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ON 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS WITH RESPECT TO SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The aim of the present paper is to study 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Firstly, we prove that extended generalized M-projective ϕ-recurrent 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection is an η-Einstein manifold with respect to Levi-Civita connection under some certain conditions. Later we study some curvature properties of 3-dimensional trans-Sasakian manifold admitting the above connection.

1. Introduction

Let (M, ϕ, ξ, η, g) be a $(2n + 1)$-dimensional almost contact metric manifold. Then the product $\bar{M} = M \times R$ has a natural almost complex structure J with the product metric G being Hermitian metric. The geometry of the almost Hermitian manifold $(\bar{M}, J, G)$ gives the geometry of the almost contact metric manifold (M, ϕ, ξ, η, g). Sixteen different types of structures on M like Sasakian manifold, Kenmotsu manifold etc are given by the almost Hermitian manifold $(\bar{M}, J, G)$. The notion of trans-Sasakian manifolds were introduced by Oubina [10] in 1985 . Then J. C. Marrero [7] has studied the local structure of trans-Sasakian manifolds. In general a trans-Sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type (α, β). Trans-Sasakian manifold of type $(0, 0)$, $(\alpha, 0)$, $(0, \beta)$ are called cosymplectic, α-Sasakian and β-Kenmotsu manifold respectively. Marrero has proved that trans-Sasakian structures are generalized quasi-Sasakian structure. He has also proved that

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a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu manifold. So, we have considered here three dimensional trans-Sasakian manifold.

The notion of a semi-symmetric linear connection on a differential manifold has been first introduced by Friedmann and Schouten [4] in 1924. In 1932 Hayden has given the idea of metric connection with torsion on Riemannian manifold in [5]. Yano [15] has given a systematic study of semi-symmetric connection on Riemannian manifold in 1970. Later K. S. Amur and S. S. Pujar [1], C. S. Bagewadi [3], Sharafuddin and Hussian (1976) [13] and others have also studied semi-symmetric connection on Riemannian manifold. S. Pahan, A. Bhattacharyya studied some curvature properties of projective curvature tensor with respect to semi-symmetric connection on a three dimensional trans-Sasakian manifold in [8].

Our aim is to study different types of curvature tensors on 3-dimensional tran-Sasakian manifold and their properties under certain condition with respect to semi symmetric metric connection.

2. Preliminaries

An n (=$2m$+1) dimensional Riemannian manifold (M, g) is called an almost contact manifold if there exists a (1,1) tensor field ϕ, a vector field ξ and a 1-form η on M such that

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \eta(\phi X) = 0, \tag{2.2}$$

$$\phi\xi = 0, \tag{2.3}$$

$$\eta(X) = g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.5}$$

$$g(X, \phi Y) + g(Y, \phi X) = 0, \tag{2.6}$$

for any vector fields X, Y on M. An odd dimensional almost contact metric manifold M is called a trans-Sasakian manifold if it satisfies the following condition

$$(\nabla_X\phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.7}$$

for some smooth functions α, β on M and we say that the trans-Sasakian structure is of type (α, β). For an n-dimensional trans-Sasakian manifold [9], from (2.7) we have

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.8}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.9}$$

In an n-dimensional trans-Sasakian manifold, we have

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X), \tag{2.10}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.11}$$

$$\begin{aligned} S(X, \xi) &= [(n-1)(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) \\ &\quad -((\phi X)\alpha) - (n-2)(X\beta). \end{aligned} \tag{2.12}$$

For $\alpha, \beta =$ constants then the above equations reduce to

$$R(\xi, X)Y = (\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X), \tag{2.13}$$

$$R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y), \tag{2.14}$$

$$S(X, Y) = (\frac{r}{2} - (\alpha^2 - \beta^2))g(X, Y) - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y), \tag{2.15}$$

$$S(X, \xi) = 2(\alpha^2 - \beta^2)\eta(X), \tag{2.16}$$

$$S(\phi X, \phi Y) = (\frac{r}{2} - (\alpha^2 - \beta^2))g(X, Y), \tag{2.17}$$

$$S(\phi X, Y) = -S(X, \phi Y). \tag{2.18}$$

Definition 2.1. A trans-Sasakian manifold M^n is said to be η- Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions.

Let (M^n, g) be a Riemannian manifold with the Levi-Civita connection ∇. A linear connection $\tilde{\nabla}$ on (M^n, g) is said to be semi-symmetric ([13], [15]) if its torsion tensor T can be written as

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{2.19}$$

where π is an $1-$ form on M^n and the associated vector field ρ defined by $\pi(X) = g(X,\rho)$, for all vector fields $X \in \chi(M^n)$.

A semi-symmetric connection $\tilde{\nabla}$ is called semi-symmetric metric connection if $\tilde{\nabla}g = 0$.

In an almost contact manifold, semi-symmetric metric connection is defined by identifying the 1-form π of the above equation with the contact 1-form η, i.e., by setting [13]

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{2.20}$$

with

$$g(X,\rho) = \eta(X), \forall X \in \chi(M^n).$$

K. Yano has obtained the relation between semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of M^n in [15] and it is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{2.21}$$

where $g(X,\xi) = \eta(X)$.

Further, a relation between the curvature tensors R and $\tilde{R}$ of type (1,3) of the connections ∇ and $\tilde{\nabla}$, respectively is given by [15],

(2.22)
$$\tilde{R}(X,Y)Z = R(X,Y)Z - K(Y,Z)X + K(X,Z)Y - g(Y,Z)FX + g(X,Z)FX,$$

where K is a tensor field of type $(0,2)$ and F is a (1,1) tensor field defined by

$$K(Y,Z) = g(FY,Z) = (\nabla_Y \eta)(Z) - \eta(X)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y,Z). \tag{2.23}$$

In this paper, we have considered that M^3 is 3-dimensional trans-Sasakian manifold. So, using (2.9), (2.19), (2.23) it follows that

$$K(Y,Z) = -\alpha g(\phi Y, Z) - (\beta+1)\eta(Y)\eta(Z) + (\beta + \frac{1}{2})g(Y,Z). \tag{2.24}$$

Using (2.22), from above equation we get

$$FY = -\alpha\phi Y - (\beta+1)\eta(Y)\xi + (\beta + \frac{1}{2})Y. \tag{2.25}$$

Now, by using above two equations we get

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z &- \alpha(g(\phi X,Z)Y - g(\phi Y,Z)X) - \alpha(g(X,Z)\phi Y - g(Y,Z)\phi X)\\ &-(\beta+1)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X)\\ &-(\beta+1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi\\ &+(2\beta+1)(g(X,Z)Y - g(Y,Z)X).\end{aligned} \tag{2.26}$$

In the view of (2.26) we get

$$\tilde{S}(Y,Z) = S(Y,Z) + \alpha g(\phi Y,Z) + (\beta+1)\eta(Y)\eta(Z) - (3\beta+1)g(Y,Z), \tag{2.27}$$

where $\tilde{S}$ and S are Ricci tensors of M^3 with respect to semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇, respectively.

From above, we have

$$\tilde{r} = r - 8\beta - 2, \tag{2.28}$$

where $\tilde{r}$ and r are scalar curvature of M^3 with respect to semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇, respectively.

We obtain from (2.15) and (2.27) that

$$\tilde{Q}\xi = 2(\alpha^2 - \beta^2 - \beta)\xi, \tag{2.29}$$

where $\tilde{Q}$ is the Ricci operator with respect to semi-symmetric metric connection $\tilde{\nabla}$.

3. Extended Generalized M-Projective ϕ-Recurrent 3-Dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

Definition 3.1. [6] A 3-dimensional trans-Sasakian manifold is said to be a generalized M-projective ϕ- recurrent manifold if the M-projective curvature tensor M^* satisfies the relation

$$\phi^2((\nabla_W M^*)(X,Y)Z) = A(W)M^*(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \tag{3.1}$$

where A and B are two 1-forms, B is non zero and these are defined by $g(W,\rho_1) = A(W)$ and $g(W,\rho_2) = B(W)$, $\forall W \in \chi(M)$.

And

$$M^*(X,Y)Z = R(X,Y)Z - \frac{1}{4}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \tag{3.2}$$

for all vector fields X, Y, Z and ρ_1 and ρ_2 being the vector fields associated to the 1-form A and B respectively.

Next we define extended generalized M-projective ϕ- recurrent manifold in the following way.

Definition 3.2. A 3-dimensional trans-Sasakian manifold is said to be an extended generalized M-projective ϕ- recurrent manifold if the M-projective curvature tensor M^* satisfies the relation

$$\phi^2((\nabla_W M^*)(X,Y)Z) = A(W)\phi^2(M^*(X,Y)Z) + B(W)\phi^2([g(Y,Z)X - g(X,Z)Y]), \tag{3.3}$$

where A and B are two 1-forms, B is non zero and these are defined by $g(W,\rho_1) = A(W)$ and $g(W,\rho_2) = B(W)$, $\forall W \in \chi(M)$.

And

$$M^*(X,Y)Z = R(X,Y)Z - \frac{1}{4}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \tag{3.4}$$

for all vector fields X, Y, Z and ρ_1 and ρ_2 being the vector fields associated to the 1-form A and B respectively.

THEOREM 3.3. *An extended generalized M-projective ϕ- recurrent three dimensional trans-Sasakian manifold with respect to semi symmetric metric connection is an η- Einstein manifold with respect to Levi-Civita connection with $\beta \neq -1$ and more over, the 1-forms A and B are related as $A(W)[\frac{r-8\beta-2}{4} - \frac{3}{2}(\alpha^2 - \beta^2 - \beta)] - 2B(W) = \frac{1}{4}dr(W)$, where r is the scalar curvature of trans-Sasakian manifold.*

Proof. Let us assume an extended generalized ϕ- recurrent trans-Sasakian manifold $(M^3, \phi, \eta, \xi, g)$ with respect to semi-symmetric connection. Then we have

$$\phi^2((\tilde{\nabla}_W M^*)(X,Y)Z) = A(W)\phi^2(M^*(X,Y)Z) + B(W)\phi^2([g(Y,Z)X - g(X,Z)Y]). \tag{3.5}$$

Taking inner product with U and then from the equations (2.1), (3.4), (3.5) we get

$$\begin{aligned}
&-g((\tilde{\nabla}_W \tilde{R})(X,Y)Z,U) + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\eta(U) + \frac{1}{4}[(\tilde{\nabla}_W \tilde{S})(Y,Z)g(X,U) \\
&-(\tilde{\nabla}_W \tilde{S})(X,Z)g(Y,U) + (\tilde{\nabla}_W \tilde{S})(X,U)g(Y,Z) - (\tilde{\nabla}_W \tilde{S})(Y,U)g(X,Z)] \\
&-\frac{1}{4}[(\tilde{\nabla}_W \tilde{S})(Y,Z)\eta(X) - (\tilde{\nabla}_W \tilde{S})(X,Z)\eta(Y) + (\tilde{\nabla}_W \tilde{S})(X,\xi)g(Y,Z) \\
&-(\tilde{\nabla}_W \tilde{S})(Y,\xi)g(X,Z)]\eta(U) = A(W)[-g(\tilde{M}^*(X,Y)Z,U) + \eta(\tilde{M}^*(X,Y)Z)\eta(U)] \\
&+B(W)[g(Y,Z)(-g(X,U) + \eta(X)\eta(U)) + g(X,Z)(g(Y,U) - \eta(Y)\eta(U))].
\end{aligned} \tag{3.6}$$

Putting $Z = \xi$ the equation (3.6), we have

$$\begin{aligned}
&-g((\tilde{\nabla}_W \tilde{R})(X,Y)\xi,U) + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)\xi)\eta(U) + \frac{1}{4}[(\tilde{\nabla}_W \tilde{S})(Y,\xi)g(X,U) \\
&-(\tilde{\nabla}_W \tilde{S})(X,\xi)g(Y,U) + (\tilde{\nabla}_W \tilde{S})(X,U)g(Y,\xi) - (\tilde{\nabla}_W \tilde{S})(Y,U)g(X,\xi)] \\
&-\frac{1}{4}[(\tilde{\nabla}_W \tilde{S})(Y,\xi)\eta(X) - (\tilde{\nabla}_W \tilde{S})(X,\xi)\eta(Y) + (\tilde{\nabla}_W \tilde{S})(X,\xi)g(Y,\xi) \\
&-(\tilde{\nabla}_W \tilde{S})(Y,\xi)g(X,\xi)]\eta(U) = A(W)[-g(\tilde{M}^*(X,Y)\xi,U) + \eta(\tilde{M}^*(X,Y)\xi)\eta(U)] \\
&+B(W)[g(Y,\xi)(-g(X,U) + \eta(X)\eta(U)) + g(X,\xi)(g(Y,U) - \eta(Y)\eta(U))].
\end{aligned} \tag{3.7}$$

Let $\{e_1, e_2, e_3 = \xi\}$ be an orthonormal basis for the tangent space of M^3 at a point $p \in M^3$. Putting $X = U = e_i$ in (3.7) and taking summation over i, we get

$$
\begin{aligned}
&-(\tilde{\nabla}_W \tilde{S})(Y,\xi) + \sum_{i=1}^{3} \eta((\tilde{\nabla}_W \tilde{R})(e_i,Y)\xi)\eta(e_i) + \frac{1}{4}[3(\tilde{\nabla}_W \tilde{S})(Y,\xi) \\
&-(\tilde{\nabla}_W \tilde{S})(e_i,\xi)g(Y,e_i) + (d\tilde{r})(W)g(Y,\xi) - (\tilde{\nabla}_W \tilde{S})(Y,e_i)g(e_i,\xi)] - \frac{1}{4}[(\tilde{\nabla}_W \tilde{S})(Y,\xi)\eta(e_i) \\
&-(\tilde{\nabla}_W \tilde{S})(e_i,\xi)\eta(Y) + (\tilde{\nabla}_W \tilde{S})(e_i,\xi)g(Y,\xi) - (\tilde{\nabla}_W \tilde{S})(Y,\xi)g(e_i,\xi)]\eta(e_i) \\
&= A(W)[-g(\tilde{M}^*(e_i,Y)\xi,e_i) + \eta(\tilde{M}^*(e_i,Y)\xi)\eta(e_i)] + B(W)[g(Y,\xi)(-g(e_i,e_i) \\
&\qquad +\eta(e_i)\eta(e_i)) + g(e_i,\xi)((g(Y,e_i) - \eta(Y)\eta(e_i))]. \qquad (3.8)
\end{aligned}
$$

Now,

$$
\begin{aligned}
g((\tilde{\nabla}_W \tilde{R})(e_i,Y)\xi,\xi) &= g((\nabla_W \tilde{R})(e_i,Y)\xi,\xi) + \eta(\tilde{R}(e_i,Y)\xi)g(W,\xi) \\
&\quad -g(W,\tilde{R}(e_i,Y)\xi)g(\xi,\xi). \qquad (3.9)
\end{aligned}
$$

We have

$$
\begin{aligned}
g((\nabla_W \tilde{R})(e_i,Y)\xi,\xi) &= g(\nabla_W \tilde{R}(e_i,Y)\xi,\xi) - g(\tilde{R}(\nabla_W e_i,Y)\xi,\xi) \\
&\quad -g(\tilde{R}(e_i,\nabla_W Y)\xi,\xi) - g(\tilde{R}(e_i,Y)\nabla_W \xi,\xi). \qquad (3.10)
\end{aligned}
$$

at $p \in M^3$. Since e_i is an orthonormal basis, so $\nabla_W e_i = 0$ at p.

Also,

$$g(\tilde{R}(e_i,Y)\xi,\xi) = -g(\tilde{R}(\xi,\xi)Y,e_i) = 0. \qquad (3.11)$$

Since $\nabla_W g = 0$, we obtain

$$g((\nabla_W \tilde{R})(e_i,Y)\xi,\xi) + g(\tilde{R}(e_i,Y)\xi,\nabla_W \xi) = 0, \qquad (3.12)$$

which implies that

$$g((\nabla_W \tilde{R})(e_i, Y)\xi, \xi) = 0. \tag{3.13}$$

Since $\eta(\tilde{R}(e_i, Y)\xi) = 0$, we have from (3.10) that

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = -g(W, \tilde{R}(e_i, Y)\xi). \tag{3.14}$$

Therefore,

$$\sum_{i=1}^{3} \eta((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi)\eta(e_i) = \alpha g(\phi Y, W) - (\alpha^2 - \beta^2 - \beta)g(\phi W, \phi Y). \tag{3.15}$$

Again, from (2.28), (3.3), (3.4) (3.15) in (3.8) we have

$$\begin{aligned} -\frac{3}{4}(\tilde{\nabla}_W \tilde{S})(Y, \xi) + \frac{1}{4}dr(W)\eta(Y) + \alpha g(\phi Y, W) - (\alpha^2 - \beta^2 - \beta)g(\phi W, \phi Y) \\ = (A(W)[\frac{r - 8\beta - 2}{4} - \frac{3}{2}(\alpha^2 - \beta^2 - \beta)] - 2B(W))\eta(Y). \end{aligned} \tag{3.16}$$

Now,

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \tilde{\nabla}_W \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_W Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi).$$

After brief calculations we obtain

$$\begin{aligned} (\tilde{\nabla}_W \tilde{S})(Y, \xi) &= 2(\alpha^2 - \beta^2 - \beta)\alpha g(\phi Y, W) + 2\beta(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W) \\ &+ \alpha S(\phi Y, W) + \alpha^2 g(\phi Y, \phi W) - \alpha(3\beta + 1)g(\phi Y, W) - (\beta + 1)S(Y, W) \\ &- \alpha(\beta+1)g(\phi Y, W) - (\beta+1)^2\eta(Y)\eta(W) + 2(\beta+1)(\alpha^2 - \beta^2 - \beta)\eta(Y)\eta(W) \\ &+ (3\beta + 1)(\beta + 1)g(Y, W) + 2(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W). \end{aligned} \tag{3.17}$$

From the equation (3.16) and (3.17)we get

$$\alpha g(\phi Y, W) - (\alpha^2 - \beta^2 - \beta)g(\phi W, \phi Y) + \frac{1}{4}dr(W)\eta(Y) - \frac{3}{4}[2(\alpha^2 - \beta^2 - \beta)\alpha g(\phi Y, W)$$
$$+ 2\beta(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W) + \alpha S(\phi Y, W) + \alpha^2 g(\phi Y, \phi W) - \alpha(3\beta+1)g(\phi Y, W)$$

$$-(\beta+1)S(Y,W)-\alpha(\beta+1)g(\phi Y,W)-(\beta+1)^2\eta(Y)\eta(W)+(3\beta+1)(\beta+1)g(Y,W)$$
$$+2(\beta+1)(\alpha^2-\beta^2-\beta)\eta(Y)\eta(W)+2(\alpha^2-\beta^2-\beta)g(\phi Y,\phi W)]$$
$$= (A(W)[\frac{r-8\beta-2}{4}-\frac{3}{2}(\alpha^2-\beta^2-\beta)]-2B(W))\eta(Y). \tag{3.18}$$

Replacing $Y=\xi$ in (3.18) we obtain

$$A(W)[\frac{r-8\beta-2}{4}-\frac{3}{2}(\alpha^2-\beta^2-\beta)]-2B(W)=\frac{1}{4}dr(W). \tag{3.19}$$

From, (3.18) we have

$$\alpha g(\phi Y,W)-(\alpha^2-\beta^2-\beta)g(\phi W,\phi Y)-\frac{3}{4}[2(\alpha^2-\beta^2-\beta)\alpha g(\phi Y,W)$$
$$+2\beta(\alpha^2-\beta^2-\beta)g(\phi Y,\phi W)+\alpha S(\phi Y,W)+\alpha^2 g(\phi Y,\phi W)-\alpha(3\beta+1)g(\phi Y,W)$$
$$-(\beta+1)S(Y,W)-\alpha(\beta+1)g(\phi Y,W)-(\beta+1)^2\eta(Y)\eta(W)+(3\beta+1)(\beta+1)g(Y,W)$$
$$+2(\beta+1)(\alpha^2-\beta^2-\beta)\eta(Y)\eta(W)+2(\alpha^2-\beta^2-\beta)g(\phi Y,\phi W)]=0. \tag{3.20}$$

Interchanging Y and W and then adding with the equation (3.20) the Ricci tensor is of the form

$$S(Y,W)=ag(Y,W)+b\eta(Y)\eta(W),$$

where a,b are scalar functions. Hence M^3 is an η-Einstein manifold. □

4. ϕ-$\tilde{W}_2$ flat 3-dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

DEFINITION 4.1. Let M be 3-dimensional trans-Sasakian manifold with respect to semi-Symmetric metric connection. The $\tilde{W}_2$-curvature tensor of M is defined by

$$\tilde{W}_2(X,Y)Z=\tilde{R}(X,Y)Z-\frac{1}{2}(g(Y,Z)\tilde{Q}X-g(X,Z)\tilde{Q}Y).$$

DEFINITION 4.2. A trans-Sasakian manifold M^3 is said to be $\phi-W_2$ flat with respect to semi-symmetric metric connection if

$$\phi^2(\tilde{W}_2(\phi X,\phi Y)\phi Z))=0. \tag{4.1}$$

Theorem 4.3. *Let M be a ϕ-$\tilde{W}_2$ flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then the manifold is an η-Einstein manifold with respect to Levi-Civita connection.*

Proof. Let M^3 be a ϕ-$\tilde{W}_2$ trans-Sasakian manifold with respect to semi-symmetric metric connection. It is easy to see that $\phi^2(\tilde{W}_2(\phi X, \phi Y)\phi Z)) = 0$ holds iff

$$g(\tilde{W}_2(\phi X, \phi Y)\phi Z, \phi V)) = 0, \forall X, Y, Z, V \in \chi(M^3). \tag{4.2}$$

Therefore, we get

$$g(\tilde{W}_2(\phi X, \phi Y)\phi Z, \phi V)) = g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi V)) - \frac{1}{2}(\tilde{S}(\phi X, \phi V)g(\phi Y, \phi Z) - \tilde{S}(\phi Y, \phi V)g(\phi X, \phi Z)). \tag{4.3}$$

Using the equation (4.2) in the equation (4.3), we get

$$g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi V)) = \frac{1}{2}(\tilde{S}(\phi X, \phi V)g(\phi Y, \phi Z) - \tilde{S}(\phi Y, \phi V)g(\phi X, \phi Z)). \tag{4.4}$$

Let $\{e_1, e_2, e_3 = \xi\}$ be a local orthonormal basis of vector fields in M^3. Then $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis of vector fields in M^3. Putting $X = V = e_i$ in the equation (4.4) and taking summation over i, $1 \leq i \leq 2$, we get,

$$\sum_{i=1}^{3} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i)) = \frac{1}{2}\sum_{i=1}^{3}[(\tilde{S}(\phi e_i, \phi e_i)g(\phi Y, \phi Z) - \tilde{S}(\phi Y, \phi e_i)g(\phi e_i, \phi Z))]. \tag{4.5}$$

Also,

$$\sum_{i=1}^{3} g(\tilde{R}(\phi e_i, Y)Z, \phi e_i)) = \tilde{S}(\phi Y, \phi Z) + (\beta - \alpha^2 + \beta^2)g(\phi Y, \phi Z) + \alpha g(\phi Z, Y), \tag{4.6}$$

$$\sum_{i=1}^{3} \tilde{S}(\phi e_i, Z)g(Y, \phi e_i) = \tilde{S}(Y, Z), \tag{4.7}$$

$$\sum_{i=1}^{3} g(\phi e_i, \phi e_i) = 2, \tag{4.8}$$

$$\sum_{i=1}^{3} g(\phi e_i, Z) g(Y, \phi e_i) = g(Y, Z), \tag{4.9}$$

$$\sum_{i=1}^{3} g(\phi e_i, e_i) = 0. \tag{4.10}$$

Therefore, using the equations (4.6), (4.7), (4.8), (4.9) and (4.10) we have

$$\tilde{S}(\phi Y, \phi Z) + (\beta - \alpha^2 + \beta^2) g(\phi Y, \phi Z) + \alpha g(\phi Z, Y) = \frac{1}{2}[g(\phi Y, \phi Z)\tilde{r} - \tilde{S}(\phi Y, \phi Z)]. \tag{4.11}$$

Then, we have

$$3\tilde{S}(\phi Y, \phi Z) = -2\alpha g(\phi Z, Y) + g(\phi Y, \phi Z)(\tilde{r} - 2(\beta - \alpha^2 + \beta^2)). \tag{4.12}$$

Hence from the above equation (4.12) we get

$$\begin{aligned} 3S(\phi Y, \phi Z) + 3\alpha g(\phi^2 Y, \phi Z) - 3(3\beta+1) g(\phi Y, \phi Z) &= -2\alpha g(\phi Z, Y) \\ &+ g(\phi Y, \phi Z)(r - 8\beta - 2 - 2(\beta - \alpha^2 + \beta^2)), \end{aligned} \tag{4.13}$$

Therefore, we get

$$3S(Y, Z) = (r - 9\beta + 1 + 2\alpha^2 - 2\beta^2) g(Y, Z) + (4\alpha^2 - 4\beta^2 + 9\beta - 1 - r)\eta(Y)\eta(Z) + \alpha g(Y, \phi Z). \tag{4.14}$$

Interchanging Y with Z in (4.14) we get

$$3S(Z, Y) = (r - 9\beta + 1 + 2\alpha^2 - 2\beta^2) g(Z, Y) + (4\alpha^2 - 4\beta^2 + 9\beta - 1 - r)\eta(Z)\eta(Y) + \alpha g(Z, \phi Y). \tag{4.15}$$

Then adding above two equations and using skew-symmetric property of ϕ we have

(4.16)
$$S(Y,Z)=\frac{1}{3}(r-9\beta+1+2\alpha^2-2\beta^2)g(Y,Z)+\frac{1}{3}(4\alpha^2-4\beta^2+9\beta-1-r)\eta(Y)\eta(Z).$$

This proves that M^3 is an η-Einstein manifold. □

Corollary 4.4. *Let M be a ϕ-$\tilde{W}_2$ flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then the manifold is a η-Einstein manifold with respect to semi-symmetric metric connection if $\alpha=0$ i.e. if M is a β-Kenmotsu manifold.*

5. Conharmonically Flat 3-dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

A Conharmonic curvature tensor has been studied by Ozgur [11], Siddiqui and Ahsan [12] amd many other authors. In almost contact manifold M of dimension $n \geq 3$, the conharmonic curvature tensor $\tilde{K}$ with respect to semi-symmetric connection $\tilde{\nabla}$ is given by

(5.1)
$$\tilde{K}(X,Y)Z=\tilde{R}(X,Y)Z-\frac{1}{n-2}[\tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y+g(Y,Z)\tilde{Q}X-g(X,Z)\tilde{Q}Y],$$

for $X,Y,Z \in \chi(M)$, where $\tilde{R},\tilde{S},\tilde{Q}$ are the Riemannian curvature tensor, Ricci tensor and the Ricci operator with respect to semi-symmetric connection $\tilde{\nabla}$ respectively.

A conharmonic curvature tensor $\tilde{K}$ with respect to semi-symmetric connection $\tilde{\nabla}$ is said to flat if it vanishes identically with respect to semi-symmetric connection $\tilde{\nabla}$.

Now, we prove the following theorem.

Theorem 5.1. *Let M be a conharmonically flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then the manifold is a η-Einstein manifold with respect to Levi-Civita connection.*

Proof. Assume that M is a conharmonically flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then from the equation (5.1) we get

$$\tilde{R}(X,Y)Z = [\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y]. \tag{5.2}$$

Then we have

$$\begin{aligned} g(\tilde{R}(X,Y)Z,U) = [&\tilde{S}(Y,Z)g(X,U) - \tilde{S}(X,Z)g(Y,U) \\ &+g(Y,Z)\tilde{S}(X,U) - g(X,Z)\tilde{S}(Y,U)]. \end{aligned} \tag{5.3}$$

Putting $X = U = \xi$ in the above the equation (5.3) we get

(5.4)
$$\tilde{S}(Y,Z) = [-(\alpha^2-\beta^2-\beta)]g(Y,Z)+[3(\alpha^2-\beta^2-\beta)]\eta(Y)\eta(Z)+\alpha g(\phi Y,Z).$$

Hence from the above the equation and using the equation (2.27) we get

(5.5)
$$S(Y,Z) = (4\beta+1-\alpha^2+\beta^2)g(Y,Z)+(3\alpha^2-3\beta^2-4\beta-1)\eta(Y)\eta(Z)-2\alpha g(\phi Y,Z).$$

Interchanging Y with Z in (5.5) we get

(5.6)
$$S(Z,Y) = (4\beta+1-\alpha^2+\beta^2)g(Z,Y)+(3\alpha^2-3\beta^2-4\beta-1)\eta(Z)\eta(Y)-2\alpha g(\phi Z,Y).$$

Adding the above two equations we obtain

$$S(Y,Z) = (4\beta+1-\alpha^2+\beta^2)g(Y,Z)+(3\alpha^2-3\beta^2-4\beta-1)\eta(Y)\eta(Z). \tag{5.7}$$

Hence, the manifold is an η-Einstein manifold with respect to Levi-Civita connection.
Therefore, the theorem is proved. $\square$

COROLLARY 5.2. *Let M be a conharmonically flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then the manifold is an η-Einstein manifold with respect to semi-symmetric metric connection if $\alpha = 0$ i.e. if M is a β-Kenmotsu manifold.*

6. ϕ- Conharmonically Flat flat 3-dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

DEFINITION 6.1. A trans-Sasakian manifold M^3 is said to be ϕ-conharmonically flat with respect to semi-symmetric connection if

$$\phi^2(\tilde{K}(\phi X, \phi Y)\phi Z)) = 0, \tag{6.1}$$

where $\tilde{K}$ is the conharmonic curvature tensor with respect to semi-symmetric metric connection.

THEOREM 6.2. *Let M be a ϕ-conharmonically flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then the manifold is a η-Einstein manifold with respect to Levi-Civita connection.*

Proof. Let M^3 be a $\phi-\tilde{K}$ be a 3-dimensional trans-Sasakian manifold manifold with respect to semi-symmetric metric connection. It is easy to see that $\phi^2(\tilde{K}(\phi X, \phi Y)\phi Z) = 0$ holds iff

$$g(\tilde{K}(\phi X, \phi Y)\phi Z, \phi V) = 0, \forall X, Y, Z, V \in \chi(M^3). \tag{6.2}$$

Now, from the definition of conharmonic curvature tensor on 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection, we get

$$\begin{aligned} g(\tilde{K}(\phi X, \phi Y)\phi Z, \phi V)) &= g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi V)) - [\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi V) \\ &- \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi V) + g(\phi Y, \phi Z)\tilde{S}(\phi X, \phi V) - g(\phi X, \phi Z)\tilde{S}(\phi Y, \phi V)]. \end{aligned} \tag{6.3}$$

Using the equation (6.2) in the equation (6.3), we get

$$\begin{aligned} g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi V)) &= [\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi V) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi V) \\ &+ g(\phi Y, \phi Z)\tilde{S}(\phi X, \phi V) - g(\phi X, \phi Z)\tilde{S}(\phi Y, \phi V)]. \end{aligned} \tag{6.4}$$

Let $\{e_1, e_2, e_3 = \xi\}$ be a local orthonormal basis of vector fields in M^3. Then $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis of vector fields in M^3. Putting $X = V = e_i$ in the equation (6.4) and taking summation over i, $1 \leq i \leq 3$, we get

$$g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi g_i)) = [\tilde{S}(\phi Y, \phi Z)g(\phi e_i, \phi e_i)$$

(6.5)
$$-\tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i) + g(\phi Y, \phi Z)\tilde{S}(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)\tilde{S}(\phi Y, \phi e_i)].$$

Therefore, using the equations (4.6)-(4.10) we have

$$\tilde{S}(\phi Y, \phi Z) + \alpha g(Y, \phi Z) = [\tilde{r} + (\beta - \alpha^2 + \beta^2)]g(\phi Y, \phi Z). \tag{6.6}$$

Hence from the equations (2.13), (2.17), (2.27) we get

(6.7)
$$S(Y, Z) = [r - 6\beta - 1 + \alpha^2 - \beta^2]g(Y, Z) + (\alpha^2 - \beta^2 + 6\beta - r + 1)\eta(Y)\eta(Z).$$

This proves that M^3 is an η-Einstein manifold with respect to Levi-Civita connection. □

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