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TOPOLOGICAL ENTROPY OF EXPANSIVE FLOW ON TVS-CONE METRIC SPACES

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Abstract. We shall study the following. Let ϕ be an expansive flow on a compact TVS-cone metric space (X, d).

First, we give some equivalent ways of defining expansiveness. Second, we show that expansiveness is conjugate invariance. Finally, we prove that $\limsup \frac{1}{t} \log v(t) \leq h(\phi)$, where $v(t)$ denotes the number of closed orbits of ϕ with a period $\tau \in [0, t]$ and $h(\phi)$ denotes the topological entropy.

Remark that in 1972, R. Bowen and P. Walters had proved this three statements for an expansive flow on a compact metric space [**?**].

1. Introduction and Preliminaries

TVS-cone metric space is a generization of metric space. In this paper, we shall define expansive flow on TVS-cone metric space and find equivalent characterization of expansive flow. Also, we study conjugate invariance and topological entropy of expansive flow. To do this, we first introduce some definitions and results. [**?**],[**?**].

Let E be a topological vector space. A subset P of E is called a topological vector space cone (abbr. TVS-cone) if the following are satisfied

(1) P is closed and $int(P) \neq \emptyset$
(2) If $u, v \in P$ and $a, b \geq 0$, then $au + bv \in P$
(3) If $u, -u \in P$, then $u = 0$.

Let P be a TVS-cone of a topological vector space E. Some partial orderings $\leq$, $<$, $\ll$ on E with respect to P are defined as followings respectively

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(1) $u \leq v$ if $v - u \in P$
(2) $u < v$ if $u \leq v$, $u \neq v$
(3) $u \ll v$ or $v \gg u$ if $v - u \in \mathrm{Int}P$

Lemma 1.1. [?] *Let P be a TVS-cone of a topological vector space E. Then the following hold.*
(1) If $u \gg 0$, then $ru \gg 0$ for all $r > 0$
(2) If $u_1 \gg v_1$, $u_2 \gg v_2$, then $u_1 + u_2 \gg v_1 + v_2$
(3) If $u \gg 0$, $v \gg 0$, then there exists $w \gg 0$ such that $w \ll u$, $w \ll v$

Let E be a topological space with cone P. A map $d : X \times X \to E$ is called a TVS-cone metric on X and (X, d) called a TVS-cone metric space if the following are satisfied.
(a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$.
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon) = \{y \in X | d(x, y) \ll \epsilon\}$, for all $x \in X$ and $\epsilon \gg 0$, is a basis for a topology $\Im$ on X. [2]

In this paper, we always suppose that a cone P is a TVS-cone of a topological vector space E and a TVS-cone metric space (X, d) is a topological space with the above topology $\Im$.

Let (X, d) be a TVS-cone metric space over topological vector space E.

A flow on X is a continuous map $\phi : X \times \mathbb{R} \to X$ satisfying
(1) $\phi(x, 0) = x$ for all $x \in X$.
(2) $\phi(\phi(x, s), t) = \phi(x, s + t)$ for all $x \in X$ and $s, t \in \mathbb{R}$.
For $t \in \mathbb{R}$, let ϕ_t be a homeomorphism of X defined by $\phi_t(x) = \phi(x, t)$ for all $x \in X$.

Denoted by $C_0(\mathbb{R})$ the set of all continuous functions $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$.

A flow ϕ on X is said to be expansive if for every $\epsilon > 0$ there exists $u \gg 0$ such that if $x, y \in X$ satisfy $d(\phi_t(x), \phi_{h(t)}(y)) \ll u$ for all $t \in \mathbb{R}$ and some $h \in C_0(\mathbb{R})$ then $y = \phi_r(x)$ where $|r| < \epsilon$.

2. Some equivalent definitions of expansive flows

Our first result shows that the study of expansive flows reduced to those without fixed points.

Proposition 2.1. *If a flow ϕ on X is expansive, then each fixed point of ϕ is an isolated point on X.*

Proof. Suppose $\phi_t(x) = x$ for all $t \in \mathbb{R}$. Let $\epsilon > 0$ be given and let $u \gg 0$ be the expansive vector. If $d(x, y) \ll u$, then choosing $h(t) = 0$ for all $t \in \mathbb{R}$, and hence $y = \phi_r(x) = x$, where $|r| < \epsilon$. Therefore $B(x, u) = \{x\}$ and x is an isolated point of X. □

By Proposition 2.1, if ϕ is an expansive flow on X, then $X = F \cup X'$ where F is the set of fixed points of ϕ and $\phi|_{X' \times \mathbb{R}}$ has no fixed points. From now on we shall always assume has no fixed points.

LEMMA 2.2. *Let X be a compact TVS-cone metric space. If a flow ϕ on X has no fixed points, then there exists $T_0 > 0$ such that every T with $0 < T < T_0$, there is a vector $u \gg 0$ with $d(\phi_T(x), x) > u$ for all $x \in X$, where $v > w \Leftrightarrow w - v \notin P$.*

Proof. If ϕ has no periodic points, let $T_0 = 1$. If ϕ has some periodic points, let A be the set of periods of periodic points. Then we claim that $\inf A > 0$.

Assume that $\inf A = 0$. Then there exists a sequence (τ_n) in A such that $\tau_n \to 0$. Let x_n be the periodic point with period τ_n. By the compactness of X, the sequence (x_n) in X has a convergent subsequence. Put $x_n \to x$. For any $t \in \mathbb{R}$, we can write $t = q_n\tau_n + r_n$ for some $q_n \in \mathbb{Z}$ and $0 \leq r_n < \tau_n$. Therefore we get that $\tau_n \to 0$ and so $\phi_t(x_n) = \phi_{q_n\tau_n + r_n}(x_n) = \phi_{r_n}(\phi_{q_n\tau_n}(x_n)) = \phi_{r_n}(x_n) \to \phi_0(x) = x$. Since $\phi_t(x_n) \to \phi_t(x)$, we have $\phi_t(x) = x$, contradicting the fact that ϕ has no fixed points. Take $T_0 = \inf A$. Let $u \gg 0$. Assume that it is false. Then there exists $0 < t < T_0$ such that for each positive integer n, we can choose $x_n \in X$ satisfying $d(\phi_t(x_n), x_n) \leq \frac{1}{n}u$. By the compactness of X, the sequence (x_n) in X has a convergent subsequence. Let $x_n \to x$. Then $d(\phi_t(x), x) = 0$ and conclude that $\phi_t(x) = x$, contradicting the choice of T_0. □

LEMMA 2.3. *Let T_0 be the number determined by Lemma 2.2. For every $T \in (0, \frac{T_0}{2})$, there exists vector $u_T \gg 0$ such that if $x, y \in X$ satisfy $d(\phi_t(x), \phi_{h(t)}(y)) \ll u$ for all $t \in \mathbb{R}$ and some $h \in C_0(\mathbb{R})$, then $h(T) \geq 0$.*

Proof. Arguing by contradiction, assume that there is $T \in (0, \frac{T_0}{2})$ such that for each vector $u \gg 0$ there exist $x_u, y_u \in X$ and $h_u \in C_0(\mathbb{R})$ such that $d(\phi_t(x_u), \phi_{h_u(t)}(y_u)) \ll u$ for all $t \in \mathbb{R}$ but $h_u(T) < 0$. Fix $u \gg 0$. For each positive integer n, there exist $x_n, y_n \in X$ and $h_n \in C_0(\mathbb{R})$ such that $d(\phi_t(x_n), \phi_{h_n(t)}(y_n)) \ll \frac{1}{n}u$ for all $t \in \mathbb{R}$, but $h_n(T) < 0$.

We may assume that $x_n \to x$ and $y_n \to y$. Since $d(x_n, y_n) = d(\phi_0(x_n), \phi_{h_n(0)}(y_n)) \ll \frac{1}{n}u$ for all n, letting $n \to \infty$, we get $x = y$.

Case 1. If $h_n(T) \geq -T$, since $[-T, 0]$ is compact, we can suppose $h_n(T) \to -L$ for some $0 \leq L \leq T$. Since $d(\phi_T(x_n), \phi_{h_n(T)}(y_n)) \ll \frac{1}{n}u$, letting $n \to \infty$, we get $\phi_T(x) = \phi_{-L}(x)$. Thus $\phi_{T+L}(x) = x$, contradicting the fact that T_0 is the smallest period of ϕ.

Case 2. If $h_n(T) < -T$, then there exists $t_n \in (0, T)$ such that $h_n(t_n) = T$ by intermediate value theorem. Since $[0, T]$ is compact, we can suppose $t_n \to t \in [0, T]$. Since $d(\phi_{t_n}(x_n), \phi_{h_n(t_n)}(y_n)) = d(\phi_{t_n}(x_n), \phi_{-T}(y_n)) \ll \frac{1}{n}u$, letting $n \to \infty$, we get $\phi_t(x) = \phi_{-T}(x)$. Thus $\phi_{T+t}(x) = x$, contradicting the fact that T_0 is the smallest period of ϕ. □

LEMMA 2.4. *Let T_0 be the number determined by Lemma 2.2, For every $T \in (0, \frac{T_2}{2})$, there exist a vector $u_T \gg 0$ and a number $\tau_T > 0$ such that if $d(\phi_t(x), \phi_{h(t)}(y)) \ll u_T$ for all $t \in \mathbb{R}$ and some $h \in C_0(\mathbb{R})$ then $h(t+T) - h(t) \geq \tau_T$ for all $t \in \mathbb{R}$.*

Proof. Take a vector $u \gg 0$ determined by Lemma2.2. Let $u_T = \frac{1}{3}u$. Since $\phi : X \times [0, 1] \to X$ is uniformly continuous, there exist vector $v \gg 0$ and number $0 < \tau < 1$ such that if $d(p, q) \ll v$ and $s, t \in [0, 1]$ with $|s - t| \leq \tau$, then $d(\phi_s(p), \phi_t(q)) \ll u$. Let $s, t \in \mathbb{R}$ with $|s - t| \leq \tau$ and $x \in X$.

Case 1. If $s \geq t$, since $0 \leq s - t \leq \tau < 1$ and $d(\phi_t(x), \phi_t(x)) = 0 \ll v$, we obtain $d(\phi_{s-t}(\phi_t(x)), \phi_0(\phi_t(x))) = d(\phi_s(x), \phi_t(x)) \ll u$.

Case 2. If $s < t$, since $0 < t - s \leq \tau < 1$ and $d(\phi_s(x), \phi_s(x)) = 0 \ll v$, we obtain $d(\phi_0(\phi_s(x)), \phi_{t-s}(\phi_s(x))) = d(\phi_s(x), \phi_t(x)) \ll u$.

Thus there is a number $\tau > 0$ such that if $s, t \in \mathbb{R}$ with $|s - t| \leq \tau$ then $d(\phi_s(x), \phi_t(x)) \ll u$ for all $x \in X$. Let $h \in C_0(\mathbb{R})$, $x, y \in X$ and $d(\phi_t(x), \phi_{h(t)}(y)) \ll u_T$ for all $t \in \mathbb{R}$.

We claim that $d(\phi_{h(t)}(y), \phi_{h(t+T)}(y)) \geq u_T$. $d(\phi_{h(t)}(y), \phi_{h(t+T)}(y)) \geq u_T$ where $v \geq w \Leftrightarrow w - v \notin \mathrm{Int}P$. To see this, if not, since $d(\phi_t(x), \phi_{h(t)}(y)) \ll u_T$ and $d(\phi_{t+T}(x), \phi_{h(t+T)}(y)) \ll u_T$, we get $d(\phi_t(x), \phi_{t+T}(x)) \ll 3u_T = u$ contradiction the choice of u. This proves the claim. Then, by the continuity of ϕ, there exists $\tau_T > 0$ such that $|h(t+T) - h(t)| \geq \tau_T$ for all $t \in \mathbb{R}$.

Now we shall prove $h(t+T) - h(t) \geq \tau_T$ for all $t \in \mathbb{R}$. Putting $J = \{t \in \mathbb{R} | h(t+t) - h(t) \geq \tau_T\}$. By Lemma 2.3, $h(T) = |h(T)| \geq \tau_T$ and so $0 \in J$. If $t \in J$, then $(t - s, t + s) \subset J$ for some $s > 0$ by $h(t+T) \geq h(t) + \tau_T$. If $t \in J^c$, then $(t - s, t + s) \subset J^c$ for some $s > 0$ by $h(t) \geq h(t+T) + \tau_T$. Thus J is open and also closed set. So $J = \mathbb{R}$. Therefore we obtain that $h(t+T) - h(t) \geq \tau_T$ for all $t \in \mathbb{R}$. □

Let (X, d) be a TVS-cone metric space over topological vector space E and define $\rho : (X \times X)^2 \to E$ by $\rho((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) +$

$d(y_1, y_2)$. Then ρ is a metric on $X \times X$. Since $B_d(x, \frac{1}{2}u) \times B_d(y, \frac{1}{2}u) \subset B_\rho((x, y), u)$ and $B_\rho((x, y), u) \subset B_d(x, u) \times B_d(y, u)$, the product topology on $X \times X$ is the metric topology induced by ρ.

The following theorem contains several characterizations of a expansive flow.

THEOREM 2.5. *Let (X, d) be a compact TVS-cone metric space and ϕ be a flow on (X, d) without fixed points. Then the following are equivalent.*

(1) ϕ is expansive flow.

(2) For every $\epsilon > 0$ there exists $u \gg 0$ such that if $x, y \in X$ satisfy $d(\phi_t(x), \phi_{h(t)}(y)) \ll u$ for all $t \in \mathbb{R}$ and some increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$, then $y = \phi_t(x)$ for $|t| < \epsilon$.

(3) For any $u \gg 0$ there is $v \gg 0$ such that if $x, y \in X$ satisfy $d(\phi_t(x), \phi_{h(t)}(y)) \ll v$ for all $t \in \mathbb{R}$ and some $h \in C_0(\mathbb{R})$, then y is on the same orbit as x and the orbit from x to y lies inside $B(x, u)$.

(4) For any $\epsilon > 0$ there exist $u \gg 0$ and $\tau > 0$ such that if $t = \{t_i\}, s = \{s_i\}$ are bisequences in $\mathbb{R}$ with $t_0 = s_0 = 0$, $0 < t_{i+1} - t_i \le \tau$, $|s_{i+1} - s_i| \le \tau$, $t_i \to \infty$ and $t_{-i} \to -\infty$ as $i \to \infty$ and if $x, y \in X$ satisfy $d(\phi_{t_i}(x), \phi_{s_i}(y)) \ll u$ for all $i \in \mathbb{Z}$, then $y = \phi_t(x)$ for $|t| < \epsilon$.

Proof. We first shall show that (1), (2), and (4) are equivalent.

To show that (2) $\Longrightarrow$ (1), let T_0 be the number determined by Lemma 2.2 and $\epsilon > 0$ be given. Let $u \gg 0$ be the corresponding vector given by (2). We can choose $T \in (0, \frac{T_0}{2})$ satisfying that if $0 \le t \le T$, then $d(x, \phi_t(x)) \ll \frac{1}{2}u$ for all $x \in X$. Indeed, there exist vector $v \gg 0$ and number $T \in (0, \frac{T_0}{2})$ such that if $s, t \in [0, T_0]$, $|s - t| \le T$ and $d(p, q) \ll u$, then $d(\phi_s(p), \phi_t(q)) \ll \frac{1}{2}u$ by the uniform continuity of $\phi : X \times [0, T_0] \to X$. Since $d(x, x) = 0 \ll v$, we have $d(x, \phi_t(x)) \ll \frac{1}{2}u$. By Lemma 2.4, there are vectors $u_T \gg 0$ and number $\tau_T > 0$ such that for any $h \in C_0(\mathbb{R})$ if $x, y \in X$ satisfy $d(\phi_t(x), \phi_{h(t)}(y)) \ll u_T$ for all $t \in \mathbb{R}$, then $h(t + T) - h(t) \ge \tau_T$ for all $t \in \mathbb{R}$. We choose vector $v \gg 0$ such that $v \ll \frac{1}{2}u$ and $v \ll u_T$. Suppose $d(\phi_t(x), \phi_{h(t)}(y)) \ll v \ll u_T$ for all $t \in \mathbb{R}$ and some $h \in C_0(\mathbb{R})$. Then $h(t + T) - h(t) \ge \tau_T$ for all $t \in \mathbb{R}$. Define $h_T : \mathbb{R} \to \mathbb{R}$ by $h_T(nT) = h(nT)$, $n \in \mathbb{Z}$ and by linearity on each interval $[nT, (n + 1)T]$. Then h_T is an increasing homeomorphism with $h_T(0) = 0$. If $t \in [nT, (n + 1)T]$, there exists $s \in [nt, (n + 1)T]$ such that $h_T(t) = h(s)$. Then $d(\phi_s(x), \phi_{h_T(t)}(y)) = d(\phi_s(x), \phi_{h(s)}(y)) \ll v \ll \frac{1}{2}u$. Since $d(\phi_t(x), \phi_s(x)) \ll \frac{1}{2}u$, we get $d(\phi_t(x), \phi_{h_r(t)}(y)) \ll u$. By (2) $y = \phi_t(x)$ for some $|t| < \epsilon$ and hence ϕ is expansive.

To show that (1) $\Longrightarrow$ (4), given $\epsilon > 0$ and let $u \gg 0$ be the corresponding expansive vector given by (1). Take number $\tau > 0$ such that if $|t-s| < \tau$ then $d(\phi_t(x), \phi_s(x)) \ll \frac{1}{3}u$ for all $x \in X$. Let $\{t_i\}_{i=-\infty}^{\infty}$, $\{s_i\}_{i=-\infty}^{\infty}$ and $x, y \in X$ satisfy the hypotheses of (4). Define $h : \mathbb{R} \to \mathbb{R}$ by $h(t_i) = s_i$ and by extending linearly on each inteval $[t_t, t_{i+1}]$. Since $h(0) = h(t_0) = s_0 = 0$, we have $h \in C_0(\mathbb{R})$. For any $t \in \mathbb{R}$, there exists i such that $t_i \le t < t_{i+1}$. Since $d(\phi_t(x), \phi_{t_i}(x)) \ll \frac{1}{3}u$, $d(\phi_t(x), \phi_{s_i}(y)) \ll \frac{1}{3}u$ and $d(\phi_{s_i}(y), \phi_{h(t)}(y)) \ll \frac{1}{3}u$, we get $d(\phi_t(x), \phi_{h(t)}(y)) \ll u$ for all $t \in \mathbb{R}$. By expansiveness of ϕ, $y = \phi_t(x)$ for some $|t| < \epsilon$.

To prove that (4) $\Longrightarrow$ (2), let $\epsilon > 0$ be given and let $u \gg 0$ and $\tau > 0$ be chosen as in (4). Suppose $d(\phi_t(x), \phi_{h(t)}(y)) \ll u$ for all $t \in \mathbb{R}$ and some increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$. Put $t_0 = 0$ and for $i \in \mathbb{N}$ define

$$t_{i+1} = \begin{cases} h^{-1}(h(t_i) + \tau), & \text{if } h^{-1}(h(t_i) + \tau) \le t_i + \tau \\ t_i + \tau, & \text{if } h^{-1}(h(t_i) + \tau) > t_i + \tau \end{cases} \text{ and}$$

$$t_{-i-1} = \begin{cases} h^{-1}(h(t_{-i}) - \tau), & \text{if } h^{-1}(h(t_{-i}) - \tau) \ge t_{-i} - \tau \\ t_{-i} - \tau, & \text{if } h^{-1}(h(t_{-i}) - \tau) < t_{-i} - \tau \end{cases}.$$

Put $s_i = h(t_i)$. Then $0 < s_{i+1} - t_i \le \tau$ and $0 < s_{i+1} - s_i \le \tau$ for all $i \in \mathbb{Z}$. Moreover, $t_i \to \infty$ and $t_{-i} \to -\infty$ as $i \to \infty$. Apply to (4), we obtain $y = \phi_t(x)$ for some $|t| < \epsilon$.

To show that (1) $\Longrightarrow$ (3), for $u \gg 0$, there exists an $\epsilon > 0$ such that $\phi_t(x) \in B(x, u)$ for all $|t| < \epsilon$. By (1), there is $v \gg 0$ such that if $d(\phi_t(x), \phi_{h(t)}(y)) \ll v$ for all $t \in \mathbb{R}$ and some $h \in C_0(\mathbb{R})$, then $y = \phi_t(x)$ for some $|t| < \epsilon$. Therefore y is on the same orbit as x and the orbit from x to y lies inside $B(x, u)$.

Finally, to show that (3) $\Longrightarrow$ (1), let $\epsilon > 0$ be given. Since $\bigcup_{x \in X}\{(x, \phi_t(x))| - \epsilon < t < \epsilon\}$ is a neighborhood of $\triangle_X$ and $\triangle_X$ is compact, there is $u \gg 0$ such that $B_\rho(\triangle_X, u) \subset \bigcup_{x \in X}\{(x, \phi_t(x))| - \epsilon < t < \epsilon\}$. By (3), we can choose $v \gg 0$ such that if $d(\phi_t(x), \phi_{h(t)}(y)) \ll v$ for all $t \in \mathbb{R}$ and some $h \in C_0(\mathbb{R})$, then y is on the same orbit as x and the orbit from x to y lies inside $B_d(x, u)$. Since $\rho((x, x), (x, y)) = d(x, y) \ll u$, we get $(x, y) \in B_\rho(\triangle_X, u) \subset \bigcup_{x \in X}\{(x, \phi_t(x))| - \epsilon < t < \epsilon\}$. Therefore, we conclude that $y = \phi_t(x)$ for some $|t| < \epsilon$. $\square$

3. Conjugate invariance

Let (X, d) and (Y, ρ) be compact TVS-cone metric spaces over topological vector space E. We recall that the flows ϕ on X and ψ on Y are said to be *conjugate* if there is a homeomorphism from X to Y mapping the orbits of ϕ onto the orbits of ψ. In other words, there are a homeomorphism $f : X \to Y$ and a continuous function $h : X \times \mathbb{R} \to \mathbb{R}$ satisfying $f(\phi_t(x)) = \psi_{h(x,t)}(f(x))$ for all $(x, t) \in X \times \mathbb{R}$.

THEOREM 3.1. *Expansiveness is conjugate invariance.*

Proof. Suppose that $f : X \to Y$ is a homeomorphism which maps the orbits of ϕ onto the orbits of ψ. Then $f^{-1} \circ \psi_t \circ f$ is a flow on X with the same orbits as ϕ_t. Fix $x \in X$.

Case 1. If the orbit of x is not periodic under ϕ, then the map $\sigma_x : \mathbb{R} \to \mathbb{R}$ given by $(f^{-1} \circ \psi_t \circ f)(x) = \phi_{\sigma_x(t)}(x)$ is a well-defined bijection. Also $\sigma_x(0) = 0$ and σ_x is either strictly increasing or strictly decreasing. Hence σ_x is a homeomorphism of $\mathbb{R}$.

Case 2. Let x be a periodic point of ϕ and let $\tau > 0$ be the period of x under ϕ and $\kappa > 0$ be the period of $f(x)$ under ψ. Then σ_x is well-defined on $[0, \kappa]$ given by $(f^{-1} \circ \psi_t \circ f)(x) = \phi_{\sigma_t(x)}(x)$ and takes values either in $[0, \tau]$ or $[-\tau, 0]$. Moreover σ_x is easily shown to be continuous on $[0, \kappa]$. Similarly σ_x can be defined on $[n\kappa, (n+1)\kappa]$ and so σ_x becomes a homeomorphism of $\mathbb{R}$.

Suppose ϕ is expansive and let $u_2 \gg 0$ be given. Since f is uniformly continuous, we can choose $u_1 \gg 0$ so that $f(B_d(x, u_1)) \subset B_\rho(f(x), u_2)$ for all $x \in X$. Let $v_1 \gg 0$ be the vector given by (3) in Theorem 2.5 to correspond to u_1 and choose $v_2 \gg 0$ so that $\rho(y_1, y_2) \ll v_2$ implies $d(f^{-1}(y_1), f^{-1}(y_2)) \ll v_1$. If $\rho(\psi_t(f(x_1)), \psi_{h(t)}(f(x_2))) \ll v_2$ for all $t \in \mathbb{R}$ and some $h \in C_0(\mathbb{R})$, then $d((f^{-1} \circ \psi_t \circ f)(x_1), (f^{-1} \circ \psi_{h(t)} \circ f)(x_2)) = d(\phi_{\sigma_{x_1(t)}}(x_1), \phi_{\sigma_{x_2(t)}(x_2)}) \ll v_1$ for all $t \in \mathbb{R}$. Putting $\sigma_{x_1}(t) = s$. Then $t = \sigma_{x_1}^{-1}(s)$ and it follows that $d(\phi_s(x_1), \phi_{\sigma_{x_2} \circ h \circ \sigma_{x_1}^{-1}(s)}(x_2)) \ll v_1$ for all $s \in \mathbb{R}$. Since $\sigma_{x_2} \circ h \circ \sigma_{x_1}^{-1} \in C_0(\mathbb{R})$, x_2 is in the same ϕ-orbit as x_1 and the ϕ-orbit from x_1 to x_2 belongs to $B_d(x_1, u_1)$. Therefore $f(x_2)$ is in the same ψ-orbit as $f(x_1)$ and the ψ-orbit from $f(x_1)$ to $f(x_2)$ belongs to $B(f(x_1), u_2)$. By Lemma 2.5.(3), ψ is expansive. $\square$

4. Topological entropy of expansive flows

We shall define topological entropy of homeomorphism of TVS-cone metric spaces. Let (X, d) be a TVS-cone metric space and f be a homeomorphism of X. Let $n \in \mathbb{N}$ and $u \gg 0$ be a vector. For $E, F \subset X$, we say that E (n, u)-spans F with respect to f if for each $x \in F$ there is $y \in E$ such that $d(f^k(x), f^k(y)) \ll u$ for all $0 \leq k < n$. We let

$\gamma_n(F,u,f)$ denote the minimum cardinality of a set which (n,u)-spans F with respect to f. If F is compact, then the continuity of f guarantess $\gamma_n(F,f) < \infty$. For a compact subset $F \subset X$ we define

$\overline{\gamma_f}(F,u) = \limsup \frac{1}{n} \log \gamma_n(F,u,f)$ and $h(f,F) = \sup \overline{\gamma_f}(F,u)$.

For $u \gg 0$ and $x \in X$ define $\Gamma_u(x,f) = \{y \in X | d(f^n(x), f^n(y)) \ll u$ for all $n \in \mathbb{Z}\}$. f is called h-expansive if there exists $u \gg 0$ such that $h(f, \Gamma_u(x,f)) = 0$ for all $x \in X$. A flow ϕ on X is called h-expansive if a homeomorphism $\phi_t : X \to X$ is h-expansive for all $t > 0$.

We shall prove the following

Theorem 4.1. *Let (X,d) be a compact TVS-cone metric space. Every expansive flow ϕ on X is h-expansive.*

Proof. Let $t > 0$. For $u \gg 0$, we define $\Gamma_u(x,\phi) = \{y \in X | d(\phi_s(x), \phi_s(y)) \ll u$ for all $s \in \mathbb{R}\}$. For any $\tau > 0$, since ϕ is expansive, there is $u \gg 0$ such that $\Gamma_u(x,\phi) \subset \phi_{[-\tau,\tau]}(x)$. By the integral continuity theorem, there exists $v \gg 0$ such that if $d(x,y) \ll v$ then $d(\phi_s(x), \phi_s(y)) \ll u$ for all $0 \leq s \leq t$. Let $y \in \Gamma_v(x,\phi_t)$. Then $d(\phi_{nt}(x), \phi_{nt}(y)) = d(\phi_t^n(x), \phi_t^n(y)) \ll v$ for all $n \in \mathbb{Z}$. For any $s \in \mathbb{R}$, there is $n \in \mathbb{Z}$ such that $nt \leq s < (n{+}1)t$. Since $d(\phi_{nt}(x), \phi_{nt}(y)) \ll v$ and $0 \leq s{-}nt < t$, we get $d(\phi_{s-nt}(\phi_{nt}(x)), \phi_{s-nt}(\phi_{nt}(y))) = d(\phi_s(x), \phi_s(y)) \ll u$. Therefore, $\Gamma_v(x,\phi_t) \subset \Gamma_u(x,\phi) \subset \phi_{[-\tau,\tau]}(x)$. Since $\phi : X \times [0,1] \to X$ uniformly continuous, there exist $w \gg 0$ and $c \in (0,1)$ such that if $d(p,q) \ll w$ and $a, b \in [0,1]$ with $|a-b| < c$, then $d(\phi_a(p), \phi_b(q)) \ll v$. We claim that if $a, b \in \mathbb{R}$ and $|a-b| < c$, then $d(\phi_a(x), \phi_b(x)) \ll v$ for all $x \in X$.

(1) When $a < b$, since $d(\phi_a(x), \phi_a(x)) = 0 \ll w$ and $0 \leq b-a < c$, we have $d(\phi_a(x), \phi_{b-a}(\phi_a(x))) = d(\phi_a(x), \phi_b(x)) \ll v$.

(2) When $a \geq b$, since $d(\phi_b(x), \phi_b(x)) = 0 \ll w$ and $0 \leq a-b < c$, we have $d(\phi_{a-b}(\phi_b(x)), \phi_b(x)) = d(\phi_a(x), \phi_b(x)) \ll v$.

Since $\{(s-c, s+c) | s \in [-\tau,\tau]\}$ is an open cover of $[-\tau,\tau]$ and $[-\tau,\tau]$ is compact, there exist finitely many $s_1, s_2, \cdots, s_m$ in $[-\tau,\tau]$ such that $[-\tau,\tau] \subset \cup_{i=1}^m (s_i - c, s_i + c)$. Let $n \in \mathbb{N}$. We assert that $\{\phi_{s_i}(x) | i = 1, 2, \cdots, m\}$ (n,v)-spans $\phi_{[-\tau,\tau]}(x)$ with respect to ϕ_t. For any $s \in [-\tau,\tau]$, there is i such that $s \in (s_i - c, s_i + c)$. For $0 \leq k < n$, since $|kt{+}s{-}(kt{+}s_i)| = |s{-}s_i| < c$, we obtain $d(\phi_t^k(\phi_s(x)), \phi_t^k(\phi_{s_i}(x))) = d(\phi_{kt+s}(x), \phi_{kt+s_i}(x)) \ll v$. Thus, $\{\phi_{s_i}(x) | i = 1, 2, \cdots, m\}$ (n,v)-spans $\phi_{[-\tau,\tau]}(x)$ with respect to ϕ_t. Therefore, $\gamma_n(\phi_{[-\tau,\tau]}(x), v, \phi) \leq m$ for all $n \in \mathbb{N}$. Then $\overline{\gamma_{\phi_t}}(\phi_{[-\tau,\tau]}(x), v) = 0$. We conclude that $h(\phi_t, \Gamma_v(x,\phi_t)) \leq h(\phi_t, \phi_{[-\tau,\tau]}(x)) = 0$. Thus ϕ is h-expansive. $\square$

We shall now discuss topological entropy of an expansive flow ϕ on a TVS-cone metric space. Let $t > 0$ and $u \gg 0$. For E, $F \subset X$ we say that E (t,u)-spans F with respect to ϕ if for each $x \in F$ there is $y \in E$ such that $d(\phi_s(x), \phi_s(y)) \ll u$ for all $0 \leq s \leq t$.

Let $\gamma_t(F, u, \phi)$ denote the minimum cardinality of a set which (t,u)-spans F with respect to ϕ. We claim that if F is compact, then $\gamma_t(F, u, \phi) < \infty$. There exists $v \ll \frac{1}{2}u$. Let $x \in F$. By the integral continuity theorem, there exists a neighborhood U_x of x such that if $y \in U_x$, then $d(\phi_s(x), \phi_s(y)) \ll u$ for all $0 \leq s \leq t$. Since $\{U_x | x \in F\}$ is an open cover of F and F is compact, there exists finitely many $x_1, x_2, \cdots, x_n \in F$ such that $F \subset \cup_{i=1}^{n} U_{x_i}$. For any $x \in F$, there is i such that $x \in U_{x_i}$. Then

$$d(\phi_s(x), \phi_s(x_i)) \ll u \text{ for all } 0 \leq s \leq t.$$

Thus $\{x_1, x_2, \cdots, x_n\}$ (t,u)-spans F with respet to ϕ and so $r_t(F, u, \phi) \leq n$. We define $\overline{r_\phi}(F, u) = \limsup \frac{1}{t} \log r_t(F, u, \phi)$.

Let $F \subset X$. We say that $E \subset F$ is a (t,u)-separated subset of F with respect to ϕ if for any $x, y \in E$ with $x \neq y$ we have $d(\phi_s(x), \phi_s(y)) \geq u$ for some $0 \leq s \leq t$. Let $S_t(F, u, \phi)$ denote the maximum cardinality of a set which is a (t,u)-separated subset of F. We claim that if F is compact, then $S_t(F, u, \phi) < \infty$. There exists $v \ll \frac{1}{2}u$. For any $x \in X$, by integral continuity theorem, there exists a neighborhood U_x of x such that if $u \in U_x$ then $d(\phi_s(x), \phi_s(y)) \ll v$ for all $0 \leq s \leq t$. Since$\{U_x | x \in F\}$ is an open cover of F and F is compact, there exist finitely many $x_1, x_2, \cdots, x_n \in F$ such that $F \subset \bigcup_{i=1}^{n} U_{x_i}$. If $E \subset F$ with $Card E \geq n+1$, then there exist $x, y \in E$ and i such that $x, y \in U_{x_i}$. Then we obtain $d(\phi_s(x), \phi_s(y)) \leq d(\phi_s(x), \phi_s(x_i)) + d(\phi_s(x_i), \phi_s(y)) \ll 2v \ll u$ for all $0 \leq s \leq t$. Thus E is not (t,u)-separated set. Therefore, $S_t(F, u, \phi) \leq n$. We define $\overline{S}_\phi(F, u) = \limsup \frac{1}{t} \log S_t(F, u)$ and topological entropy by $h(\phi, F) = \sup \overline{S}_\phi(F, u) = \sup \overline{\gamma}_\phi(F, u)$. These limits exist and are equal by following proposition.

Proposition 4.2. *(1)* $\gamma_t(F, u, \phi) \leq S_t(F, u, \phi) \leq \gamma_t(F, v, \phi)$, *where* $v \ll \frac{1}{2}u$.

(2) If $u \ll v$, *then we have* $\overline{\gamma}_\phi(F, v) \leq \overline{\gamma}_\phi(F, u)$ *and* $\overline{S}_\phi(F, v) \leq \overline{S}_\phi(F, u)$.

Proof. (1) Let E be a maximal (t,u)-separated subset of F. For any $x \in F - E$, since $E \cup \{x\}$ is not a (t,u)-separated subset of F, there exists $y \in E$ such that $d(\phi_s(x), \phi_s(y)) \ll u$ for all $0 \leq s \leq t$. Thus E (t,u)-spans F. Hence $\gamma_t(F, u, \phi) \leq S_t(F, u, \phi)$. Let E_1 be a (t,u)-separated subset of F and let $E_2(t,v)$-spans F. For any $x \in E_1 \subset F$, there exists $f(x) \in E_2$ such that $d(\phi_s(x), \phi_s(f(x))) \ll v$ for all $0 \leq s \leq t$. If $f(x) = f(y)$, then we have $d(\phi_s(x), \phi_s(y)) \leq d(\phi_s(x), \phi_s(f(x))) +$

$d(\phi_s(f(x)), \phi_s(y)) \ll u$ for all $0 \leq s \leq t$. Since E_1 is (t,u)-separated, we have $x = y$ and so f is injective. Thus, we conclude that $CardE_1 \leq CardE_2$. Therefore, $S_t(F,u,\phi) \leq \gamma_t(F,v,\phi)$.

(2) The proof is trivial. □

For $t > 0$, let $v(t)$ denote the number of closed orbits of ϕ with a period $\tau \in [0,t]$ and $v_c(t)$ be the number of closed orbits of ϕ with a period $\tau \in [t-c, t+c]$.

THEOREM 4.3. *Let ϕ be an expansive flow on a compact TVS-cone metric space X. Then the topological entropy $h(\phi)$ satisfies* $\limsup \frac{1}{t} \log v(t) \leq h(\phi) \equiv h(\phi, X)$.

Proof. Let $\epsilon > 0$. By Theorem 2.5 (4), there exist $u \gg 0$ and $\tau > 0$ such that if $(t_i), (u_i)$ are bi-sequences with $t_0 = u_0 = 0$, $0 < t_{i+1} - t_i \leq 2\tau$, $|u_{i+1} - u_i| \leq 2\tau$ for all $i \in \mathbb{Z}$, $t_i \to \infty$ and $t_{-i} \to -\infty$ as $i \to \infty$ and if $d(\phi_{t_i}(x), \phi_{u_t}(y)) \ll u$ for all $i \in \mathbb{Z}$, then $y = \phi_s(x)$ for some $|s| < \epsilon$. Let $x, y \in X$ be distinct periodic points with periods $a, b \in [t - \frac{\tau}{2}, t + \frac{\tau}{2}]$ respectively. Let $m = [\frac{t - \frac{\tau}{2}}{\tau}] + 1$ and put $t_{pm+q} = pa + q\tau$, $u_{pm+q} = pb + q\tau$ for $(p,q) \in \mathbb{Z} \times \{0, 1, \cdots, m-1\}$. Then $t_0 = u_0 = 0$, $t_{pm+q+1} - t_{pm+q} = u_{pm+q+1} - u_{pm+q} = \tau$ for $(p,q) \in \mathbb{Z} \times \{0, 1, 2, \cdots, m-2\}$ and $0 < t_{(p+1)m} - t_{pm+m-1} \leq 2\tau$, $0 < u_{(p+1)m} - u_{pm+m-1} \leq 2\tau$, $t_i \to \infty$ and $t_{-i} \to -\infty$ as $i \to \infty$. Suppose that x, y are not (t,u)-separated. Then $d(\phi_s(x), \phi_s(y)) \ll u$ for all $0 \leq s \leq t$. Since $d(\phi_{t_{pm+q}}(x), \phi_{u_{pm+q}}(y)) = d(\phi_{qr}(\phi_{pa}(x)), \phi_{qr}(\phi_{pb}(y))) = d(\phi_{qr}(x), \phi_{qr}(y))$ and $0 \leq qr \leq (m-1)\tau < t$, we have $d(\phi_{t_i}(x), \phi_{u_i}(y)) \ll u$ for all $i \in \mathbb{Z}$. Thus $y = \phi_s(x)$ for some $|s| < \epsilon$ and so we have a contradiction. Hence x and y are (t,u)-separated. Hence $v_{\frac{\tau}{2}}(t) \leq \gamma_t(\phi, u)$. Let $t_1 < t_2$. If E is (t_1, u)-separated, then E is (t_2, u)-separated. Therefore, $\gamma_{t_1}(\phi, u) \leq \gamma_{t_2}(\phi, u)$, i.e., $\gamma_t(\phi, u)$ increases with t. On the other hand, let $t = m\tau + s$ where $m = [\frac{t}{\tau}]$, $0 \leq s < \tau$. From $[\frac{t}{\tau}] = m$, it follows that $m \leq \frac{t}{\tau} < m+1$. We consider two cases.

Case 1. $s \leq \frac{\tau}{2}$. By $(0,t] \subset [0,\tau] \cup \bigcup_{n=1}^{m} [n\tau - \frac{\tau}{2}, n\tau + \frac{\tau}{2}]$, we get $v(t) \leq v_{\frac{\tau}{2}}(\frac{\tau}{2}) + \sum_{n=1}^{m} v_{\frac{\tau}{2}}(n\tau) \leq (m+1)\gamma_t(\phi, u) \leq (\frac{t}{\tau} + 2)\gamma_t(\phi, u)$.

Case 2. $\frac{\tau}{2} < s < \tau$. From $(0,t] \subset [0,\tau] \cup \bigcup_{n=1}^{m} [n\tau - \frac{\tau}{2}, n\tau + \frac{\tau}{2}] \cup [m\tau, (m+1)\tau]$, we obtain $v(t) \leq v_{\frac{\tau}{2}}(\frac{\tau}{2}) + \sum_{n=1}^{m} v_{\frac{\tau}{2}}(n\tau) + v_{\frac{\tau}{2}}((m + \frac{1}{2})\tau) \leq (m+2)\gamma_t(\phi, u) \leq (\frac{t}{\tau} + 2)\gamma_t(\phi, u)$. Hence $\frac{1}{t} \log v(t) \leq \frac{1}{t} \log(\frac{t}{\tau} + 2) + \frac{1}{t} \log \gamma_t(\phi, u)$. Since $\lim_{t \to \infty} \frac{1}{t} \log(\frac{t}{\tau} + 2) = 0$, $\limsup \frac{1}{t} \log v(t) \leq \limsup \frac{1}{t} \log \gamma_t(\phi, u) = h(\phi, u) \leq h(\phi)$. This completes the proof. □

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