J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2021.28.3.217 ISSN(Online) 2287-6081
Volume 28, Number 3 (August 2021), Pages 217–234

# CONSTRUCTION OF AN HV-BE-ALGEBRA FROM A BE-ALGEBRA BASED ON "BEGINS LEMMA"

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Abstract. In this paper, first we introduce the new class of HV-BE-algebra as a generalization of a (hyper) BE-algebra and prove some basic results and present several examples. Then, we construct the HV-BE-algebra associated to a BE-algebra (namely BL-BE-algebra) based on "Begins lemma" and investigate it.

## 1. Introduction

The class of BCK-algebras was introduced in 1978 by Y. Imai and K. Iseki [17]. Then in 1998, Y. B. Jun et al. [18] introduced a new notion, called a BH-algebra, which is a generalization of a BCK-algebra, i.e., $x*x = 0$; $x*0 = x$ and $x*y = 0$ and $y*x = 0$ imply $x = y$ for any $x, y \in X$. In 1999, J. Neggers et al. [22] introduced the notion of a $d$-algebra which is another generalization of a BCK-algebra. Also, in 2007, H. S. Kim and Y. H. Kim [20] introduced the notion of a BE-algebra, as a generalization of a BCK-algebra, and using the notion of a upper set they gave an equivalent condition of a filter in a BE-algebra.

In 2012 and 2013, A. Rezaei et al. [30, 31] studied commutative ideals in BE-algebras and gave some properties. Also, they showed a commutative implicative BE-algebra is equivalent to a commutative self distributive BE-algebra. Moreover, they proved every Hilbert algebra is a self distributive BE-algebra and a commutative self distributive BE-algebra is a Hilbert algebra and showed one can not remove the conditions of commutativity and self distributivity. In [1], S. S. Ahn et al. introduced the notion of a terminal section of a BE-algebra and gave some characterization of

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Received by the editors November 05, 2020. Accepted August 17, 2021.
2010 *Mathematics Subject Classification.* 06F35-14L17.
*Key words and phrases.* BE-algebra, BE-hyperalgebra, Ends lemma, HV-BE-algebra, BL-BE-algebra.
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a commutative BE-algebra in terms of lattice order relations and terminal sections. Recently, R. A. Borzooei et al. introduced the notion of a pseudo BE-algebra which is a generalization of a BE-algebra. They defined the basic concepts of a pseudo subalgebra and a pseudo filter and proved that under some conditions, a pseudo subalgebra can be a pseudo filter [2].

The algebraic hyperstructure theory as a generalization of the algebraic structure was first introduced in 1934, by French mathematician F. Marty at the 8th congress of Scandinavian mathematicians [21]. A *hypergroupoid* is a non-empty set $H$ with a *hyperoperation* $\circ$ defined on $H$, that is, a mapping of $H \times H$ into the family of non-empty subsets of $H$. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$. If $A, B$ are non-empty subsets of $H$ then $A \circ B$ is given by $A \circ B = \bigcup\{x \circ y \mid x \in A,\ y \in B\}$.

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$ and it is called a *hypergroup* if it is a semihypergroup and $a \circ H = H \circ a = H$ for all $a \in H$. For instance, if $x \circ y = \{x, y\}$ for all $x, y \in H$, then $(H, \circ)$ is a hypergroup. Afterward, because of many applications of this theory in applied sciences, many authors study in this context. Some reviews of the hyperstructure theory can be found in [6, 8, 38]. Corsini's book on hyperstructures [4] points out their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. In [19], Y. B. Jun et al. applied the hyperstructure to a BCK-algebra and introduced the notion of a hyper BCK-algebra which is a generalization of the BCK-algebra and investigated some related properties. A. Radfar et al. defined the notion of a hyper BE-algebra, some types of hyper filters in this structure and described the relationship between them [29].

An *HV-structure* as a generalization of the hyperstructure was first introduced by Vougiouklis at the Forth AHA congress in 1990 [39]. There are some important reasons for introducing and investigation of so called HV-structures, that is an HV-group, an HV-ring, and so on, which are defined from the well known classes of hyperstructures in a certain simple way. The idea consists in replacing some axioms, such as the associative law, the distributive law, and others by the corresponding weak ones. The hyperstructure $(H, \circ)$ is called an *HV-semigroup* if $a \circ (b \circ c) \cap (a \circ b) \circ c \neq \phi$ for all $a, b, c \in H$. The hyperstructure $(H, \circ)$ is called an *HV-group* if $(H, \circ)$ is an HV-semigroup and $a \circ H = H \circ a = H$ for all $a \in H$. Since a quotient of an HV-structure with respect to a fundamental equivalence relation ($\beta^*, \gamma^*, \epsilon^*$, ets.) is always an ordinary structure and this is why it is called an HV-structure. Many authors have published papers relating different "HV-structures". In particular, a variety of HV-structures theory have been defined such as: partial abelian HV-monoids [9], HV-semigroups [33], HV-groups [34], HV-rings [35], HV-modules [10] and HV-vector spaces [37]. In [7] Davvaz surveyed the theory of HV-structures.

The relation of ordered sets and algebraic hyperstructures was first studied by Vougiouklis in 1987 [36]. Then the connection between hyperstructures and ordered sets has been analyzed by many researchers such as Corsini [5], Omidi and Davvaz [28] Hoskova [16], Heidari and Davvaz [14], and others. One special aspect of this issue, known as EL-hyperstructures, which was first introduced by Chavlina in [3] are hypercompositional structures constructed from a partially (semi)group using a construction known as Ending lemma or Ends lemma. Lots of papers regarding this topic have been written by number of authors like Hoskova [15, 16], Novak [23, 26, 27], Rosenberg [32], and others [11, 12, 13]. Among them, Novak in [23] studied subhyperstructures of EL-hyperstructures and in [24] he discussed some interesting results of important elements in this family of hyperstructures. Then, in [24] Novak studied some basic properties of EL-hyperstructures like invertibility, normality, property of being closed and ultra closed, regularity, and etc. Now, there arises a natural question that "Is it possible to go further to stronger hyperstructure like BE-algebras, B-algebras, etc?"

In this paper, first we define the concept of an HV-BE-algebra and prove some basic results, then we apply "Ends lemma" on a BE-algebra and achieve the new HV-BE-algebra associated to it.

## 2. Basic Definitions and Results

The notion of a *BE-algebra*, as a generalization of a *BCK-algebra*, was introduced by H. S. Kim and Y. H. Kim [20]. The aim of this section is to introduce an HV-BE-algebra, give some examples, and find some of their properties. Let $X$ be a nonempty set, $*: X \times X \to X$ be a binary operation and "1" be constant. The triple $(X, *, 1)$ is called a *BE-algebra* if for all $x \in X$ we have $x * x = 1$, $x * 1 = 1$ and $1 * x = x$, where a relation "$\leq$" is defined by $x \leq y$ if and only if $x * y = 1$ and for all $x, y, z \in X$, we have $x * (y * z) = y * (x * z)$. A nonempty subset $Y$ of a BE-algebra $(X, *, 1)$ is said to be a BE-*subalgebra* of $X$, if $1 \in Y$ and $x * y \in Y$, for all $x, y \in Y$. A BE-algebra $(X, *, 1)$ is said to be *commutative*, if $(x * y) * y = (y * x) * x$ for any $x, y \in X$ [20].

**Definition 2.1** ([29]). Let $H$ be a nonempty set, $\circ : H \times H \to \wp^*(H)$ be a hyperoperation and "1" be constant. The triple $(H, \circ, 1)$ is called a *hyper BE-algebra* if for all $x, y, z \in H$ we have $x \leq 1$, $x \leq x$, $x \circ (y \circ z) = y \circ (x \circ z)$, $x \in 1 \circ x$ and $1 \leq x$ implies $x = 1$, where the relation "$\leq$" is defined by $x \leq y$ if and only if $1 \in x \circ y$. For any two nonempty subsets $X$ and $Y$ of $H$, $X \leq Y$ if and only if there exist $x \in X$ and $y \in Y$ such that $x \leq y$. A nonempty subset $S$ of a hyper BE-algebra $H$ is said to be a hyper BE-*subalgebra* of $H$, if $1 \in S$ and $x \circ y \subseteq S$, for all $x, y \in S$.

**Example 1.** Let $H = \{1, a, b\}$ be a set with the following table:

| $\circ$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a, b\}$ | $\{b\}$ |
| $a$ | $\{1\}$ | $\{1, a\}$ | $\{1, b\}$ |
| $b$ | $\{1\}$ | $\{1, a, b\}$ | $\{1\}$ |

Then it follows that $(H, \circ, 1)$ is a hyper BE-algebra.

**Example 2.** It is obvious that $\{1\}$ and $H$ are hyper BE-subalgebras of a hyper BE-algebra of $H$. In Example 1, $\{1, a\}$ is not a hyper BE-subalgebra of the hyper BE-algebra $(H, \circ, 1)$. Also, $\{1\}$ and $\{1, b\}$ are hyper BE-subalgebras of the hyper BE-algebra $(H, \circ, 1)$.

**Definition 2.2.** $(H, \circ, 1)$ is called an *HV-BE-algebra*, if it satisfies the following axioms:
(HVBE1) $x \leq 1$, $x \leq x$,
(HVBE2) $x \circ (y \circ z) \cap y \circ (x \circ z) \neq \phi$,
(HVBE3) $x \in 1 \circ x$,
(HVBE4) $1 \leq x$ implies $x = 1$, for all $x, y, z \in H$,
where the relation "$\leq$" is defined by $x \leq y$ if and only if $1 \in x \circ y$. For any two nonempty subsets $X$ and $Y$ of $H$, $X \leq Y$ if and only if there exist $x \in X$ and $y \in Y$ such that $x \leq y$. An HV-BE-algebra $(H, \circ, 1)$ is said to be *commutative* if

$$(x \circ y) \circ y \cap (y \circ x) \circ x \neq \phi$$

for all $x, y \in H$.

It is obvious that a hyper BE-algebra is an HV-BE-algebra.

**Example 3.** Let $H = \{1, a, b, c\}$ and define a hyperoperation "$\circ$" as follows:

| $\circ$ | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ |
| $a$ | $\{1\}$ | $\{1\}$ | $\{b, c\}$ | $\{b, c\}$ |
| $b$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{c\}$ |
| $c$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |

Then by examining the HV-BE-algebra's properties we conclude that $(H, \circ, 1)$ is an HV-BE-algebra. Since $a \circ (b \circ c) = \{b, c\}$ and $b \circ (a \circ c) = \{1, c\}$, then $(H, \circ, 1)$ is not a hyper BE-algebra.

**Example 4.** (i) Let $H = \{1, a\}$. Define hyperoperations "$\circ_1$" and "$\circ_2$" as follows:

| $\circ_1$ | $1$ | $a$ |
|---|---|---|
| $1$ | $\{1, a\}$ | $\{a\}$ |
| $a$ | $\{1\}$ | $\{1\}$ |

| $\circ_2$ | $1$ | $a$ |
|---|---|---|
| $1$ | $\{1, a\}$ | $\{a\}$ |
| $a$ | $\{1, a\}$ | $\{1\}$ |

Then $1 \in x \circ_1 1, 1 \in x \circ_1 x, 1 \in x \circ_2 1$ and $1 \in x \circ_2 x$ for all $x \in H$. By examining the other properties of this algebra, we conclude $(H, \circ_1, 1)$ and $(H, \circ_2, 1)$ are HV-BE-algebras. Since $1 \circ_1 (a \circ_1 a) = \{1, a\}$ and $a \circ_1 (1 \circ_1 a) = \{1\}$, then $(H, \circ_1, 1)$ is not a hyper BE-algebra. Also, $(H, \circ_2, 1)$ is an HV-BE-algebra. Since $1 \circ_2 (a \circ_2 a) = \{1, a\}$ and $a \circ_2 (1 \circ_2 a) = \{1\}$, then $(H, \circ_2, 1)$ is not a hyper BE-algebra.

(ii) Let $H = \{1, a, b\}$. Define hyperoperations "$\circ_3$" to "$\circ_6$" as follows:

| $\circ_3$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a, b\}$ | $\{b\}$ |
| $a$ | $\{1, b\}$ | $\{1\}$ | $\{1, a, b\}$ |
| $b$ | $\{1\}$ | $\{1, b\}$ | $\{1, b\}$ |

| $\circ_4$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{b\}$ |
| $a$ | $\{1, b\}$ | $\{1\}$ | $\{1, a, b\}$ |
| $b$ | $\{1\}$ | $\{1, b\}$ | $\{1, b\}$ |

| $\circ_5$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{b\}$ |
| $a$ | $\{1\}$ | $\{1, b\}$ | $\{1, b\}$ |
| $b$ | $\{1, a\}$ | $\{1\}$ | $\{1, a, b\}$ |

| $\circ_6$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{b\}$ |
| $a$ | $\{1\}$ | $\{1, a\}$ | $\{1, a\}$ |
| $b$ | $\{1, a\}$ | $\{1\}$ | $\{1, a, b\}$ |

Then by calculating the properties of this algebra, it follows that $(H, \circ_3)$, $(H, \circ_4), (H, \circ_5)$ and $(H, \circ_6)$ are HV-BE-algebras which are not hyper BE-algebras.

(iii) Let $H = \{1, 2, ...\}$ and the operation "$\circ$" be defined as follows:

$$x \circ y = \begin{cases} \{1\} & \text{if } y \leq x \\ \{h \in H | h \geq y\} & \text{otherwise,} \end{cases}$$

for any $x, y \in H$. Then it can be verified that $(H, \circ)$ is an HV-BE-algebra. Since $1 \circ (2 \circ 2) = \{1\}$ and $2 \circ (1 \circ 2) = \{1, 3, 4, ...\}$, then $(H, \circ, 1)$ is not a hyper BE-algebra.

**Example 5.** (i) Let $H = \{1, a, b\}$ and define a hyperoperation "$\circ$" as follows:

| $\circ$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a, b\}$ | $\{b\}$ |
| $a$ | $\{1, b\}$ | $\{1\}$ | $\{1, a, b\}$ |
| $b$ | $\{1\}$ | $\{1, b\}$ | $\{1, b\}$ |

Then it can be checked that $(H, \circ, 1)$ is a commutative HV-BE-algebra.

(ii) In Example 3, the HV-BE-algebra $(H, \circ, 1)$ is not commutative, since $(a \circ b) \circ b \cap (b \circ a) \circ a = \phi$.

**Theorem 2.3.** *Let $(H, \circ, 1)$ be an HV-BE-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets $A$ and $B$ of $H$ the following statements hold:*

*(i)* $x \circ (y \circ z) \leq y \circ (x \circ z)$ *and* $y \circ (x \circ z) \leq x \circ (y \circ z)$,

*(ii)* $A \circ (B \circ C) \cap B \circ (A \circ C) \neq \phi$,

*(iii)* $A \circ (B \circ C) \leq B \circ (A \circ C)$ *and* $B \circ (A \circ C) \leq A \circ (B \circ C)$,

*(iv)* $x \leq y \circ y$,

*(v)* $x \leq x \circ x$,

*(vi)* $A \leq B \circ B$,

*(vii)* $A \leq A \circ A$,

*(viii)* $A \leq A$,

*(ix)* $1 \leq A$ *implies* $1 \in A$,

*(x)* $A \leq B$ *if and only if* $1 \in A \circ B$,

*(xi)* $A \subseteq 1 \circ A$,

*(xii)* $A \subseteq B$ *implies* $A \leq B$,

*(xiii)* $1 \in x \circ (x \circ 1)$.

*Proof.* (i) By (HVBE2), there exists $d \in x \circ (y \circ z) \cap y \circ (x \circ z)$. Then there exists $d \in x \circ (y \circ z)$ and $d \in y \circ (x \circ z)$ such that $d \leq d$.

(ii) There exist $a \circ (b \circ c) \subseteq A \circ (B \circ C)$ and $b \circ (a \circ c) \subseteq B \circ (A \circ C)$ for all $a \in A, b \in B$ and $c \in C$. Then by (HVBE2), there exists $d \in a \circ (b \circ c) \cap b \circ (a \circ c)$ and so there exists $d \in A \circ (B \circ C) \cap B \circ (A \circ C)$, i.e., $A \circ (B \circ C) \cap B \circ (A \circ C) \neq \phi$.

(iii) By (ii), there exists $d \in A \circ (B \circ C) \cap B \circ (A \circ C)$. Then there exists $d \in A \circ (B \circ C)$ and $d \in B \circ (A \circ C)$ such that $d \leq d$.

(iv) By (HVBE1), $1 \in x \circ 1 \subseteq x \circ (y \circ y)$ and so $1 \in x \circ (y \circ y)$, i.e., $x \leq y \circ y$.

(v) If $y = x$, by (iv), we have $x \leq x \circ x$.

(vi) There exist $a \in A$ and $b \circ b \subseteq B \circ B$ such that $a \leq b \circ b$ by (iv), i.e., $A \leq B \circ B$.

(vii) There exist $a \in A$ and $a \circ a \subseteq A \circ A$ such that $a \leq a \circ a$ by (v), i.e., $A \leq A \circ A$.

(viii) By (HVBE1), there exists $a \in A$ such that $a \leq a$. It means $A \leq A$.

(ix) Let $1 \leq A$. It means that there exists $a \in A$ such that $1 \leq a$. By (HVBE4), $a = 1$ and so $1 \in A$.

(x)

$$\begin{aligned} A \leq B &\Leftrightarrow \exists a \in A, \exists b \in B \;\; s.t. \;\; a \leq b \\ &\Leftrightarrow \exists a \in A, \exists b \in B \;\; s.t. \;\; 1 \in a \circ b \\ &\Leftrightarrow 1 \in \bigcup_{a \in A, b \in B} a \circ b \\ &\Leftrightarrow 1 \in A \circ B. \end{aligned}$$

(xi) Since $1 \circ A = \bigcup_{a \in A} 1 \circ a$ and $a \in 1 \circ a$, we have $A \subseteq 1 \circ A$.

(xii) Let $x \in A$, then $x \in B$. Hence $1 \in x \circ x$, which implies $1 \in A \circ B$. By (x), we have $A \leq B$.

(xiii) By (HVBE1), $1 \in x \circ 1 \subseteq x \circ (x \circ 1)$ and so $1 \in x \circ (x \circ 1)$.

□

**Theorem 2.4.** *Let $(H_1, \circ_1, 1_1)$ and $(H_2, \circ_2, 1_2)$ be HV-BE-algebras and $H = H_1 \times H_2$. We define a hyperoperation "$\circ$" on $H$ as follows,*

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 \circ_1 a_2, b_1 \circ_2 b_2)$$

*for all $(a_1, b_1), (a_2, b_2) \in H$, where for $A \subseteq H_1$ and $B \subseteq H_2$ by $(A, B)$, we mean $(A, B) = \{(a, b) \mid a \in A, b \in B\}, 1 = (1_1, 1_2)$. Then $(H, \circ, 1)$ is an HV-BE-algebra, and it is called the HV-BE-product of $H_1$ and $H_2$.*

*Proof.* Let $(x, y) \in H$. By $H_1$VBE1 and $H_2$VBE1, $1_1 \in x \circ_1 1_1$ and $1_2 \in y \circ_2 1_2$. Since $(x, y) \circ (1_1, 1_2) = (x \circ_1 1_1, y \circ_2 1_2), 1 \in (x, y) \circ 1$. Then $(x, y) \leq 1$. The proof of $(x, y) \leq (x, y)$ is obtained by $x \leq x$ and $y \leq y$. Therefore HVBE1 is valid.

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in H$. Then

$$\begin{aligned} (x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) &= (x_1, y_1) \circ (x_2 \circ_1 x_3, y_2 \circ_2 y_3) \\ &= \bigcup \{(x_1, y_1) \circ (a, b) \mid a \in x_2 \circ_1 x_3, b \in y_2 \circ_2 y_3\} \end{aligned}$$

$$= \bigcup\{(x_1 \circ_1 a, y_1 \circ_2 b \mid a \in x_2 \circ_1 x_3, b \in y_2 \circ_2 y_3\}$$
$$= (x_1 \circ_1 (x_2 \circ_1 x_3), y_1 \circ_2 (y_2 \circ_2 y_3)).$$

By $H_1$VBE2 and $H_2$VBE2, $x_1 \circ_1 (x_2 \circ_1 x_3) \cap x_2 \circ_1 (x_1 \circ_1 x_3) \neq \phi$, $y_1 \circ_2 (y_2 \circ_2 y_3) \cap y_2 \circ_2 (y_1 \circ_2 y_3) \neq \phi$ and so $(x_1 \circ_1 (x_2 \circ_1 x_3), y_1 \circ_2 (y_2 \circ_2 y_3)) \cap (x_2 \circ_1 (x_1 \circ_1 x_3), y_2 \circ_2 (y_1 \circ_2 y_3)) \neq \phi$. On the other hand $(x_2 \circ_1 (x_1 \circ_1 x_3), y_2 \circ_2 (y_1 \circ_2 y_3)) = (x_2, y_2) \circ (x_1 \circ_1 x_3, y_1 \circ_2 y_3) = (x_2, y_2) \circ ((x_1, y_1) \circ (x_3, y_3))$. Therefore $(x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) \cap (x_2, y_2) \circ ((x_1, y_1) \circ (x_3, y_3)) \neq \phi$ and HVBE2 is valid.

Let $(x, y) \in H$. By $H_1$VBE3 and $H_2$VBE3, $x \in 1_1 \circ_1 x, y \in 1_2 \circ_2 y$. Then $(x, y) \in (1_1 \circ_1 x, 1_2 \circ_2 y) = (1_1, 1_2) \circ (x, y) = 1 \circ (x, y)$. Therefore $(x, y) \in 1 \circ (x, y)$ and HVBE3 is valid.

Let $(x, y) \in H$ and $(1_1, 1_2) \leq (x, y)$. By $H_1$VBE4 and $H_2$VBE4, $1_1 \leq x$ and $1_2 \leq y$ implies $x = 1_1$ and $y = 1_2$. Then $(x, y) = (1_1, 1_2) = 1$ and so HVBE4 is valid. Therefore $(H, \circ, 1)$ is an HV-BE-algebra. □

**Example 6.** Consider two HV-BE-algebras $(H, \circ_3, 1)$ and $(H, \circ_4, 1)$ in Example 4. By calculating the properties of the HV-BE-product, we conclude $(H \times H, \circ, (1, 1))$ with the following table is an HV-BE-algebra of $(H, \circ_3, 1)$ and $(H, \circ_4, 1)$.

| $\circ$ | $(1,1)$ | $(1,a)$ | $(1,b)$ | $(a,1)$ | $(a,a)$ | $(a,b)$ | $(b,1)$ | $(b,a)$ | $(b,b)$ |
|---|---|---|---|---|---|---|---|---|---|
| $(1,1)$ | $\{(1,1)\}$ | $\{(1,a)\}$ | $\{(1,b)\}$ | $A$ | $B$ | $C$ | $\{(b,1)\}$ | $\{(b,a)\}$ | $\{(b,b)\}$ |
| $(1,a)$ | $D$ | $\{(1,1)\}$ | $E$ | $F$ | $A$ | $G$ | $H$ | $\{(b,1)\}$ | $I$ |
| $(1,b)$ | $\{(1,1)\}$ | $D$ | $D$ | $A$ | $F$ | $F$ | $\{(b,1)\}$ | $H$ | $H$ |
| $(a,1)$ | $J$ | $K$ | $L$ | $\{(1,1)\}$ | $\{(1,a)\}$ | $\{(1,b)\}$ | $M$ | $N$ | $O$ |
| $(a,a)$ | $P$ | $J$ | $Q$ | $D$ | $\{(1,1)\}$ | $E$ | $R$ | $M$ | $S$ |
| $(a,b)$ | $J$ | $P$ | $P$ | $\{(1,1)\}$ | $D$ | $D$ | $M$ | $R$ | $R$ |
| $(b,1)$ | $\{(1,1)\}$ | $\{(1,a)\}$ | $\{(1,b)\}$ | $J$ | $K$ | $L$ | $J$ | $K$ | $L$ |
| $(b,a)$ | $D$ | $\{(1,1)\}$ | $E$ | $P$ | $J$ | $Q$ | $P$ | $J$ | $Q$ |
| $(b,b)$ | $\{(1,1)\}$ | $D$ | $D$ | $J$ | $P$ | $P$ | $J$ | $P$ | $P$ |

where $A = \{(a, 1), (b, 1)\}$, $B = \{(a, a), (b, a)\}$, $C = \{(a, b), (b, b)\}$, $D = \{(1, 1), (1, b)\}$, $E = \{(1, 1), (1, a), (1, b)\}$, $F = \{(a, 1), (b, 1), (a, b), (b, b)\}$, $G = \{(a, 1), (b, 1), (a, a), (b, a), (a, b), (b, b)\}$, $H = \{(b, 1), (b, b)\}$, $I = \{(b, 1), (b, a), (b, b)\}$, $J = \{(1, 1), (b, 1)\}$, $K = \{(1, a), (b, a)\}$, $L = \{(1, b), (b, b)\}$, $M = \{(1, 1), (a, 1), (b, 1)\}$, $N = \{(1, a), (a, a), (b, a)\}$, $O = \{(1, b), (a, b), (b, b)\}$, $P = \{(1, 1), (b, 1), (1, b), (b, b)\}$, $Q = \{(1, 1), (b, 1), (1, a), (b, a), (1, b), (b, b)\}$, $R = \{(1, 1), (a, 1), (b, 1), (1, b), (a, b), (b, b)\}$ and $S = \{(1, 1), (a, 1), (b, 1), (1, a), (a, a), (b, a), (1, b), (a, b), (b, b)\}$.

**Theorem 2.5.** *Let $(H_1, \circ_1, 1)$ and $(H_2, \circ_2, 1)$ be HV-BE-algebras such that $H_1 \cap H_2 = \{1\}$, $H = H_1 \cup H_2$ and $x \circ_2 y \cap y \circ_2 x \neq \phi$, for all $x, y \in H_2$. Then $(H, \circ, 1)$ is* *an HV-BE-algebra, where the hyperoperation "$\circ$" on $H$ is defined as follows:*

$$x \circ y := \begin{cases} x \circ_1 y & \text{if } x, y \in H_1, \\ x \circ_2 y & \text{if } x, y \in H_2, \\ \{t | t = x \text{ or } t = y \text{ and } t \in H_2\} & \text{otherwise}, \end{cases}$$

*for all $x, y \in H$.*

*Proof.* (1) If $x, y \in H_1$ or $H_2$, then $(H, \circ, 1)$ is an HV-BE-algebra. (2) If $x \in H_1$ and $y \in H_2$. Then HVBE1, HVBE3 and HVBE4 is valid, Since $(H_1, \circ_1, 1)$ is an HV-BE-algebra. For checking HVBE2, we have two states:

(i) Let $z \in H_1$. Then $x \circ (y \circ z) \cap y \circ (x \circ z) \neq \phi$ and HVBE2 is valid.
(ii) Let $z \in H_2$. Then $x \circ (y \circ z) \cap y \circ (x \circ z) \neq \phi$ and HVBE2 is valid.

(3) If $x \in H_2$ and $y \in H_1$. The proof is similar to (2). □

**Definition 2.6.** A nonempty subset $S$ of an HV-BE-algebra $(H, \circ, 1)$ is said to be an HV-BE-*subalgebra* of $H$, if $1 \in S$ and $x \circ y \subseteq S$, for all $x, y \in S$.

**Example 7.** (i) Let $H = \{1, a, b\}$. Define a hyperoperation "$\circ_1$" as follows:

| $\circ_1$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{b\}$ |
| $a$ | $\{1\}$ | $\{1\}$ | $\{a, b\}$ |
| $b$ | $\{1, b\}$ | $\{1\}$ | $\{1\}$ |

Then by examining the properties of the HV-BE-algebra, it follows that $(H, \circ, 1)$ is an HV-BE-algebra and $S = \{1, a\}$ is an HV-BE-subalgebra of $H$.

(ii) Let $H = \{1, a, b, c\}$. Define a hyperoperation "$\circ_2$" on $H$ as follows:

| $\circ_2$ | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ |
| $a$ | $\{1\}$ | $\{1\}$ | $\{b, c\}$ | $\{b, c\}$ |
| $b$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{c\}$ |
| $c$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |

Then by checking the properties of the HV-BE-algebra it follows that $(H, \circ_2, 1)$ is an HV-BE-algebra and $S = \{1, b, c\}$ is an HV-BE-subalgebra of $H$.

**Example 8.** Let $H = \{1, a, b, c, d\}$ be a set. Then we can check that $(H, \circ, 1)$ with the following table is an HV-BE-algebra.

| $\circ$ | 1 | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| 1 | $\{1\}$ | $\{a,b\}$ | $\{b\}$ | $\{c\}$ | $\{d\}$ |
| $a$ | $\{1,b\}$ | $\{1\}$ | $\{1,a,b\}$ | $\{c\}$ | $\{d\}$ |
| $b$ | $\{1\}$ | $\{1,b\}$ | $\{1,b\}$ | $\{c\}$ | $\{d\}$ |
| $c$ | $\{1\}$ | $\{c\}$ | $\{c\}$ | $\{1\}$ | $\{1,c,d\}$ |
| $d$ | $\{1\}$ | $\{d\}$ | $\{d\}$ | $\{1\}$ | $\{1\}$ |

Then they can be verified that $S = \{1, a, b\}$ is an HV-BE-subalgebra of $H$, but $T = \{1, a, b, d\}$ is not an HV-BE-subalgebra of $H$ since $d\circ(a\circ a) = 1$ and $a\circ(d\circ a) = d$.

**Remark 1.** By Theorem 2.5, we can see the HV-BE-algebra $(H, \circ, 1)$ in Example 8 is obtained from two HV-BE-algebras as follows:

| $\circ_1$ | 1 | $a$ | $b$ |
|---|---|---|---|
| 1 | $\{1\}$ | $\{a,b\}$ | $\{b\}$ |
| $a$ | $\{1,b\}$ | $\{1\}$ | $\{1,a,b\}$ |
| $b$ | $\{1\}$ | $\{1,b\}$ | $\{1,b\}$ |

| $\circ_2$ | 1 | $c$ | $d$ |
|---|---|---|---|
| 1 | $\{1\}$ | $\{c\}$ | $\{d\}$ |
| $c$ | $\{1\}$ | $\{1\}$ | $\{c,d\}$ |
| $d$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |

## 3. Some Types of HV-BE-algebras

Radfar and et. al. in [29] introduced some types of hyper BE-algebras. In this section, we introduce them for HV-BE-algebras and give an example for each of them.

**Definition 3.1.** We say an HV-BE-algebra is:

(i) a *row HV-BE-algebra* (briefly, an R-HV-BE-algebra), if $1 \circ x = \{x\}$, for all $x \in H$,

(ii) a *column HV-BE-algebra* (briefly, a C-HV-BE-algebra), if $x \circ 1 = \{1\}$, for all $x \in H$,

(iii) a *diagonal HV-BE-algebra* (briefly, a D-HV-BE-algebra), if $x \circ x = \{1\}$, for all $x \in H$,

(iv) a *thin HV-BE-algebra* (briefly, a T-HV-BE-algebra), if it is an R-HV-BE-algebra and a C-HV-BE-algebra (or an RC-HV-BE-algebra),

(v) a *very thin HV-BE-algebra* (briefly, a V-HV-BE-algebra), if it is an R-HV-BE-algebra, a C-HV-BE-algebra and a D-HV-BE-algebra (or an RCD-HV-BE-algebra).

(vi) a *CD-HV-BE-algebra*, if it is a C-HV-BE-algebra and a D-HV-BE-algebra.

**Example 9.** (i) In Example 4, $(H, \circ_4, 1)$ is an R-HV-BE-algebra.

(ii) Let $H = \{1, a, b\}$. Define hyperoperations "$\circ_1$" to "$\circ_6$" as follows:

| $\circ_1$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{a, b\}$ |
| $a$ | $\{1\}$ | $\{1\}$ | $\{1, a, b\}$ |
| $b$ | $\{1\}$ | $\{1, b\}$ | $\{1, b\}$ |

| $\circ_2$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{a, b\}$ |
| $a$ | $\{1, b\}$ | $\{1\}$ | $\{1, a, b\}$ |
| $b$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |

| $\circ_3$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{b\}$ |
| $a$ | $\{1\}$ | $\{1\}$ | $\{1, a, b\}$ |
| $b$ | $\{1\}$ | $\{1, b\}$ | $\{1, b\}$ |

| $\circ_4$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{b\}$ |
| $a$ | $\{1\}$ | $\{1\}$ | $\{1, a, b\}$ |
| $b$ | $\{1\}$ | $\{1, b\}$ | $\{1\}$ |

| $\circ_5$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{b\}$ |
| $a$ | $\{1, b\}$ | $\{1\}$ | $\{1, a, b\}$ |
| $b$ | $\{1\}$ | $\{1\}$ | $\{1, b\}$ |

| $\circ_6$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $\{1\}$ | $\{a\}$ | $\{a, b\}$ |
| $a$ | $\{1\}$ | $\{1\}$ | $\{1, a, b\}$ |
| $b$ | $\{1\}$ | $\{1, b\}$ | $\{1\}$ |

Then they can be checked that $(H, \circ_1, 1)$ is a C-HV-BE-algebra, $(H, \circ_2, 1)$ is a D-HV-BE-algebra, $(H, \circ_3, 1)$ is a T-HV-BE-algebra, $(H, \circ_4, 1)$ is a V-HV-BE-algebra, $(H, \circ_5, 1)$ is an RD-HV-BE-algebra and $(H, \circ_6, 1)$ is a CD-HV-BE-algebra.

**Theorem 3.2.** *Let H be a D-HV-BE-algebra. Then*

*(i) there exists $a \in 1 \circ x$ such that $1 \in x \circ a$,*

*(ii) $x \circ 1 \cap y \circ (x \circ y) \neq \phi$,*

*(iii) $x \circ 1 \cap 1 \circ (x \circ 1) \neq \phi$.*

*Proof.* (i) By Definition 3.1, $\{1\} = 1 \circ 1 = 1 \circ (x \circ x)$ and by (HVBE2), $1 \in x \circ (1 \circ x)$. Then there exists $a \in 1 \circ x$ such that $1 \in x \circ a$.

(ii) By (HVBE2), $y \circ (x \circ y) \cap x \circ (y \circ y) \neq \phi$ and by Definition 3.1, $x \circ (y \circ y) = x \circ 1$. Then $x \circ 1 \cap y \circ (x \circ y) \neq \phi$.

(iii) By (HVBE2), $1 \circ (x \circ 1) \cap x \circ (1 \circ 1) \neq \phi$ and by Definition 3.1, $x \circ (1 \circ 1) = x \circ 1$. Then $x \circ 1 \cap 1 \circ (x \circ 1) \neq \phi$.

□

**Theorem 3.3.** *Let H be a CD-HV-BE-algebra. Then*

*(i) $1 \in x \circ (y \circ x)$,*

*(ii)* $z \in y \circ x$ *implies* $x \leq z$*, for all* $x, y, z \in H$.

*Proof.* (i) By Definition 3.1, $y \circ (x \circ x) = y \circ 1 = \{1\}$ and by (HVBE2), $1 \in x \circ (y \circ x)$.

(ii) By (i), $1 \in x \circ z$ for some $z \in y \circ x$, then $x \leq z$. □

## 4. BL-BE-algebras

Can we arrive to hyper BE-algebras (HV-BE-algebras) from BE-algebras based on "Ends lemma"? In the following theorem, we are going to answer this question by changing it.

**Theorem 4.1.** *Let* $(X, *, 1)$ *be a BE-algebra. Then the binary hyperoperation* $\bullet : X \times X \to \wp^*(X)$ *defined by*

$$x \bullet y = (x * y]_{\leq}, \text{ for all } x, y \in X,$$

*is an HV-BE-algebra. Moreover, if the BE-algebra* $(X, *, 1)$ *is commutative, then the HV-BE-algebra* $(X, \bullet, 1)$ *is commutative.*

*Proof.* Let $x \in X$. First, we show that (HVBE1) is valid. Since $x \bullet 1 = (x*1]_{\leq} = \{t \in X \mid t \leq x*1\}$, then $1 \in x \bullet 1$ and so $x \leq 1$. Also, we have $x \bullet x = (x*x]_{\leq} = \{t \in X \mid t \leq x*x\}$, then $1 \in x \bullet x$ and so $x \leq x$. Then, we show that (HVBE2) is valid. We have $x \bullet (y \bullet z) = \{x \bullet t \mid t \in y \bullet z\} = \{x \bullet t \mid t \leq y*z\} = \{t' \in X \mid t' \leq x*t, t \leq y*z\}$. Then there exist $x * (y * z) \in x \bullet (y \bullet z)$ and $y * (x * z) \in y \bullet (x \bullet z)$. Since $x * (y * z) = y * (x * z)$, we have $x \bullet (y \bullet z) \cap y \bullet (x \bullet z) \neq \phi$. Now we check that (HVBE3) is valid. Since $1 \bullet x = (1*x]_{\leq} = \{t \in X \mid t \leq 1*x\}$, then $x \in 1 \bullet x$. Finally for checking (HVBE4), let $1 \leq x$. Since $1 \in 1 \bullet x = (1*x]_{\leq} = \{t \in X \mid t \leq x\}$, we have $1 \leq x$ and so $1 * x = 1$. On the other hand, $1 * x = x$. Therefore $x = 1$.

Suppose that $(x * y) * y = (y * x) * x$. Then $(x \bullet y) \bullet y = \{t \bullet y \mid t \leq x * y\} = \{t' \in X \mid t' \leq t * y, t \leq x * y\}$. Therefore, $(x * y) * y \in (x \bullet y) \bullet y$ and similarly $(y * x) * x \in (y \bullet x) \bullet x$. Since the BE-algebra $(X, *, 1)$ is commutative, we have $(x \bullet y) \bullet y \cap (y \bullet x) \bullet x \neq \phi$ and the HV-BE-algebra $(H, \bullet, 1)$ constructed in this way is commutative. □

The HV-BE-algebra $(X, \bullet, 1)$ constructed in this way, we call *the associated HV-BE-algebra to the BE-algebra* $(X, *, 1)$ or *"Begins lemma" based on HV-BE-algebras*, or *BL-BE-algebras* for short.

**Example 10.** Let $X = \{1, a\}$ be a set with the following table:

| $*$ | 1 | $a$ |
|---|---|---|
| 1 | 1 | $a$ |
| $a$ | 1 | 1 |

Then it is easy to see that $(X, *, 1)$ is a BE-algebra and $(X, \bullet, 1)$ is a BL-BE-algebra with the following:

| $\bullet$ | 1 | $a$ |
|---|---|---|
| 1 | $X$ | $\{a\}$ |
| $a$ | $X$ | $X$ |

**Example 11.** Let $X := \{1, a, b, c, d, 0\}$ be a set with the following table:

| $*$ | 1 | $a$ | $b$ | $c$ | $d$ | 0 |
|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | 0 |
| $a$ | 1 | 1 | $a$ | $c$ | $c$ | $d$ |
| $b$ | 1 | 1 | 1 | $c$ | $c$ | $c$ |
| $c$ | 1 | $a$ | $b$ | 1 | $a$ | $b$ |
| $d$ | 1 | 1 | $a$ | 1 | 1 | $a$ |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |

Then $(X, *, 1)$ is a BE-algebra [20]. We can check that $(X, \bullet, 1)$ is a commutative BL-BE-algebra with the following table:

| $\bullet$ | 1 | $a$ | $b$ | $c$ | $d$ | 0 |
|---|---|---|---|---|---|---|
| 1 | $X$ | $\{a, b, d, 0\}$ | $\{b, 0\}$ | $\{c, d, 0\}$ | $\{d, 0\}$ | $\{0\}$ |
| $a$ | $X$ | $X$ | $\{a, b, d, 0\}$ | $\{c, d, 0\}$ | $\{c, d, 0\}$ | $\{d, 0\}$ |
| $b$ | $X$ | $X$ | $X$ | $\{c, d, 0\}$ | $\{c, d, 0\}$ | $\{c, d, 0\}$ |
| $c$ | $X$ | $\{a, b, d, 0\}$ | $\{b, 0\}$ | $X$ | $\{a, b, d, 0\}$ | $\{b, 0\}$ |
| $d$ | $X$ | $X$ | $\{a, b, d, 0\}$ | $X$ | $X$ | $\{a, b, d, 0\}$ |
| 0 | $X$ | $X$ | $X$ | $X$ | $X$ | $X$ |

**Theorem 4.2.** *Let $(X, \bullet, 1)$ be a BL-BE-algebra. Then for any $x, y \in X$ and for all nonempty subsets $A$ and $B$ of $X$ the following statements holds:*

*(i)* $x \bullet (y \bullet y) = X$,

*(ii)* $x \bullet (x \bullet x) = X$,

*(iii)* $A \bullet (B \bullet B) = X$,

*(iv)* $A \bullet (A \bullet A) = X$,

*(v)* $A \bullet A = X$,

*(vi)* $x \bullet (x \bullet 1) = X$.

*Proof.* It is straightforward. □

**Theorem 4.3.** *Let $(X_1, \bullet_1, 1_1)$ and $(X_2, \bullet_2, 1_2)$ be BL-BE-algebras and $X = X_1 \times X_2$. We define a hyperoperation "$\bullet$" on $X$ as follows,*

$$(x_1, y_1) \bullet (x_2, y_2) = (x_1 \bullet_1 x_2, y_1 \bullet_2 y_2)$$

*for all $(x_1, y_1), (x_2, y_2) \in H$, where for $A \subseteq X_1$ and $B \subseteq X_2$ by $(A, B)$ we mean $(A, B) = \{(a, b) \mid a \in A, b \in B\}, 1 = (1_1, 1_2)$. Then $(X, \bullet, 1)$ is a BL-BE-algebra, and it is called the BL-BE-product of $X_1$ and $X_2$.*

*Proof.* It is similar to the proof of Theorem 2.4. □

**Example 12.** Let $X_1 = \{1, a, b\}$ and $X_2 = \{1, c, d\}$ be two sets and $(X_1, \bullet_1, 1)$ and $(X_2, \bullet_2, 1)$ be two BL-BE-algebras as follows:

| $\bullet_1$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $X_1$ | $\{a, b\}$ | $\{b\}$ |
| $a$ | $X_1$ | $X_1$ | $\{a, b\}$ |
| $b$ | $X_1$ | $X_1$ | $X_1$ |

| $\bullet_2$ | $1$ | $c$ | $d$ |
|---|---|---|---|
| $1$ | $X_2$ | $\{c, d\}$ | $\{d\}$ |
| $c$ | $X_2$ | $X_2$ | $\{c, d\}$ |
| $d$ | $X_2$ | $X_2$ | $X_2$ |

It can be verified that $(X_1 \times X_2, \bullet, (1, 1))$ is a BL-BE-algebra by Theorem 4.1.

**Theorem 4.4.** *Let $(X_1, \bullet_1, 1)$ and $(X_2, \bullet_2, 1)$ be BL-BE-algebras such that $X_1 \cap X_2 = \{1\}$ and $X = X_1 \cup X_2$. Then $(X, \bullet, 1)$ is a BL-BE-algebra, where the hyperoperation "$\bullet$" on $H$ is defined as follows:*

$$x \bullet y := \begin{cases} x \bullet_1 y & \text{if } x, y \in X_1, \\ x \bullet_2 y & \text{if } x, y \in X_2, \\ X & \text{otherwise}, \end{cases}$$

*for all $x, y \in X$.*

*Proof.* It is similar to the proof of Theorem 2.5. □

We use the notation $X_1 \oplus X_2$ for the union of two BL-BE-algebras $X_1$ and $X_2$.

**Example 13.** Consider two BL-BE-algebras $(X_1, \bullet_1, 1)$ and $(X_2, \bullet_2, 1)$ in Example 12. By Theorem 2.5, it can be verified that $(X, \bullet, 1)$ with the following table is a BL-BE-algebra.

| $\bullet$ | $1$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $1$ | $X_1$ | $\{a, b\}$ | $\{b\}$ | $\{c, d\}$ | $\{d\}$ |
| $a$ | $X_1$ | $X_1$ | $\{a, b\}$ | $X$ | $X$ |
| $b$ | $X_1$ | $X_1$ | $X_1$ | $X$ | $X$ |
| $c$ | $X_2$ | $X$ | $X$ | $X_2$ | $\{c, d\}$ |
| $d$ | $X_2$ | $X$ | $X$ | $X_2$ | $X_2$ |

Now, we give the concept of a principal beginning generated by an element, which lies in the subset in H:= X.

Suppose an HV-BE-algebra $(H, \bullet, 1)$ associated to the BE-algebra $(H, *, 1)$ and a nonempty subset $G$ of $H$. For an arbitrary element $g \in G$, we may write

$$(g]_{\leq G} = \{x \in G \mid x \leq g\}$$

as well as

$$(g]_{\leq H} = \{x \in H \mid x \leq g\}.$$

Given this notation we may distinguish between $(G, \bullet_G, 1)$ based on the hyperoperation $\bullet_G$ such that for an arbitrary pair of elements $a, b \in G$ we set

$$a \bullet_G b = (a * b]_{\leq G} = \{x \in G \mid x \leq a * b\}$$

and $(G, \bullet_H, 1)$, where $a \bullet_H b$ is defined by

$$a \bullet_H b = (a * b]_{\leq H} = \{x \in H \mid x \leq a * b\}.$$

Obviously, properties of $(G, \bullet_G, 1)$ and $(G, \bullet_H, 1)$ will not be the same.

**Theorem 4.5.** *Let $(H, \bullet, 1)$ be the associated HV-BE-algebra to the BE-algebra $(H, *, 1)$ and $(G, *, 1)$ its BE-subalgebra of $H$. Then*

*(i) $x \in y \bullet_H G$, for all $x, y \in H$,*

*(ii) $x \in y \bullet_G G$, for all $x, y \in G$.*

*Proof.* (i) Let $x, y \in H$. Then $y \bullet_H G = \bigcup_{g \in G} y \bullet_H g = y \bullet_H 1 \cup ... = (y * 1]_{\leq H} \cup ... = (1]_{\leq H} \cup ... = \{t \in H \mid t * 1 = 1\} \cup ... = H \cup ... = H$ and so $x \in y \bullet_H G$.

(ii) Let $x, y \in G$. Then $y \bullet_G G = \bigcup_{g \in G} y \bullet_G g = y \bullet_G 1 \cup ... = G$ and so $y \in G$.

□

**Remark 2.** In general, in every BL-BE-algebra, Theorem 4.5 is valid. For see, in Example 13, $a \in c \bullet_X X$ in the union of two BL-BE-algebras $X_1$ and $X_2$ i.e., $X$.

**Theorem 4.6.** *Let $(H, \bullet, 1)$ be the associated HV-BE-algebra of a BE-algebra $(H, *, 1)$. If $(G, *, 1)$ is a subalgebra of a BE-algebra $(H, *, 1)$, then $(G, \bullet_G, 1)$ is an HV-BE-algebra.*

*Proof.* Let $x \in G$. Since $x \bullet_G 1 = (x * 1]_{\leq G} = \{t \in G \mid t \leq 1\}$, then $1 \in x \bullet_G 1$ and so $x \leq 1$. we have $x \bullet_G x = (x * x]_{\leq G} = \{t \in G \mid t \leq 1\}$, then $1 \in x \bullet_G x$ and so $x \leq x$. Also, there exist $x * (y * z) \in x \bullet_G (y \bullet_G z)$, $y * (x * z) \in y \bullet_G (x \bullet_G z)$ and

$x * (y * z) = y * (x * z)$, then $x \bullet_G (y \bullet_G z) \cap y \bullet_G (x \bullet_G z) \neq \phi$. Moreover, since $1 \bullet_G x = (1 * x]_{\leq_G} = \{t \in G \mid t \leq x\}$, then $x \in 1 \bullet_G x$. Finally, let $1 \leq x$. Since $1 \in 1 \bullet_G x = (1 * x]_{\leq_G} = \{t \in G \mid t \leq x\}$, we have $1 \leq x$ and so $1 * x = 1$. On the other hand $1 * x = x$. Therefore $x = 1$. □

**Remark 3.** In Theorem 4.6, $(G, \bullet_H, 1)$ is not an HV-BE-algebra. Since $G$ is not closed with respect to the hyperoperation $\bullet_H$.

**Example 14.** Let $H = \{1, a, b\}$ and define the operation "$*$" on $H$ by the following table:

| $*$ | 1 | $a$ | $b$ |
|---|---|---|---|
| 1 | 1 | $a$ | $b$ |
| $a$ | 1 | 1 | $b$ |
| $b$ | 1 | $a$ | 1 |

It can be easily verified that $(H, *, 1)$ is a BE-algebra. Further define the hyperoperation in the usually "Begins lemma" way, i.e. for an arbitrary pair $x, y \in H$ define $x \bullet y = (x * y]_{\leq}$. Then $(H, \bullet, 1)$ is an HV-BE-algebra with the following table:

| $\bullet$ | 1 | $a$ | $b$ |
|---|---|---|---|
| 1 | $H$ | $\{a\}$ | $\{b\}$ |
| $a$ | $H$ | $H$ | $\{b\}$ |
| $b$ | $H$ | $\{a\}$ | $H$ |

$G = \{1, a\}$ is a BE-subalgebra of a BE-algebra $(H, *, 1)$ and by Theorem 4.6, $(G, \bullet_G, 1)$ is an HV-BE-algebra. But, $(G, \bullet_H, 1)$ is not an HV-BE-algebra.

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