $-\frac{1}{r}\tan u$ of multiplicity $2p$, $\frac{1}{r}\cot u$ of multiplicity $2q$ and $\alpha = \frac{2}{r}\cot 2u$ of multiplicity 1, where $p > 0, q > 0$, and $p + q = n - 1$.*

Lemma 3.3. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$ and ξ be a principal curvature vector with corresponding principal curvature α. If X and ϕX are principal vector fields with principal curvatures $\{\lambda, \mu\} = \{0, -\frac{c}{2\alpha}\}$, then M does not exist in $M_n(c)$.*

Using Lemmas above, we get the following important tool of this paper;

Lemma 3.4. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If M satisfies $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$, then M is a Hopf hypersurface in $M_n(c)$.*

Proof. We assume that $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$ for any vector fields X and Y. By using (2.1) and symmetric properties of the shape operator, we have

$$\begin{aligned}(\nabla_X\phi)Y + (\nabla_Y\phi)X &= \eta(Y)AX - g(AX, Y)\xi + \eta(X)AY - g(AY, X)\xi \\ &= \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi.\end{aligned}$$

From the our assumption and the above equation, it follows that

$$\eta(Y)AX + \eta(X)AY = 2g(AX, Y)\xi. \tag{3.2}$$

If we put $Y = \xi$ into (3.2) and make use of (2.2), then we have

$$AX = \{\alpha\eta(X) + 2\mu g(X, W)\}\xi - \mu\eta(X)W. \tag{3.3}$$

If we substitute $X = W$ into (3.3), then we obtain

$$AW = 2\mu\xi. \tag{3.4}$$

Taking inner product of (3.4) with ξ and using (2.2), we have $\mu = 0$ on Ω, then it is a contradiction. Thus the set Ω is empty and hence M is a Hopf hypersurface. □

4. Non-existence of Real Hypersurfaces

In this section, we shall study a real hypersurface M in $M_n(c)$ which satisfies $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$.

Theorem 4.1. *Let M be a real hypersurface in $M_n(c)$, $c \neq 0$. If $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$, then we obtain $\lambda = 0$ and $\mu = -\frac{c}{2\alpha}$.*

Proof. By Lemma 3.4, the real hypersurface M satisfying $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$ is a Hopf hypersurface in $M_n(c)$, that is, $A\xi = \alpha\xi$. Since ξ is a Reeb vector field, the assumption $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$ is given by

$$\eta(Y)AX + \eta(X)AY = 2g(AX, Y)\xi. \tag{4.1}$$

To find the principal curvatures, we can divide equation (4.1) into three cases.

In the first case, if we put $Y = \xi$ into (4.1), then we have

$$AX + \eta(X)A\xi = 2\eta(AX)\xi. \tag{4.2}$$

For any vector field $X \perp \xi$ on M such that $AX = \lambda X$, the principal value $\lambda = 0$ follows from (4.2). From the equation (3.1), we obtain

$$-\frac{\alpha}{2}A\phi X = \frac{c}{4}\phi X. \tag{4.3}$$

If $\alpha = 0$, then $c = 0$, and there is no real hypersurface. Now, we assume that α is not zero. Then it follows from (4.3) that ϕX is a principal direction, say $A\phi X = -\frac{c}{2\alpha}\phi X$. Therefore, we see that the principal curvatures are constant α, $\lambda = 0$ and $\mu = -\frac{c}{2\alpha}$.

In the second case, if we substitute $X = \xi$ into (4.1), then we obtain the principal curvature $\{\lambda, \mu\} = \{0, -\frac{c}{2\alpha}\}$ in a similar way to the first case.

In the last case, if any vector field $X = Y$ is orthogonal to ξ on M and $AX = \lambda X$, then we get the principal value $\{\lambda, \mu\} = \{0, -\frac{c}{2\alpha}\}$ by using (4.2) and (4.3).

From the above three cases, we conclude that the principal curvatures are $\lambda = 0$ and $\mu = -\frac{c}{2\alpha}$. □

Theorem 4.2. *Let M be a real hypersurface in $M_n(c)$, $c \neq 0$. If $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$, then M does not exist in $M_n(c)$.*

Proof. By Lemma 3.4 and Theorem 4.1, we know that the real hypersurface M is a Hopf hypersurface and the principal curvatures have values of 0 and $-\frac{c}{2\alpha}$. By Lemma 3.3, M does not exist in $M_n(c)$ and the proof is completed. □

REFERENCES

1. J. Berndt: Real hypersurfaces with constant principal curvatures in complex hyperbolic space. J. Reine Angew. Math. **395** (1989), 132-141.
2. U-H. Ki & Y.J. Suh: On real hypersurfaces of a complex space form. J. Okayama Univ. **32** (1990), 207-221.
3. D.H. Lim & H.S. Kim: Non-existence real hypersurfaces in a nonflat complex space form with Codazzi type of structure tensor field. J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **28** (2021), 61-69.
4. D.H. Lim: Characterizations of real hypersurfaces in a nonflat complex space form with respect to structure tensor field. Far East. J. Math. Scie. **104** (2018), 277-284.
5. S. Maeda & S. Udagawa: Real hypersurfaces of a complex projective space in terms of Holomorphic distribution. Tsukuba J. Math. **14** (1990), 39-52.
6. S. Montiel & A. Romero: On some real hypersurfaces of a complex hyperbolic space. Geometriae Dedicata. **20** (1986), 245-261.
7. R. Niebergall & P.J. Ryan: Real hypersurfaces in complex space forms, in Tight and Taut submanifolds. Cambridge Univ. Press. (1998), 233-305.
8. M. Okumura: On some real hypersurfaces of a complex projective space. Trans. Amer. Math. Soc. **212** (1975), 355-364.
9. R. Takagi: On homogeneous real hypersurfaces in a complex projective space. Osaka J. Math. **10** (1973), 495-506.

[a]DEPARTMENT OF MATHEMATICS EDUCATION, SEHAN UNIVERSITY, JEOLLANAM-DO 58447, REPUBLIC OF KOREA
Email address: `dhlim@sehan.ac.kr`

[b]DEPARTMENT OF MATHEMATICS EDUCATION, SEHAN UNIVERSITY, JEOLLANAM-DO 58447, REPUBLIC OF KOREA
Email address: `hoonjoo@sehan.ac.kr`