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GAUSSIAN VERTEX PRIME LABELING OF SOME GRAPHS

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ABSTRACT. In the present work we introduce Gaussian vertex prime labeling and investigate Gaussian vertex prime labeling for gear graph, book graph, wheel graph, kayak paddle, one point union of gear graph and one point union of wheel graph.

1. Introduction

In this paper only undirected and non-trivial graph $G = (V(G), E(G))$ with the vertex set $V(G)$ and edge set $E(G)$ are taken under consideration. Gross and Yellen[7] is followed for graph theoretic notations and terminology whereas D.M. Burton[4] is followed for number theoretic terminology.

The notion of a prime labeling originated with Roger Entringer and was introduced in a paper by Tout et al.[3]. A huge number of graphs are studied for prime labeling listed in [6]. Many remarkable results are given under prime labeling. And many other labelings are also introduced by inspiring from prime labeling like neighborhood-prime labeling, vertex prime labeling etc. Recently, Gaussian prime labeling is introduced in this sequence. In this paper we define one more variant of prime labeling called Gaussian vertex prime labeling.

The spiral ordering in Gaussian integers is defined by S. Klee *et al.* in [9]. Motivated by prime labeling on natural numbers they introduced Gaussian prime labeling with respect to the spiral ordering. So, Gaussian prime labeling can be seen as an extension of prime labeling in some sense.

Hunter Lehmann and Andrew Park proved that any tree with ≤ 72 vertices is Gaussian prime tree in [5]. In [9], S. Klee *et al.* proved that n-centipede tree, $(n, 2)-$centipede tree, the path graph, spider tree, star graph, $(n, 3)$ firecracker tree, (n, k, m) double star tree are Gaussian prime graph.

Deretsky, Lee and Mitchem[10] defined a dual of prime labeling named as vertex prime labeling as follows.

DEFINITION 1.1. [10] A graph with edge set E has a vertex prime labeling if its edges can be labeled with distinct integers $1, 2, ..., |E|$ such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1.

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Deretsky, Lee and Mitchem proved that forests, all connected graphs, $C_{2k} \bigcup C_n$, $C_{2m} \bigcup C_{2n} \bigcup C_{2k+1}$, $C_{2m} \bigcup C_{2n} \bigcup C_{2t} \bigcup C_k$, and $5C_{2m}$ are vertex prime graphs. They further proved that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling.

We will start our discussion by giving some definitions starting from spiral ordering of Gaussian integers and its properties given by Steven Klee et al. [9] with some basic results of Gaussian integers before the main results.

2. Spiral ordering of the Gaussian integers

Gaussian integers are complex numbers of the form $\gamma = x + iy$ where x and y are integers and $i^2 = -1$. The set of Gaussian integers is usually denoted by $\mathbb{Z}[i]$. A Gaussian integer γ is called even if $1 + i$ divides γ. Otherwise, it is an odd Gaussian integer. The norm on $\mathbb{Z}[i]$ is defined as $d(x + iy) = x^2 + y^2$. $\pm 1, \pm i$ are the only units of $\mathbb{Z}[i]$. The associates of γ are unit multiples of γ. In $\mathbb{Z}[i]$, γ and γ' are relatively prime if the common divisors of γ and γ' are the only units of $\mathbb{Z}[i]$. A γ is said to be prime Gaussian integer if and only if $\pm 1, \pm i, \pm\gamma, \pm i\gamma$ are the only divisors of γ.

In [9] S. Klee *et al.* introduced the Spiral ordering of the Gaussian integer. The recursion relation of Spiral ordering of the Gaussian integers starting with $\gamma_1 = 1$ is defined as follows:

(2.1)

$$\gamma_{n+1} = \begin{cases} \gamma_n + i & \text{if } Re(\gamma_n) \equiv 1 \pmod 2, \quad Re(\gamma_n) > Im(\gamma_n) + 1 \\ \gamma_n - 1 & \text{if } Im(\gamma_n) \equiv 0 \pmod 2, \quad Re(\gamma_n) \leq Im(\gamma_n) + 1, \ \ Re(\gamma_n) > 1 \\ \gamma_n + 1 & \text{if } Im(\gamma_n) \equiv 1 \pmod 2, \quad Re(\gamma_n) < Im(\gamma_n) + 1 \\ \gamma_n + i & \text{if } Im(\gamma_n) \equiv 0 \pmod 2, \quad Re(\gamma_n) = 1 \\ \gamma_n - i & \text{if } Re(\gamma_n) \equiv 0 \pmod 2, \quad Re(\gamma_n) \geq Im(\gamma_n) + 1, \ \ Im(\gamma_n) > 0 \\ \gamma_n - i & \text{if } Re(\gamma_n) \equiv 0 \pmod 2, \quad Im(\gamma_n) = 0 \end{cases}$$

The notation γ_n is used to denote the n^{th} Gaussian integer under the above ordering. We symbolically write first $'n'$ Gaussian integers by $[\gamma_n]$.
Under the spiral ordering of Gaussian integers following results are proved in [9]:

- Any two consecutive Gaussian integer are relatively prime.
- Any two consecutive odd Gaussian integer are relatively prime.
- γ and $\gamma + \mu$ are relatively prime, if γ is a Gaussian integer and μ is a unit.
- γ and $\gamma + \mu(1 + i)^k$ are relatively prime, if γ is an odd Gaussian integer and μ is a unit. where k is a positive integer.
- γ and $\gamma + \pi$ are relatively prime if and only if π does not divides γ where γ is Gaussian integer and π is a prime Gaussian integer.

DEFINITION 2.1. [9] Let G be a graph having n vertices. A bijective function $g : V(G) \to [\gamma_n]$ is called Gaussian prime labeling, if the images of adjacent vertices are relatively prime. A graph which admits Gaussian prime labeling is known as Gaussian prime graph.

Here, we introduce Gaussian vertex prime labeling with respect to above defined spiral ordering motivated by Gaussian prime labeling and vertex prime labeling.

DEFINITION 2.2. A graph with edge set E has a Gaussian vertex prime labeling if its edges can be labeled with first $|E|$ Gaussian integers $\gamma_1, \gamma_2, ..., \gamma_{|E|}$ such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is unit. A graph which admits Gaussian vertex prime labeling is known as Gaussian vertex prime graph.

3. Preliminaries

Now, let us give definitions of graphs which are studied in this paper.

DEFINITION 3.1. [1] A wheel graph W_n is obtained by taking a cycle C_n and a new vertex w outside of C_n. w is joined to each vertex of C_n by an edge. It has $2n$ edges and $n+1$ vertices.

DEFINITION 3.2. [1] A Gear Graph G_n is obtained from W_n by adding a new vertex on each edge of cycle C_n of W_n. It has $2n+1$ vertices and $3n$ edges.

One can easily show that path graph, cycle graph, wheel graph, helm graph and gear graph are Gaussian vertex prime.

DEFINITION 3.3. [6] A book graph B^n_m is made from n copies of cycle C_m that share an edge in common. It is n page book with each page as a $m-$gone.

DEFINITION 3.4. [6] Irregular book graph $B^n_{m_1,m_2,...,m_n}$ is a book on n pages that share an edge in common containing pages as cycles of length $m_1, m_2, ..., m_n$.

DEFINITION 3.5. [2] A kayak paddle $KP(k, m, t)$ is a graph obtained by joining cycles C_k and C_m by a path of length t.

DEFINITION 3.6. [8] The graph $G^{(k)}$ (where $k \geq 2$) is known as the one point union of k copies of the graph G obtained from the k copies of the graph G by identifying exactly one vertex of each of these k copies of G.

Throughout this paper, we will understand that the graph is Gaussian vertex prime graph means to be a Gaussian vertex prime graph with respect to the spiral ordering of Gaussian integers.

4. Main theorems

THEOREM 4.1. *One point union of k copies of wheel graph $W_n^{(k)}$ is a Gaussian vertex prime graph.*

Proof. Let $G = W_n^{(k)}$ be a graph with $nk+1$ vertices and $2nk$ edges. For $k = 1$, $G = W_n$ is a Gaussian vertex prime graph. The set of vertices of i^{th} copy of W_n in G be given by $\{w^i, w_1^i, w_2^i, ..., w_n^i\}$ where w^i is hub and remaining vertices are rim vertices i.e., $w_1^i, w_2^i, ..., w_n^i$. The edge set is given by $E(G) = \{c_j^i = (w^i w_j^i); i = 1, 2, ..., k$ and $j = 1, 2, ..., n\} \cup \{e_j^i = (w_j^i w_{j+1}^i); i = 1, 2, ..., k$ and $j = 1, 2, ..., n$ and $j+1$ is taken as modulo $n\}$. Let the graph $G = W_n^{(k)}$ (where $k \geq 2$) be obtained from the k copies of the graph W_n by identifying hub of each of these k copies of W_n. Denote identifying vertex by w. That is $w = w^i$, $i = 1, 2, ..., k$. Define a bijective map $f : E(G) \to [\gamma_{2nk}]$ as follows:
$f(e_j^i) = \gamma_{2n(i-1)+j}$; $i = 1, 2, ..., k$ and $j = 1, 2, ..., n$
$f(c_j^i) = \gamma_{n(2i-1)+j}$; $j = 1, 2, ..., n$

For each vertex there is at least one pair of incident edges whose labels are consecutive Gaussian integers. Since consecutive Gaussian integers are relatively prime, the graph G is Gaussian vertex prime. Now, if we take any rim vertex of W_n as an identifying vertex of one point union of k copies of W_n then we get another structure. These two structures are the only possible non-isomorphic structures of G and above defined bijective map will work as a Gaussian vertex prime labeling function for both the non-isomorphic structures and hence the graph G is Gaussian vertex prime. □

ILLUSTRATION 4.1. Figure 1 shows Gaussian vertex prime labeling of $W_4^{(3)}$ with hub as common point.

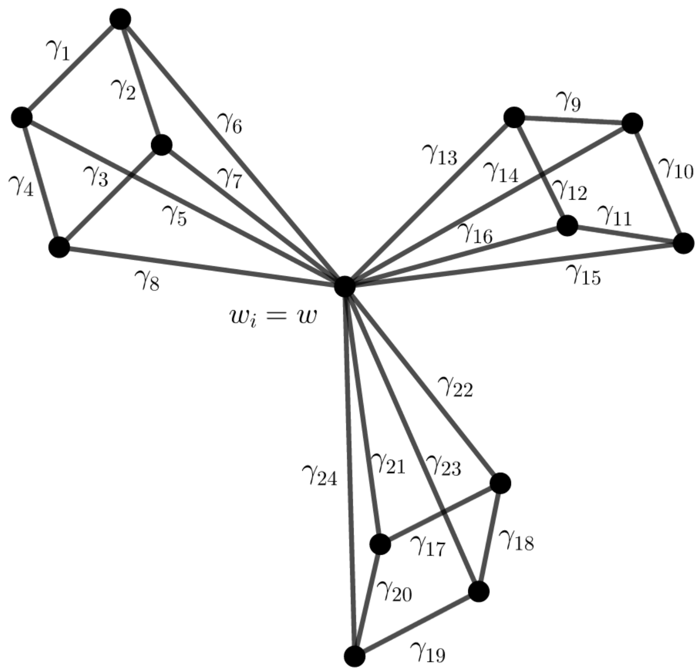

FIGURE 1. Gaussian vertex prime labeling of $W_4^{(3)}$

THEOREM 4.2. *One point union of k copies of gear graph $G_n^{(k)}$ is a Gaussian vertex prime graph.*

Proof. Since G_n has $2n+1$ vertices and $3n$ edges, the graph $G = G_n^{(k)}$ is a graph with $2nk+1$ vertices and $3nk$ edges. The set of vertices of i^{th} copy of G_n in G be given by $\{w^i, w_1^i, w_2^i, w_3^i, ..., w_{2n}^i\}$ where w^i is hub and remaining vertices are cycle vertices i.e., $w_1^i, w_2^i, w_3^i, ..., w_{2n}^i$. The edge set is given by $E(G) = \{c_j^i = (w^i w_j^i); i = 1, 2, ..., k \text{ and } j = 1, 3, 5, ..., 2n-1\} \cup \{e_j^i = (w_j^i w_{j+1}^i); i = 1, 2, ..., k \text{ and } j = 1, 2, 3, ..., 2n \text{ and } j+1 \text{ taken as } \bmod 2n\}$.

Let the graph $G_n^{(k)}$ (where $k \geq 2$) be obtained from the k copies of the graph G_n by identifying hub of each of these k copies of G_n.

Define a bijective map $f : E(G) \to [\gamma_{3nk}]$ as follows:

$f(e_j^i) = \gamma_{3n(i-1)+j} \quad ; j = 1, 2, ..., 2n \text{ and } i = 1, 2, ..., k$

$f(c_j^i) = \gamma_{\frac{24i+j+4n-25}{2}+1} \quad ; j = 1, 3, ..., 2n-1$

Here, the set of incident edges of every vertex contains at least one pair of edges whose labels are consecutive Gaussian integers. Now, by the fact that consecutive Gaussian integers are relatively prime we can say that graph G is Gaussian vertex prime. The non-isomorphic structures of one point union

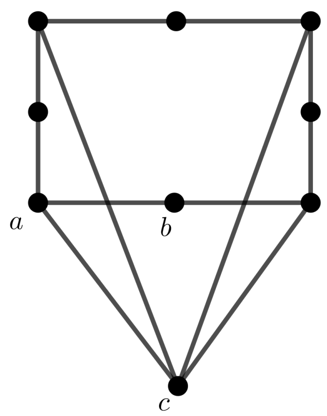

FIGURE 2. Gear Graph G_4

of G_n are obtained at points a or b or c of gear graph G_n as shown in figure. These three structures are the only possible non-isomorphic structures of G and above defined bijective map will work as a Gaussian vertex prime labeling function for all the non-isomorphic structures and hence the graph G is Gaussian vertex prime. □

ILLUSTRATION 4.2. Figure 3 shows Gaussian vertex prime labeling of one point union of three copies of gear graph G_4 i.e. $G_4^{(3)}$ with $'a'$ as common point

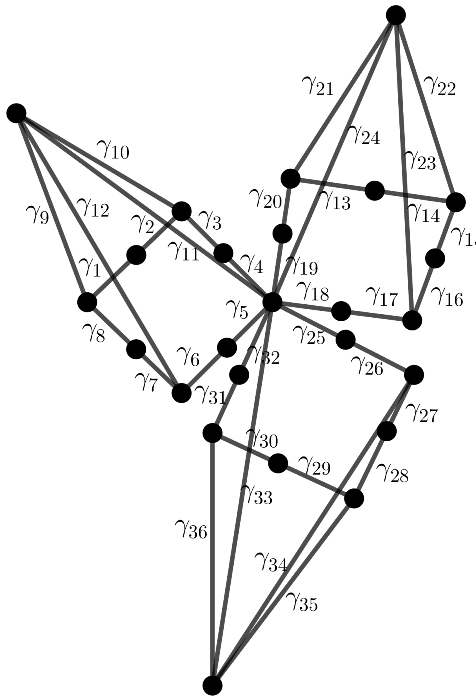

FIGURE 3. Gaussian vertex prime labeling of $G_4^{(3)}$

THEOREM 4.3. *Let G' be a graph obtained from W_n by replacing each cycle edge (edges except spokes) by a path P_m. Then $G'^{(k)}$ is Gaussian vertex prime.*

Proof. Since G' has $nm - n + 1$ vertices and nm edges, the graph $G = G'^{(k)}$ be a graph with $(nm - n)k + 1$ vertices and nmk edges. The set of vertices of i^{th} copy of G' in G be given by $\{w^i, w_1^i, w_2^i, w_3^i, ..., w_{nm-n}^i\}$ where w^i is hub and remaining vertices are rim vertices i.e., $w_1^i, w_2^i, w_3^i, ..., w_{nm-n}^i$. The edge set is given by $E(G) = \{c_j^i = (w^i w_j^i); j = 1, m, 2m - 1, 3m - 2, ..., nm - m - n + 2$ and $i = 1, 2, ..., k;\} \cup \{e_j^i = (w_j^i w_{j+1}^i); j = 1, 2, ..., nm - n, j + 1$ taken as modulo $(nm - n)$ and for $i = 1, 2, ..., k\}$.

Define a bijective map $f : E(G) \to [\gamma_{nmk}]$ as follows:

$f(e_j^i) = \gamma_{nm(i-1)+j} \quad ; i = 1, 2, ..., k$ and $j = 1, 2, ..., nm - n$

$f(c_1^i) = \gamma_{n(mi-1)+1} \quad ; i = 1, 2, ..., k$

$f(c_m^i) = \gamma_{n(mi-1)+2} \quad ; i = 1, 2, ..., k$

$\vdots$

$f(c_{nm-n-m+2}^i) = \gamma_{n(mi-1)+n} \quad ; i = 1, 2, ..., k$

Here, the set of incident edges of every vertex contains at least one pair of edges whose labels are consecutive Gaussian integers. Now, by the fact that consecutive Gaussian integers are relatively prime we can say that graph G is Gaussian vertex prime.

The non-isomorphic structures of one point union of G' are obtained at hub or at any of degree-2 or degree-3 vertices. These structures are the only possible non-isomorphic structures of G and above defined bijective map will work as a Gaussian vertex prime labeling function for all the non-isomorphic structures and hence the graph G is Gaussian vertex prime.

□

ILLUSTRATION 4.3. Figure 4 shows Gaussian vertex prime labeling of one point union of two copies of G' with each cycle edge replaced by path P_5.

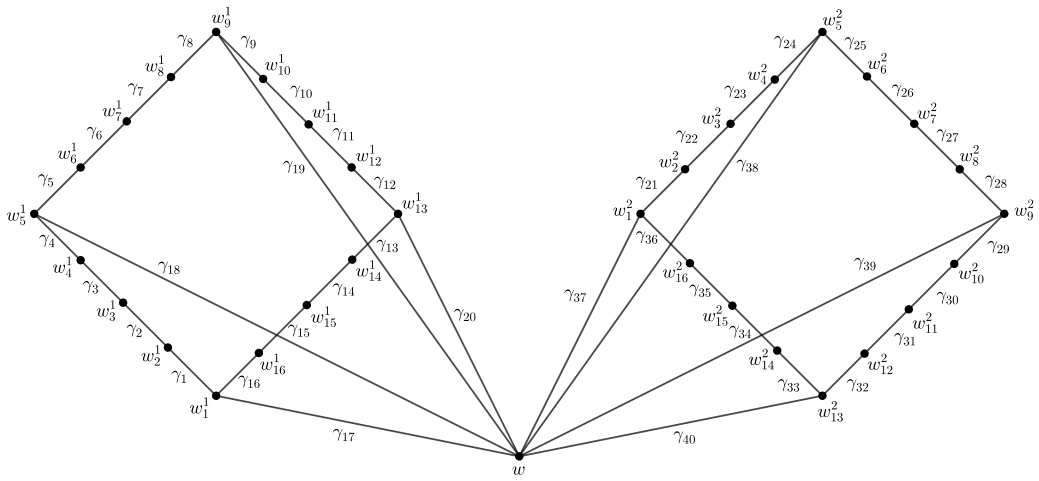

FIGURE 4. Gaussian vertex prime labeling of $G'^{(2)}$

Similarly, we can prove the following two results.

THEOREM 4.4. *Let G' be a graph obtained from W_n by replacing each spokes edge by a path P_m. Then $\bigcup\limits_{k=1}^{m} G'^{(k)}$ is Gaussian vertex prime.*

THEOREM 4.5. *Let G'_n be a graph obtained from W_n by replacing each cycle edge (these edges are other than spokes) by a path P_m and each spoke is replaced by a path P_r. Then G'_n and one point union of k copies of G'_n i.e. $G = (G'_n)^{(k)}$ with all possible structures are Gaussian vertex prime.*

THEOREM 4.6. *A book graph B_m^n is Gaussian vertex prime.*

Proof. Let $G = B_m^n$ be a book graph. It has $n(m-1)+1$ edges and $n(m-2)+2$ vertices. The n copies of cycle C_m are denoted by $C^1, C^2, ..., C^n$. Let vertex v_j^i be the j^{th} vertex on i^{th} cycle and the edge $e_j^i = (v_j^i v_{j+1}^i)$ is the j^{th} edge on i^{th} cycle, $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$ where subscripts are taken modulo n. The edge $e_1 = e_1^k$ is adjacent to e_2^k and e_m^k, $k = 1, 2, ..., n$. Define a bijective map $f : E(G) \to [\gamma_{n(m-1)+1}]$ as follows

$f(e_j^1) = \gamma_j$; $j = 1, 2, ..., m$

$f(e_j^k) = \gamma_{m+(m-1)(k-2)+j-1}$; $k = 2, 3, 4, ..., n$ and $j = 2, 3, ..., m$

Here, labels of incident edges to any vertex are either consecutive Gaussian integers or one of the label is γ_1. Thus, graph G is Gaussian vertex prime. □

ILLUSTRATION 4.4. Figure 5 shows Gaussian vertex prime labeling of book graph B_6^2.

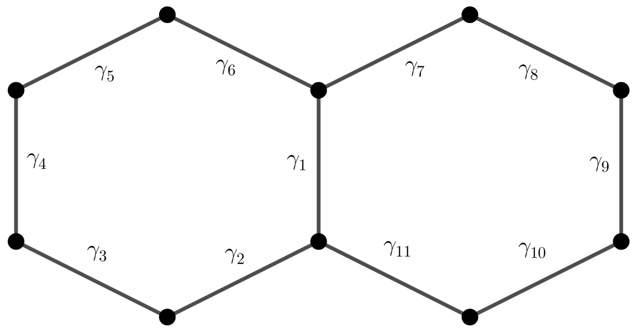

FIGURE 5. Gaussian vertex prime labeling of B_6^2

THEOREM 4.7. *Irregular book graph $B^n_{m_1,m_2,...,m_n}$ is Gaussian vertex prime.*

Proof. Let $G = B^n_{m_1,m_2,...,m_n}$ with $C_{m_1}, C_{m_2}, ..., C_{m_n}$ as different n cycles. e_1 is the edge common to all the cycles and e_i^k be the i^{th} edge on k^{th} cycle where $i = 2, 3, ..., m_k$ and $k = 1, 2, 3, ..., n$. Define a bijective map $f : E(G) \to [\gamma_{|E|}]$ as

$$f(e_i^k) = \begin{cases} \gamma_i & \text{if } k = 1 \text{ and } i = 1, 2, ..., m_1 \\ \gamma_{\sum\limits_{t=1}^{k-1} m_t - k + 1 + i} & \text{if } k = 2, 3..., n \text{ and } i = 2, 3, ..., m_k. \end{cases}$$

Here, labels of incident edges to any vertex are either consecutive Gaussian integers or one of the label is γ_1. Thus, graph G is Gaussian vertex prime.

□

ILLUSTRATION 4.5. Figure 6 shows Gaussian vertex prime labeling of irregular book graph $B^3_{3,4,6}$

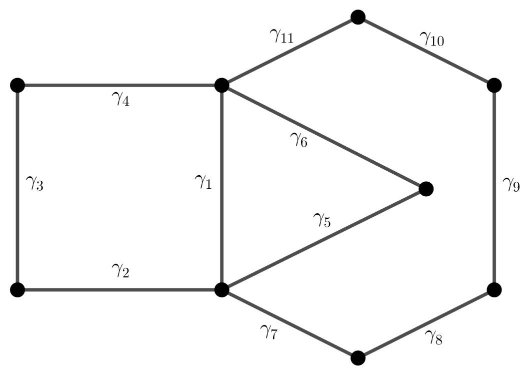

FIGURE 6. Gaussian vertex prime labeling of $B^3_{3,4,6}$

THEOREM 4.8. *A kayak paddle $G = KP(k, m, t)$ is Gaussian vertex prime.*

Proof. Let $G = KP(k, m, t)$ with vertex set $V(G) = \{v_1, v_2, ..., v_k, u_1, u_2, ..., u_{t-1}, w_1, w_2, ..., w_m\}$ where $v_1, v_2, ..., v_k$ and $w_1, w_2, ..., w_m$ are consecutive vertices of cycles C_k and C_m respectively. And $v_1, u_1, u_2, ..., u_{t-1}, w_1$ are consecutive vertices of path P_{t+1}. The edge set $E(G) = \{e_i = (v_i v_{i+1}), i = 1, 2, ..., k-1\} \cup \{e_k = (v_k v_1)\} \cup \{p_1 = (v_1 u_1), p_t = (u_{t-1} w_1)\} \cup \{p_{i+1} = (u_i u_{i+1}), i = 1, 2, ..., t-2\} \cup \{e'_i = (w_i w_{i+1}), i = 1, 2, ..., m-1\} \cup \{e'_m = (w_m w_1)\}$
Define a bijective function $f : E(G) \to [\gamma_{|E|}]$ as
$f(e_i) = \gamma_i \quad ; i = 1, 2, ..., k$
$f(p_i) = \gamma_{k+i} \quad ; i = 1, 2, ..., t$
$f(e'_i) = \gamma_{k+t+i} \quad ; i = 1, 2, ..., m.$
Here, labels of incident edges to any vertex are either consecutive Gaussian integers or one of the label is γ_1. Thus, graph G is Gaussian vertex prime. $\square$

ILLUSTRATION 4.6. Figure 7 shows Gaussian vertex prime labeling of kayak paddle $KP(6, 7, 6)$.

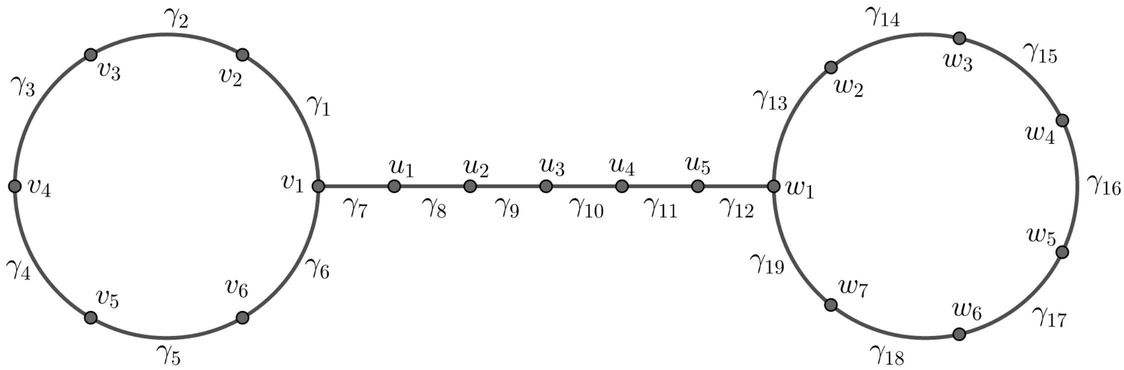

FIGURE 7. Gaussian vertex prime labeling of $KP(6, 7, 6)$

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