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FIXED POINTS OF SET-VALUED MAPPINGS IN RELATIONAL METRIC SPACES

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Abstract. In this paper, we generalize the notion of comparable set-valued mappings by introducing two types of $\mathcal{R}$-closed set-valued mappings and utilize these to obtain an analogue of celebrated Mizoguchi and Takahashi fixed point theorem in relational metric spaces. To annotate the claims and usefulness of such findings, we prove fixed point results for both set-valued and single-valued mappings and validate the assertions with the help of examples. In this way, these investigations extend, modify and generalize some prominent recent fixed point results obtained by Tiammee and Suantai [24], Amini-Harandi and Emami [4], Prasad and Dimri [19] and several others in the settings of relational metric spaces.

1. Introduction

Let (X, d) be a metric space, we denote by $CB(X)$ the collection of all non-empty closed and bounded subsets of X, and H the Hausdorff metric on $CB(X)$ with respect to d, that is,

$$H(A,B) = \max\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\} \tag{1.1}$$

for every $A, B \in CB(X)$ where $d(x,A) = \inf_{y\in A} d(x,y)$.

The existence of fixed points for set-valued mappings using the notion of Hausdorff metric was initiated by Nadler in 1969.

Theorem 1.1 ([15]). *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $k \in [0, 1)$ such that*

$$H(Tx, Ty) \le kd(x, y) \tag{1.2}$$

for all $x, y \in X$. Then there exists $z \in X$, such that $z \in Tz$.

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Afterwards, several generalizations and extensions of the Nadler's fixed point theorems were published and there exists an extensive literature on this theme, but keeping in the light of the requirements of this presentation, we merely refer to [5-12, 24]. In this continuation Mizoguchi and Takahashi [14] besides giving the partial answer to the problem posed by Reich [21], generalized Theorem 1.1 as follows :

Theorem 1.2 ([14])**.** *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \tag{1.3}$$

for all $x, y \in X$ where $\alpha : [0, \infty) \to [0, 1)$ is a function satisfying $\limsup_{s\to t^+} \alpha(s) < 1,\ t \in [0, \infty)$. *Then there exists $z \in X$ such that $z \in Tz$.*

Daffer et al. [10] proved that $\alpha(t) = 1 - at^{b-1},\ a > 0$, for some $b \in (1, 2)$ on some interval $[0, s],\ 0 < s < a^{-\frac{1}{b-1}}$, is the class of functions verifying all the assumptions of the conjecture of Reich [21], and utilize the same to obtain fixed point results. Recently, Suzuki [23] remarked that Theorem 1.2 is a real generalization of the Nadler's theorem besides presenting the simplest proof.

On another point of note, many authors followed fixed point results of Ran and Reurings [20] and Nieto and López [16] in the last fifteen years and presented some useful fixed point results in this direction (see for instance, [1, 4, 6, 7, 19, 24, 25]). In this continuation, several authors are interested to investigate the fixed point results in some more general metric settings by considering a non-empty set equipped with an arbitrary binary relation. The motivation behind this, is to relax the necessity of a partial order relation, successfully, they obtained some useful generalizations on this theme (see, [2, 3, 17, 18, 22, 24]). Among these generalizations, we must quote the one due to Alam and Imdad [3], where some relation theoretic analogues of standard metric notions (such as continuity and completeness) were utilized to prove some useful fixed point results in this direction.

Most recently, Tiammee and Suantai [24] presented the notion of two types of monotone set-valued mappings and utilize these to prove an analogue of prominent fixed point result due to Mizoguchi and Takahashi [14] in the settings of partially ordered metric spaces. Inspired by this work, Prasad and Dimri [19] introduced two types of comparable set-valued mappings which are relatively weaker notions of monotone set-valued mappings, and proved some fixed point results for these mappings in ordered metric spaces. Our aim in this work, is to give an extension of these results by introducing $\mathcal{R}$-closed set-valued mappings which are relatively weaker forms of comparable set-valued mappings in the settings of relational metric spaces.

2. Preliminaries

In this section, we recall basic definitions and introduce the notion of $\mathcal{R}$-closed set-valued mappings and validate these assertion with the help of some examples. Throughout this paper, $\mathcal{R}$ stands for a non-empty binary relation, we denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{N}_0$ the set of whole numbers ($\mathbb{N}_0 = \mathbb{N} \cup \{o\}$), and by $\mathcal{F}$ the class of functions $\alpha : [0,\infty) \to [0,1)$, such that $\limsup_{r\to t+0} \alpha(r) < 1$, for $t \in [0,\infty)$.

Definition 2.1 ([13, 25])**.** (a) Let X be a non-empty set endowed with a partial order relation (anti-symmetric, reflexive and transitive) denoted by '$\preceq$'. Then the pair $(X, \preceq)$ is partially ordered set (or an ordered set).
(b) The element x is comparable to y if either $x \preceq y$ or $x \succeq y$ and is denoted by the symbol '$\prec\succ$'. (c) X is linearly ordered or totally ordered if any two elements of X are comparable.

Beg and Butt [7] defined relations between two non empty subsets A and B of X as follows:
(1) $A \preceq^{(I)} B$ if $x \preceq y$ for all $x \in A$ and $y \in B$.
(2) $A \preceq^{(II)} B$ if for each $x \in A$ there exists $y \in B$, such that $x \preceq y$.

Definition 2.2 ([24])**.** Let $(X, d, \preceq)$ be a partially ordered metric space and $T : X \to CB(X)$. Then T is a :
(i) monotone non-decreasing of type (I) if
$x, y \in X,\ x \preceq y \implies Tx \preceq^{(I)} Ty$;
(ii) monotone non-decreasing of type (II) if
$x, y \in X,\ x \preceq y \implies Tx \preceq^{(II)} Ty$.

Prasad and Dimri [19] presented the generalizations of these monotone set-valued mappings by introducing the two types of comparable set-valued mappings, as follows:

Definition 2.3. Let $(X, d, \preceq)$ be a partially ordered metric space and $T : X \to CB(X)$. Then T is a :

(i) comparable of type (I), if
$x, y \in X,\ x \preceq y \implies Tx \prec\succ^{(I)} Ty$, that is, either $Tx \preceq^{(I)} Ty$ or $Tx \succeq^{(I)} Ty$.
(ii) comparable of type (II), if
$x, y \in X,\ x \preceq y \implies Tx \prec\succ^{(II)} Ty$, that is, either $Tx \preceq^{(II)} Ty$ or $Tx \succeq^{(II)} Ty$.

Example 2.4 ([18])**.** Let $X = [-1, 1]$ equipped with the natural ordering of real numbers. Define $T : X \to CB(X)$ by $Tx = [0, x^2]$, then T is a comparable set-valued mapping but not monotone. Also it can be easily verified that T is a comparable set-valued mapping of type (II) but not of type (I).

Noticeably, every monotone set-valued mappings of type (I) and that of type (II) are comparable set-valued mappings of type (I) and of type (II) respectively. However the converse is not true. Although, there exists set-valued mappings which do not comes under above these two types. To substantiate this view point, we furnish an example.

Example 2.5. Let $X = \{-\frac{1}{2}, -\frac{1}{4}, ... -\frac{1}{2^n}, ...\} \cup \{\frac{1}{2}, \frac{1}{4}, ... \frac{1}{2^n}, ...\}$ for all $n \in \mathbb{N}$, with usual partial ordering defined on it. Define a set-valued mapping T, such that

$$T(x) = \begin{cases} \{-\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\} & \text{if } x = \frac{1}{2^{2n}}, \\ \{\frac{1}{2^{2n}}, -\frac{1}{2^{2n+1}}\} & \text{if } \ x = -\frac{1}{2^{2n}}. \end{cases}$$

Let us consider, $-\frac{1}{2^n} \preceq -\frac{1}{2^{n+1}} \implies T(-\frac{1}{2^n}) \npreceq^{(I)} T(-\frac{1}{2^{n+1}})$ and $T(-\frac{1}{2^n}) \nsucceq^{(I)} T(-\frac{1}{2^{n+1}})$
that is, $\{\frac{1}{2^{2n}}, -\frac{1}{2^{2n+1}}\} \npreceq^{(I)} \{\frac{1}{2^{2n+2}}, -\frac{1}{2^{2n+3}}\}$ and $\{\frac{1}{2^{2n}}, -\frac{1}{2^{2n+1}}\} \nsucceq^{(I)} \{\frac{1}{2^{2n+2}}, -\frac{1}{2^{2n+3}}\}$.
This implies that T is an incomparable set-valued mapping of type (I).
Again if we consider, $\frac{1}{2^n} \succeq -\frac{1}{2^{n+1}} \implies T(\frac{1}{2^n}) \nsucceq^{(II)} T(-\frac{1}{2^{n+1}})$ and $T(\frac{1}{2^n}) \npreceq^{(II)} T(-\frac{1}{2^{n+1}})$
that is $\{-\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\} \nsucceq^{(II)} \{\frac{1}{2^{2n+2}}, -\frac{1}{2^{2n+3}}\}$ and $\{-\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\} \npreceq^{(II)} \{\frac{1}{2^{2n+2}}, -\frac{1}{2^{2n+3}}\}$.
This implies that T is an incomparable set-valued mapping of type (II) too.

Interestingly, Example 2.5 ensures the existence of incomparable set-valued mappings, motivated by this fact, we extend the notion of comparable set-valued mappings, by introducing two types of $\mathcal{R}$-closed set-valued mappings, first we recall binary relation :

Definition 2.6 ([3])**.** A binary relation on a non-empty set X is defined as a subset of $X \times X$, which will be denoted by $\mathcal{R}$. We say that x *relates to* y under $\mathcal{R}$ if and only if $(x, y) \in \mathcal{R}$.

Definition 2.7. Let $(X, d, \mathcal{R})$ be a relational metric space and $T : X \to CB(X)$. Then T is an :
(i) $\mathcal{R}$-closed of type (I) if
$x, y \in X,\ (x, y) \in \mathcal{R} \implies (Tx, Ty) \in \mathcal{R}^{(I)}$, that is, for all $x_i \in Tx$, and $y_j \in Ty$, we have $(x_i, y_j) \in \mathcal{R}$.
(ii) $\mathcal{R}$-closed of type (II) if
$x, y \in X,\ (x, y) \in \mathcal{R} \implies (Tx, Ty) \in \mathcal{R}^{(II)}$, that is, for each $x_i \in Tx$ there exists $y_j \in Ty$, such that $(x_i, y_j) \in \mathcal{R}$.

Noticeably, every comparable set-valued mappings of type (I) and that of type (II) are $\mathcal{R}$-closed set-valued mappings of type (I) and of type (II) respectively. However, the converse is not true.
The subsequent remark shows that the concept of an $\mathcal{R}$-closed set-valued mapping is weaker than that of the comparable set-valued mapping.

Remark 2.8. If we define binary relation $\mathcal{R}$ on X of above Example 2.5., such that

$$\mathcal{R} = \{(x, y) \in \mathcal{R} \text{ if } xy \leq x \ or\ -x\}.$$

Then by routine calculation one can easily verify that T is an $\mathcal{R}$-closed set-valued mappings of the type (I) and of the type (II) too. In this way, the notion of an $\mathcal{R}$-closed mapping is an extension or an improvement over the comparable set-valued mapping of the existing theory of monotone mappings.

Alam and Imdad [3] presented the relational metrical variants of continuity, completeness and some other sequence related notions as follows:

Definition 2.9 ([3])**.** Let X be a non-empty set equipped with a binary relation $\mathcal{R}$. For $x, y \in X$, x and y are called $\mathcal{R}$-*comparative* if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.

Definition 2.10 ([3])**.** Let X be a non-empty set equipped with a binary relation $\mathcal{R}$.
(1) The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, and defined by $\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$.
(2) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup R^{-1}$, that is, $\mathcal{R}^s := R \cup \mathcal{R}^{-1}$. In fact, $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Definition 2.11 ([3])**.** Let X be a non-empty set equipped with a binary relation $\mathcal{R}$ and T be a self-mapping on X. Then $\mathcal{R}$ is called T-*closed* if for any $x, y \in X, (x, y) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}$.

Theorem 2.12 ([3])**.** *Let X be a non-empty set equipped with a binary relation $\mathcal{R}$ and T be a self-mapping on X. If $\mathcal{R}$ is T-closed, then, for all $n \in \mathbb{N}_0$, $\mathcal{R}$ is T^n-closed, where T^n denotes n-th iterate of T.*

Definition 2.13 ([3])**.** Let X be a non-empty set equipped with a binary relation $\mathcal{R}$. Then a sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-*preserving* if $(x_n, x_{n+1}) \in \mathcal{R},\ n \in \mathbb{N}_0$.

Definition 2.14 ([2])**.** Let $(X, d, \mathcal{R})$ be a metric space equipped with a binary relation $\mathcal{R}$. Then (X, d) is called $\mathcal{R}$-*complete* if every $\mathcal{R}$-preserving Cauchy sequence in X converges to a point in X.

Noticeably, every complete metric space is $\mathcal{R}$-complete. In particular, the notion of $\mathcal{R}$-completeness coincides with usual completeness under the universal relation.

Definition 2.15 ([2])**.** Let X be a non-empty set equipped with a binary relation $\mathcal{R}$. A self-mapping T on X is called $\mathcal{R}$-*continuous at* x, if for any $\mathcal{R}$-preserving sequence $\{x_n\}$, such that $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. Moreover, T is called $\mathcal{R}$-*continuous* if it is $\mathcal{R}$-continuous at each point of X.

Noticeably, every continuous mapping is $\mathcal{R}$-continuous. In particular, the notion of $\mathcal{R}$-continuity coincides with usual continuity under the universal relation.
The subsequent notion is a generalization of d-self-closedness of a partial order relation $(\preceq)$ (defined by Turinici [25]).

Definition 2.16 ([25])**.** Let (X, d) be a metric space. A binary relation $\mathcal{R}$ on X is called d-*self-closed* if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$.

Definition 2.17 ([3])**.** Let $(X, d, \mathcal{R})$ be a metric space equipped with a binary relation $\mathcal{R}$. Then a subset E of X is called $\mathcal{R}$-*connected* if for each pair $x, y \in E$, there exists a path in $\mathcal{R}$ from x to y.

Definition 2.18 ([3])**.** Let $\mathcal{R}$ be a binary relation on a non-empty set X and a pair of points $x, y \in X$. If there is a finite sequence $\{z_0, z_1, z_2, ..., z_k\} \subset X$, such that $z_0 = x, z_k = y$ and $(z_i, z_{i+1}) \in \mathcal{R}$ for each $i (0 \le i \le k-1)$, then this finite sequence is called a *path of length* k (where $k \in \mathbb{N}$) in $\mathcal{R}$ from x to y in $\mathcal{R}$.

Noticeably, a path of length k involves $k+1$ elements of X, although they are not necessarily distinct.

Definition 2.19. Let $T : X \to 2^X$ be a set-valued mapping. A point $z \in X$ is called *fixed point* of T if $z \in Tz$.

Given a binary relation $\mathcal{R}$ and for a single-valued self-mappings T defined on a non-empty set X, we utilize the subsequent notations.
(i) $F(T) :=$ the set of all fixed points of T,
(ii) $X(T, \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$.

3. Main Results

In this section, we first consider the existence of fixed points for $\mathcal{R}$-closed set-valued mappings and then validates it by an illustrative example.

Theorem 3.1. *Let (X, d) be an $\mathcal{R}$-complete relational metric space and $T : X \to CB(X)$ be a set-valued mapping on X. Suppose that the subsequent assumptions hold:*
(a) T is an $\mathcal{R}$-closed of type (I),
(b) there exists $x_0 \in X$, such that $(\{x_0\}, Tx_0) \in \mathcal{R}^{(II)}$
(c) $\mathcal{R}$ is d-self-closed,
(d) there exists $\alpha \in \mathcal{F}$, such that

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$. Then there exists $x \in X$, such that $x \in Tx$.

Proof. Define a function $\beta(t) = \frac{(\alpha(t)+1)}{2}$, so that $\beta \in \mathcal{F}$. Then for all $t \in [0, \infty)$, we have $\beta(t) < \alpha(t)$ and $\limsup_{r \to t+0} \beta(r) < 1$. Also, as a property of this function for all $x, y \in X$ and $u \in Tx$, there exists an element $v \in Tx$, such that

$$d(u, v) \leq \beta(d(x, y))d(x, y). \tag{3.1}$$

In the light of assumption (b) there exists $x_1 \in Tx_0$, such that $(x_0, x_1) \in \mathcal{R}$. If $x_0 = x_1$, then x_0 is a fixed point of T and hence we are done.
Otherwise, if $x_0 \neq x_1$, we have $(x_0, x_1) \in \mathcal{R}$. By assumption (a), we have $(Tx_0, Tx_1) \in \mathcal{R}^{(I)}$, which indicate the existence of $x_2 \in Tx_1$, such that $(x_1, x_2) \in \mathcal{R}$.

As $(x_0, x_1) \in \mathcal{R}$, utilizing contractive assumption (d), in the light of defined function β, we have

$$\begin{aligned} H(Tx_0, Tx_1) &\leq \beta(d(x_0, x_1))d(x_0, x_1), \\ d(x_1, x_2) \leq H(Tx_0, Tx_1) &\leq \beta(d(x_0, x_1))d(x_0, x_1) \end{aligned}$$

$$d(x_1, x_2) \leq \beta(d(x_0, x_1))d(x_0, x_1). \tag{3.2}$$

Since $(x_1, x_2) \in \mathcal{R}$, as previously if $x_1 = x_2$, the proof is accomplished.
However, if $x_1 \neq x_1$, then exploiting the assumption (a), that is, $(Tx_1, Tx_2) \in \mathcal{R}^{(I)}$, we have $x_3 \in Tx_2$, such that $(x_2, x_3) \in \mathcal{R}$.
Again applying contractive assumption (d), we obtain

$$d(x_2, x_3) \leq \beta(d(x_1, x_2))d(x_1, x_2). \tag{3.3}$$

Continuing this process inductively, we can define an $\mathcal{R}$-preserving sequence $\{x_n\}$ in X, whose consecutive terms are related under some underlying arbitrary binary relation $\mathcal{R}$, such that $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}_0$, and

$$d(x_{n+1}, x_{n+2}) \leq \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}). \tag{3.4}$$

As $\beta(t) < 1,\ t \in [0, \infty)$ and $\{a_n\} = \{d(x_n, x_{n+1})\}$ is a non-increasing sequence of non-negative real numbers. Therefore, $\{a_n\}$ converges to some non-negative real number a.
Since $\limsup_{s \to a+0} \beta(s) < 1$ and $\beta(a) < 1$, there exist $r \in [0, 1)$ and $\epsilon > 0$, such that $\beta(s) \leq r$ for all $s \in [a, a + \epsilon]$. Now we can choose $m, n \in \mathbb{N}_0$ with $n \geq m$, such that $a \leq a_n \leq a + \epsilon$. Notice that,

$$a_{n+1} \leq \beta(a_n)a_n \leq ra_n,$$

and thus, we obtain

$$\sum_{n=1}^{\infty} a_n < \infty. \tag{3.5}$$

Hence $\{x_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence in X. Since (X, d) is an $\mathcal{R}$-complete metric space, so there exists $x \in X$, such that $x_n \xrightarrow{d} x$.
Next, owing to assumption (c), that is, $\mathcal{R}$ is d-self-closed, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that

$$(x_{n_k}, x) \in \mathcal{R},\ \text{for all } k \in \mathbb{N}_0. \tag{3.6}$$

Applying contractive assumption (d), to (3.6), we have

$$\begin{aligned} d(x,Tx) &= \lim_{n\to\infty} d(x_{n_k+1},Tx) \\ &\leq \lim_{n\to\infty} H(Tx_{n_k},Tx) \\ &\leq \lim_{n\to\infty} \beta(d(x_{n_k},x))d(x_{n_k},x) \\ &\leq \lim_{n\to\infty} d(x_{n_k},x) = 0. \end{aligned}$$

This implies that $x \in Tx$, that is, x is a fixed point of T. This completes the proof. □

Corollary 3.2. *Let (X, d) be an $\mathcal{R}$-complete relational metric space and $T : X \to CB(X)$ be a set-valued mapping on X. Suppose that the subsequent assumptions hold:*
(a) T is an $\mathcal{R}$-closed of type (II),
(b) there exists $x_0 \in X$, such that $(\{x_0\}, Tx_0) \in \mathcal{R}^{(II)}$
(c) $\mathcal{R}$ is d-self-closed,
(d) there exists $\alpha \in \mathcal{F}$ such that

$$H(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$. Then there exists $x \in X$, such that $x \in Tx$.

Proof. Since an $\mathcal{R}$-closed set-valued mapping of type (I) implies $\mathcal{R}$-closed set-valued mapping of type (II), the proof of this corollary is almost similar to that of Theorem 3.1. □

Example 3.3. Let $X = \{-\frac{1}{2}, -\frac{1}{4}, ..., -\frac{1}{2^n}, ...\} \cup \{0\} \cup \{\frac{1}{2}, \frac{1}{4}, ..., \frac{1}{2^n}, ...\}$ for all $n \in \mathbb{N}$, equipped with the binary relation

$$\mathcal{R} = \{(x,y) \in \mathcal{R} \text{ if } xy \leq x \textit{ or } -x\}$$

and usual metric d; defined by $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is $\mathcal{R}$-complete relational metric space. Define a set-valued function $T : X \to CB(X)$, such that

$$Tx = \begin{cases} \{-\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\} & \text{if } x = \frac{1}{2^n}, \\ \{\frac{1}{2^{2n}}, -\frac{1}{2^{2n+1}}\} & \text{if } x = -\frac{1}{2^n}, \\ \{0\} & \text{if } x = 0. \end{cases}$$

Noticeably, T is an $\mathcal{R}$-closed set-valued mapping of the type (I), to verify this conclusion, we refer Example 2.5. and Remark 2.8.

Next, if $-\frac{1}{2^m} \preceq -\frac{1}{2^n}$,

$$\begin{aligned} H(T(-\frac{1}{2^m}), T(-\frac{1}{2^n})) &= H(\{\frac{1}{2^{2m}}, -\frac{1}{2^{2m+1}}\}, \{\frac{1}{2^{2n}}, -\frac{1}{2^{2n+1}}\}) \\ &= |\frac{1}{2^{2m+1}} - \frac{1}{2^{2n+1}}| \\ &= \frac{1}{2}|\frac{1}{2^{2m}} - \frac{1}{2^{2n}}| \\ &< \frac{1}{2}|\frac{1}{2^m} - \frac{1}{2^n}| = \frac{1}{2}d(-\frac{1}{2^m}, -\frac{1}{2^n}), \end{aligned}$$

and

$$\begin{aligned} H(T(-\frac{1}{2^n}), T(\{0\}) &= H(\{\frac{1}{2^{2n}}, -\frac{1}{2^{2n+1}}\}, \{0\}) \\ &= \frac{1}{2^{2n+1}} \\ &< \frac{1}{2^{n+1}} = \frac{1}{2}d(-\frac{1}{2^n}, 0). \end{aligned}$$

Also, similar calculation holds for other values of $x \in X$. Thus T satisfies all the assumptions of the Theorem 3.1 and Corollary 3.2. Hence T has a fixed point.

We also, highlight the connection between the main results of this presentation and some important recent comparable ones obtained in the settings of ordered metric spaces by the subsequent remark.

Remark 3.4. By setting $\mathcal{R} = \preceq$, that is, the partial order relation instead of arbitrary binary relation in Theorem 3.1 and Corollary 3.2, we obtain Theorems 3.2 and 3.1 of Tiammee and Suantai [24]. Also if we set $\mathcal{R} = \prec\succ$, that is, the $\mathcal{R}$-comparative relation, then we obtain Theorems 3.1 and 3.2 of Prasad and Dimri [19] respectively.

4. FIXED POINT THEOREMS FOR SINGLE-VALUED MAPPINGS

In this section, we prove existence and uniqueness of fixed points for single-valued mappings.

Theorem 4.1. *Let (X, d) be an $\mathcal{R}$-complete relational metric space and T be a self-mapping on X. Suppose that the subsequent assumptions hold:*
(a) $\mathcal{R}$ is T-closed,
(b) $X(T, \mathcal{R})$ is non-empty
(c) either T is an $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed,

(d) there exists $\alpha \in \mathcal{F}$, such that

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$. Then T has a fixed point.

Proof. In the light of assumption (b) if $x_0 = Tx_0$, then x_0 is a fixed point of T and the proof is accomplished. However, if $x_0 \neq Tx_0$, then we can define a sequence of Picard iterates, that is, $x_n = T^n x_0$ for all $n \in \mathbb{N}_0$. Since $(x_0, Tx_0) \in \mathcal{R}$ and $\mathcal{R}$ is T-closed, by induction we have

$$(T^n x_0, T^{n+1} x_0) \in \mathcal{R}.$$

Notice that,

$$(x_n, x_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0, \tag{4.1}$$

such that the sequence $\{x_n\}$ is an $\mathcal{R}$-preserving. Applying the contractive assumption (d) to (4.1), we have

$$d(x_n, x_{n+1}) \leq \alpha(d(x_{n-1}, x_n))d(x_{n-1}, x_n). \tag{4.2}$$

Which is equivalent to the inequality (3.4) in the proof of Theorem 3.1.
Therefore, we recall Theorem 3.1 above and follow similar pattern of proof till we obtain that, $\{x_n\}$ is $\mathcal{R}$-preserving Cauchy sequence.
As (X, d) is $\mathcal{R}$-complete metric space, so there exists $x \in X$, such that $x_n \xrightarrow{d} x$.
Now in the light of assumption (c), firstly, assume that T is $\mathcal{R}$-continuous and then utilizing the inequality (4.2), we obtain

$$x_{n+1} = Tx_n \xrightarrow{d} Tx.$$

By uniqueness of the limit, we obtain $Tx = x$, that is, x is a fixed point of T.
On the other hand, suppose that $\mathcal{R}$ is d-self-closed, then again tracing back the proof of Theorem 3.1 along with the contractive assumptions (d), this theorem can be proved. □

Theorem 4.2. *In addition to the hypothesis of Theorem 4.1 suppose that the subsequent assumption holds :*
(e) $T(X)$ is $\mathcal{R}^s$-connected. Then T has a unique fixed point.

Proof. Let x and y be two fixed points of T, that is, $F(T) \neq \phi$ and $x, y \in F(T)$, then for all $n \in N_0$, we have

$$T^n x = x, \ T^n y = y. \tag{4.3}$$

Noticeably $x, y \in T(X)$. By assumption (e), there exists a path (say $z_0, z_1, z_2, ..., z_k$) of finite length k in $\mathcal{R}^s$ from x to y such that

$$z_0 = x,\ z_k = y \text{ and } [z_i, z_{i+1}] \in \mathcal{R} \text{ for each } i(0 \le i \le k-1). \tag{4.4}$$

As $\mathcal{R}$ is T-closed, then in the light of Theorem 2.12, we obtain

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \text{ for each } i(0 \le i \le k-1) \text{ and for each } n \in \mathbb{N}_0. \tag{4.5}$$

Now, applying the contractive assumption (d) to (4.5), we obtain

$$d(T^n z_i, T^n z_{i+1}) \le \alpha((d(T^{n-1} z_i, T^{n-1} z_{i+1}))(d(T^{n-1} z_i, T^{n-1} z_{i+1}). \tag{4.6}$$

For convenience, we put $a_n^i = d(T^n z_i, T^n z_{i+1})$.
For a fix i we shall discuss two cases: Firstly, suppose that $a_{n_0}^i = d(T^{n_0} z_i, T^{n_0} z_{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, that is, $T^{n_0} z_i = T^{n_0} z_{i+1}$, which implies that $T^{n_0+1} z_i = T^{n_0+1} z_{i+1}$. In this way $a_{n_0+1}^i = d(T^{n_0+1} z_i, T^{n_0+1} z_{i+1}) = 0$. Thus by induction, we get $a_{n_0}^i = 0$ for every $n \ge n_0$. Therefore $\lim_{n\to\infty} a_n^i = 0$.
Secondly, suppose that $a_n^i > 0$ for all $n \in \mathbb{N}_0$, then using (4.5), in light of the contractive assumption (d) and taking limit $n \to \infty$ in the last inequality (4.6) , we obtain $\lim_{n\to\infty} a_n^i = 0$ for each $i(0 \le i \le k-1)$.
Finally, utilizing the triangular inequality of metric d, in the light of above conclusion, we obtain

$$d(x, y) = d(T^n z_0, T^n z_k) \le a_n^0 + a_n^1 + ... + a_n^{k-1} \to o$$

as $n \to \infty$. Hence T has a unique fixed point. □

Remark 4.3. Theorems 4.1 and 4.2 are generalized and improved version of Theorem 2.1 presented in Amini-Harandi and Emami [4]. Therefore, Theorem 2.1 of Amini-Harandi and Emami [4] is a particular case of Theorems 4.1 and 4.2 and can be obtained by setting $\mathcal{R} = \preceq$, that is, the partial order relation instead of an arbitrary binary relation in Theorems 4.1 and 4.2.

Remark 4.4. Theorems 4.1 and 4.2 are generalized and improved version of Theorems 4.1 and 4.6 presented in Prasad and Dimri [19] for contractive type single-valued mappings and can be obtained by setting $\mathcal{R} = \prec\succ$, that is, the comparable relation instead of an arbitrary binary relation together with some other oder theoretic metrical analogous of completeness and continuity. We also highlight the fact that, the method of proof of these newly obtained fixed point results of this presentation are inspired by the proof given by T. Suzuki [23].

Conclusion

In this work, inspired by the fact that there exists an incomparable set-valued mapping, we introduced the notion of an $\mathcal{R}$-closed set-valued mapping and validated the assertion with the help of examples. For usefulness of such findings, we proved the variants of celebrated Mizoguchi and Takahashi fixed point theorems for set-valued and single-valued mappings in the settings of relational metric spaces. We verified this fact with an illustrative example too. Moreover, we considered simpler method of proof and shown the utility of these newly obtained results by providing some interesting remarks.

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