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# ON THE STABILITY OF THE GENERAL SEXTIC FUNCTIONAL EQUATION

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Abstract. The general sextic functional equation is a generalization of many functional equations such as the additive functional equation, the quadratic functional equation, the cubic functional equation, the quartic functional equation and the quintic functional equation. In this paper, motivating the method of Găvruta [J. Math. Anal. Appl., **184** (1994), 431–436], we will investigate the stability of the general sextic functional equation.

## 1. Introduction

In this paper, let $V$, $X$, and $Y$ be a real vector space, a real normed space, and a real Banach space, respectively. Ulam [25] raised the question about the stability of group homomorphisms in 1940 and Hyers [8] gave a partial answer to this question by solving the stability of the Cauchy functional equation in the following year. Since then, many mathematicians have generalized the Hyers's result [6, 7, 9, 11, 15, 18, 20, 24]. In particular, Găvruta [7] generalized the result of Hyers as follow:

Proposition 1.1. *Let $(G,+)$ be an abelian group and $\varphi : G^2 \to [0,\infty)$ be a function such that*

$$\tilde{\varphi}(x,y) := \sum_{k=0}^{\infty} 2^{-k}\varphi(x,y) < \infty$$

*for all $x, y \in G$. If $f : G \to Y$ is a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y)$$

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*for all $x, y \in G$, then there exists a unique additive mapping $T : G \to Y$ such that*

$$\|f(x) - T(x)\| \le \frac{1}{2}\tilde{\varphi}(x, x)$$

*for all $x \in G$.*

In this paper, motivating Proposition 1.1, we will study the stability of the general sextic functional equation :

$$\sum_{i=0}^{7} {}_7C_i(-1)^{7-i} f(x + iy) = 0. \tag{1.1}$$

More detailed term for the concept of a general sextic mapping can be found in Baker's paper [3] by the term, generalized polynomial mapping of degree at most 6.

Jun-Kim[10] have previously studied the stability of a general cubic functional equation and others studied the stability of a general quadratic functional equation, a general cubic functional equation, and a general quartic functional equation(refer to [13, 14, 16, 19]). Also, many mathematicians have investigated the stability of the sextic functional equation (refer to [1, 2, 4, 5, 23, 21, 22]). But they worked on a special sextic functional equation. The stability of the general sextic functional equation have been investigated in Lee [17] and Roh-Lee-Jung [12]. Roh-Lee-Jung [12] have proved the stability of the general sextic functional equation by applying the fixed point theorem in the sense of Cădariu and Radu. And Lee [17] proved the Hyers-Ulam-Rassias stability of the general sextic functional equation as follows :

Proposition 1.2. (Theorem 3 in [17]) *Let $p \neq 1, 2, 3, 4, 5, 6$ be a fixed nonnegative real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\|\sum_{i=0}^{7} {}_7C_i(-1)^{7-i} f(x + iy)\| \le \theta(\|x\|^p + \|y\|^p)$$

*for all $x, y \in X$. Then there exists a general sextic mapping $F$ with $F(0) = 0$ and a constant $K(p)$ such that*

$$\|f(x) - f(0) - F(x)\| \le K(p)\ \theta\ \|x\|^p, \quad \text{for all } x \in X.$$

In this paper, we will investigate the stability of the general sextic functional equation in sense of Găvruta[7].

In fact, Lee [17] investigated the stability of the general sextic functional equation for the mapping $f$ such that

$$Df(x,y) = \sum_{i=0}^{7} {}_7C_i(-1)^{7-i} f(x+iy) \le \theta(\|x\|^p + \|y\|^p)$$

and Roh-Lee-Jung [12] did for the mapping $f$ such that

$$Df(x,y) = \sum_{i=0}^{7} {}_7C_i(-1)^{7-i} f(x+iy) \le \varphi(x,y),$$

where $\varphi : V^2 \to [0,\infty)$ is a function for which there exists a constant $0 < L < 1$ such that

$$\varphi(2x,2y) \le (4\sqrt{21}\cos\theta - 14)L\varphi(x,y)$$

for all $x, y \in V$ and $\theta$ is a real constant such that $0 < \theta < \frac{\pi}{4}$ and $\cos(3\theta) = \frac{-17}{21\sqrt{21}}$.

In this paper, we will investigate the stability of the general sextic functional equation for the mapping $f$ such that

$$Df(x,y) = \sum_{i=0}^{7} {}_7C_i(-1)^{7-i} f(x+iy) \le \varphi(x,y),$$

where $\varphi : V^2 \to [0,\infty)$ is a function such that

$$\sum_{n=0}^{\infty} 64^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty \quad \text{or} \quad \sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in V$.

## 2. Stability of a general sextic functional equation

Throughout this section, for a given mapping $f : V \to Y$, we use the following abbreviations:

$$f_o(x) := \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2},$$

$$Df(x,y) := \sum_{i=0}^{7} {}_7C_i(-1)^{7-i} f(x+iy),$$

$$\Gamma(x) := Df_o(-6x,2x) + 6Df_o(-x,x) + 42Df_o(-2x,x) + 112Df_o(-3x,x),$$

$$\Delta f(x) := Df_e(-6x,2x) + 8Df_e(-x,x) + 56Df_e(-2x,x) + 112Df_e(-3x,x)$$

for all $x, y \in V$. If $\tilde{f}$ is a mapping defined by $\tilde{f}(x) = f(x) - f(0)$, then the mapping $\tilde{f}$ satisfies the properties $D\tilde{f}(x,y) = Df(x,y)$ and $\tilde{f}(0) = 0$. By tedious computation we can get the equalities

$$\Gamma f(x) = f_o(8x) - 42f_o(4x) + 336f_o(2x) - 512f_o(x), \tag{2.1}$$

$$\Delta f(x) = f_e(8x) - 84f_e(4x) + 1344f_e(2x) - 4096f_e(x) \tag{2.2}$$

for all $x \in V$.

Lemma 2.1. *For a given mapping $f : V \to Y$ with $f(0) = 0$, let $J_n f, J'_n f : V \to Y$ be the mappings defined by*

$$\begin{aligned} &J_n f(x) \\ &:= \frac{4^n - 20\cdot 16^n + 64\cdot 64^n}{45} f_e\left(\frac{x}{2^n}\right) - \frac{80(4^n - 17\cdot 16^n + 16\cdot 64^n)}{45} f_e\left(\frac{x}{2^{n+1}}\right) \\ &+ \frac{1024(4^n - 5\cdot 16^n + 4\cdot 64^n)}{45} f_e\left(\frac{x}{2^{n+2}}\right) + \frac{2^n - 20\cdot 8^n + 64\cdot 32^n}{45} f_o\left(\frac{x}{2^n}\right) \\ &- \frac{40(2^n - 17\cdot 8^n + 16\cdot 32^n)}{45} f_o\left(\frac{x}{2^{n+1}}\right) + \frac{256(2^n - 5\cdot 8^n + 4\cdot 32^n)}{45} f_o\left(\frac{x}{2^{n+2}}\right) \end{aligned}$$

*and*

$$\begin{aligned} &J'_n f(x) \\ &:= \left(\frac{1}{64^n} - \frac{5}{16^n} + \frac{4}{4^n}\right)\frac{f_e(2^{n+2}x)}{2880} - \left(\frac{20}{64^n} - \frac{340}{16^n} + \frac{320}{4^n}\right)\frac{f_e(2^{n+1}x)}{2880} \\ &+ \left(\frac{64}{64^n} - \frac{1280}{16^n} + \frac{4096}{4^n}\right)\frac{f_e(2^n x)}{2880} + \left(\frac{1}{32^n} - \frac{5}{8^n} + \frac{4}{2^n}\right)\frac{f_o(2^{n+2}x)}{720} \\ &- \left(\frac{10}{32^n} - \frac{170}{8^n} + \frac{160}{2^n}\right)\frac{f_o(2^{n+1}x)}{720} + \left(\frac{16}{32^n} - \frac{320}{8^n} + \frac{1024}{2^n}\right)\frac{f_o(2^n x)}{720} \end{aligned}$$

*for all $x \in V$ and all nonnegative integers $n$. Then*

$$\begin{aligned} J_n f(x) - J_{n+1} f(x) = &\left(\frac{4^n}{45} - \frac{20\cdot 16^n}{45} + \frac{64\cdot 64^n}{45}\right)\Delta f\left(\frac{x}{2^{n+3}}\right) \\ &+ \left(\frac{2^n}{45} - \frac{20\cdot 8^n}{45} + \frac{64\cdot 32^n}{45}\right)\Gamma f\left(\frac{x}{2^{n+3}}\right) \end{aligned} \tag{2.3}$$

$$\begin{aligned} J'_n f(x) - J'_{n+1} f(x) = &-\left(\frac{4}{4^{n+1}} - \frac{5}{16^{n+1}} + \frac{1}{64^{n+1}}\right)\frac{\Delta f(2^n x)}{2880} \\ &- \left(\frac{4}{2^{n+1}} - \frac{5}{8^{n+1}} + \frac{1}{32^{n+1}}\right)\frac{\Gamma f(2^n x)}{720} \end{aligned} \tag{2.4}$$

*for all $x \in V$ and all nonnegative integers $n$.*

**Proof**. From the equalities (2.1) and the definitions of $J_n f$ and $J'_n f$, we obtain the equalities

$$\begin{aligned} &J_n f(x) - J_{n+1} f(x) \\ &\quad = \left(\frac{4^n}{45} - \frac{20\cdot 16^n}{45} + \frac{64\cdot 64^n}{45}\right)\Delta f\left(\frac{x}{2^{n+3}}\right) \\ &\qquad + \left(\frac{2^n}{45} - \frac{20\cdot 8^n}{45} + \frac{64\cdot 32^n}{45}\right)\Gamma f\left(\frac{x}{2^{n+3}}\right) \end{aligned}$$

and similarly we have

$$\begin{aligned} &J'_n f(x) - J'_{n+1} f(x) \\ &\quad = -\left(\frac{4}{4^{n+1}} - \frac{5}{16^{n+1}} + \frac{1}{64^{n+1}}\right)\frac{\Delta f(2^n x)}{2880} \\ &\qquad - \left(\frac{4}{2^{n+1}} - \frac{5}{8^{n+1}} + \frac{1}{32^{n+1}}\right)\frac{\Gamma f(2^n x)}{720} \end{aligned}$$

for all $x \in V$ and all nonnegative integers $n$. □

Lemma 2.2. *If $f : V \to Y$ is a mapping such that*

$$Df(x, y) = 0$$

*for all $x, y \in V$ with $f(0) = 0$, then*

$$J_n f(x) = f(x) \quad \text{and} \quad J'_n f(x) = f(x)$$

*for all $x \in V$ and all positive integers $n$.*

**Proof**. If $f : V \to Y$ is a mapping such that

$$Df(x, y) = 0$$

for all $x, y \in V$ with $f(0) = 0$, then it follows from the definitions of $\Delta f(x)$ and $\Gamma f(x)$ that $\Delta f(x) = 0$ and $\Gamma f(x) = 0$ for all $x \in V$. Therefore, together with the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1}(J_i f(x) - J_{i+1} f(x))$ and the equality (2.3), we conclude that

$$J_n f(x) = f(x)$$

for all $x \in V$ and all positive integers $n$. In the same way we can easily show the equality $J'_n f(x) = f(x)$ for all $x \in V$. □

From Lemma 2.1 and Lemma 2.2, we can prove the following stability theorem.

THEOREM 2.3. *Let $\varphi : V^2 \to [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 64^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty \tag{2.5}$$

*for all $x, y \in V$. Suppose that $f : V \to Y$ is a mapping such that*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.6}$$

*for all $x, y \in V$. Then there exists a general sextic mapping $F$ such that*

(2.7)
$$\begin{aligned}\|f(x) - f(0) - F(x)\| \le \sum_{n=0}^{\infty}\Bigg[&\left(\frac{4^n}{45} - \frac{20\cdot 16^n}{45} + \frac{64\cdot 64^n}{45}\right)\Phi\left(\frac{x}{2^{n+3}}\right)\\ &+\left(\frac{2^n}{45} - \frac{20\cdot 8^n}{45} + \frac{64\cdot 32^n}{45}\right)\Phi'\left(\frac{x}{2^{n+3}}\right)\Bigg]\end{aligned}$$

*for all $x \in V$ and $F(0) = 0$, where $\varphi_e : V^2 \to [0,\infty)$ and $\Phi, \Phi' : V \to [0,\infty)$ are functions defined by*

$$\begin{aligned}\varphi_e(x,y) :=& \frac{\varphi(x,y) + \varphi(-x,-y)}{2}\\ \Phi(x) :=& \varphi_e(-6x, 2x) + 8\varphi_e(-x,x) + 56\varphi_e(-2x,x) + 112\varphi_e(-3x,x)\\ \Phi'(x) :=& \varphi_e(-6x, 2x) + 6\varphi_e(-x,x) + 42\varphi_e(-2x,x) + 112\varphi_e(-3x,x).\end{aligned}$$

**Proof**. If $\tilde{f}$ is a mapping defined by $\tilde{f}(x) = f(x) - f(0)$, then the mapping $\tilde{f}$ satisfies the properties $D\tilde{f}(x,y) = Df(x,y)$ and $\tilde{f}(0) = 0$. By (2.1) and the definitions of $\Gamma f$ and $\Delta f$, we have

$$\|\Gamma\tilde{f}(x)\| \le \Phi'(x) \quad \text{and} \quad \|\Delta\tilde{f}(x)\| \le \Phi(x),$$

for all $x \in V$. Hence, from (2.3) and (2.5), we have

$$\begin{aligned}&\|J_n\tilde{f}(x) - J_{n+1}\tilde{f}(x)\|\\ &\quad\le \left(\frac{4^n}{45} - \frac{20\cdot 16^n}{45} + \frac{64\cdot 64^n}{45}\right)\Phi\left(\frac{x}{2^{n+3}}\right)\\ &\quad+\left(\frac{2^n}{45} - \frac{20\cdot 8^n}{45} + \frac{64\cdot 32^n}{45}\right)\Phi'\left(\frac{x}{2^{n+3}}\right)\end{aligned}$$

for all $x \in V$. So we obtain

(2.8)
$$\|J_n\tilde{f}(x) - J_{n+m}\tilde{f}(x)\| \le \sum_{i=n}^{n+m-1}\left[\left(\frac{4^i}{45} - \frac{20\cdot 16^i}{45} + \frac{64\cdot 64^i}{45}\right)\Phi\left(\frac{x}{2^{i+3}}\right) + \left(\frac{2^i}{45} - \frac{20\cdot 8^i}{45} + \frac{64\cdot 32^i}{45}\right)\Phi'\left(\frac{x}{2^{i+3}}\right)\right]$$

for all $x \in V$ and $n, m \in \mathbb{N}\cup\{0\}$. It follows from (2.5) and (2.8) that the sequence $\{J_n\tilde{f}(x)\}$ is a Cauchy sequence for all $x \in V$. Since $Y$ is complete, the sequence $\{J_n\tilde{f}(x)\}$ converges for all $x \in V$. Hence we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n\to\infty} J_n\tilde{f}(x)$$

for all $x \in V$. Notice that $J_0\tilde{f}(x) = f(x) - f(0)$ for all $x \in V$ and $F(0) = 0$ follows from $\tilde{f}(0) = 0$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.8) we get the inequality (2.7). From the definition of $F$, we easily get

$$\begin{aligned}
\|DF(x,y)\| &= \lim_{n\to\infty}\left\|DJ_n\tilde{f}(x,y)\right\| \\
&\le \lim_{n\to\infty}\left(\frac{64^{n+1}}{45}\varphi_e\left(\frac{x}{2^n},\frac{y}{2^n}\right) + \frac{20\cdot 64^{n+1}}{45}\varphi_e\left(\frac{x}{2^{n+1}},\frac{y}{2^{n+1}}\right)\right. \\
&\quad + \frac{64^{n+1}}{45}\varphi_e\left(\frac{x}{2^{n+2}},\frac{y}{2^{n+2}}\right) + \frac{64\cdot 32^n}{45}\varphi_e\left(\frac{x}{2^n},\frac{y}{2^n}\right) \\
&\quad \left. + \frac{640\cdot 32^n}{45}\varphi_e\left(\frac{x}{2^{n+1}},\frac{y}{2^{n+1}}\right) + \frac{1024\cdot 32^n}{45}\varphi_e\left(\frac{x}{2^{n+2}},\frac{y}{2^{n+2}}\right)\right) = 0,
\end{aligned}$$

for all $x, y \in V$. To prove the uniqueness of $F$, let $F' : V \to Y$ be another general sextic mapping satisfying (2.7) and $F'(0) = 0$. Instead of the condition (2.7), it is sufficient to show that there is a unique mapping satisfying the simpler condition

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty}\left(64^{i+1}\Phi\left(\frac{x}{2^{i+3}}\right) + 64^{i+1}\Phi'\left(\frac{x}{2^{i+3}}\right)\right),$$

for all $x \in V$. By Lemma 2.2, the equality $F'(x) = J_n F'(x)$ holds for all $n \in \mathbb{N}$. So we have

$$\begin{aligned}
&\|J_n\tilde{f}(x) - F'(x)\| = \|J_n\tilde{f}(x) - J_nF'(x)\| \\
&\le 64^{n+1}\left\|\tilde{f}_e\left(\frac{x}{2^n}\right) - F'_e\left(\frac{x}{2^n}\right)\right\| + 64^{n+1}\left\|\tilde{f}_e\left(\frac{x}{2^{n+1}}\right) - F'_e\left(\frac{x}{2^{n+1}}\right)\right\| \\
&+ 64^{n+2}\left\|\tilde{f}_e\left(\frac{x}{2^{n+2}}\right) - F'_e\left(\frac{x}{2^{n+2}}\right)\right\| + 64^{n+1}\left\|\tilde{f}_o\left(\frac{x}{2^n}\right) - F'_o\left(\frac{x}{2^n}\right)\right\| \\
&+ 64^{n+1}\left\|\tilde{f}_o\left(\frac{x}{2^{n+1}}\right) - F'_o\left(\frac{x}{2^{n+1}}\right)\right\| + 64^{n+1}\left\|\tilde{f}_o\left(\frac{x}{2^{n+2}}\right) - F'_o\left(\frac{x}{2^{n+2}}\right)\right\| \\
\le& \sum_{i=n+3}^{\infty} 64^i \Phi\left(\frac{x}{2^i}\right) + \sum_{i=n+3}^{\infty} 64^i \Phi'\left(\frac{x}{2^i}\right),
\end{aligned}$$

for all $x \in V$ and all positive integer $n$. Taking the limit in the above inequality as $n \to \infty$, we obtain the equality $F'(x) = \lim_{n\to\infty} J_n\tilde{f}(x)$ for all $x \in V$, which means that $F(x) = F'(x)$ for all $x \in V$. $\square$

THEOREM 2.4. *Let $\varphi : V^2 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty \tag{2.9}$$

*for all $x, y \in V$. Suppose that $f : V \to Y$ is a mapping such that*

$$\|Df(x, y)\| \le \varphi(x, y) \tag{2.10}$$

*for all $x, y \in V$. Then there exists a general sextic mapping $F$ such that*

$$\begin{aligned}
\|f(x) - f(0) - F(x)\| \le \sum_{n=0}^{\infty}\Bigg[&\left(\frac{4}{4^{n+1}} - \frac{5}{16^{n+1}} + \frac{1}{64^{n+1}}\right)\frac{\Phi(2^n x)}{2880} \\
&+ \left(\frac{4}{2^{n+1}} - \frac{5}{8^{n+1}} + \frac{1}{32^{n+1}}\right)\frac{\Phi'(2^n x)}{720}\Bigg],
\end{aligned} \tag{2.11}$$

*for all $x \in V$ and $F(0) = 0$, where $\varphi_e, \Phi, \Phi'$ are functions defined as Theorem 2.3.*

**Proof**. If $\tilde{f}$ is a mapping defined by $\tilde{f}(x) = f(x) - f(0)$, then the mapping $\tilde{f}$ satisfies the properties $D\tilde{f}(x, y) = Df(x, y)$ and $\tilde{f}(0) = 0$. By the definitions of $\Gamma f$ and $\Delta f$, we have

$$\|\Gamma\tilde{f}(x)\| \le \Phi'(x) \quad \text{and} \quad \|\Delta\tilde{f}(x)\| \le \Phi(x),$$

for all $x \in V$. So it follows from (2.4) and (2.10) that

$$\begin{aligned}\|J'_n\tilde{f}(x) - J'_{n+1}\tilde{f}(x)\| \le& \left(\frac{4}{4^{n+1}} - \frac{5}{16^{n+1}} + \frac{1}{64^{n+1}}\right)\left\|\frac{\Delta f(2^n x)}{2880}\right\| \\ &+ \left(\frac{4}{2^{n+1}} - \frac{5}{8^{n+1}} + \frac{1}{32^{n+1}}\right)\left\|\frac{\Gamma f(2^n x)}{720}\right\| \\ \le& \left(\frac{4}{4^{n+1}} - \frac{5}{16^{n+1}} + \frac{1}{64^{n+1}}\right)\frac{\Phi(2^n x)}{2880} \\ &+ \left(\frac{4}{2^{n+1}} - \frac{5}{8^{n+1}} + \frac{1}{32^{n+1}}\right)\frac{\Phi'(2^n x)}{720}\end{aligned}$$

for all $x \in V$. Together with the equality

$$J'_n\tilde{f}(x) - J'_{n+m}\tilde{f}(x) = \sum_{i=n}^{n+m-1} (J'_i\tilde{f}(x) - J'_{i+1}\tilde{f}(x))$$

for all $x \in V$, we obtain that

(2.12)

$$\begin{aligned}\|J'_n\tilde{f}(x) - J'_{n+m}\tilde{f}(x)\| \le \sum_{i=n}^{n+m-1} \Bigg[&\left(\frac{4}{4^{i+1}} - \frac{5}{16^{i+1}} + \frac{1}{64^{i+1}}\right)\frac{\Phi(2^i x)}{2880} \\ &+ \left(\frac{4}{2^{n+1}} - \frac{5}{8^{i+1}} + \frac{1}{32^{i+1}}\right)\frac{\Phi'(2^i x)}{720}\Bigg]\end{aligned}$$

for all $x \in V$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from (2.9) and (2.12) that the sequence $\{J'_n\tilde{f}(x)\}$ is a Cauchy sequence for all $x \in V$. Since $Y$ is complete, the sequence $\{J'_n\tilde{f}(x)\}$ converges for all $x \in V$. Hence we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n\to\infty} J'_n\tilde{f}(x)$$

for all $x \in V$. Note that $F(0) = 0$ follows from $\tilde{f}(0) = 0$. Notice that $J'_0\tilde{f}(x) = f(x) - f(0)$ for all $x \in V$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (2.12) we get the inequality (2.11). From the definition of $F$, we easily get

$$
\begin{aligned}
\|DF(x,y)\| &= \lim_{n\to\infty} \left\|DJ'_n f(x,y)\right\| \\
&\le \lim_{n\to\infty} \bigg( \frac{2\varphi_e(2^n x, 2^n y) + \varphi_e(2^{n+1}x, 2^{n+1}y) + +\varphi_e(2^{n+2}x, 2^{n+2}y)}{4^n} \\
&\qquad + \frac{2\varphi_e(2^n x, 2^n y) + \varphi_e(2^{n+1}x, 2^{n+1}y) + +\varphi_e(2^{n+2}x, 2^{n+2}y)}{2^n} \bigg) \\
&= 0
\end{aligned}
$$

for all $x, y \in V$. To prove the uniqueness of $F$, let $F' : V \to Y$ be another general sextic mapping satisfying (2.11) and $F'(0) = 0$. Instead of the condition (2.11), it is sufficient to show that there is a unique mapping satisfying the simpler condition

$$
\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{\Phi(2^i x) + \Phi'(2^i x)}{2^i}
$$

for all $x \in V$. By Lemma 2.2, the equality $F'(x) = J'_n F'(x)$ holds for all $n \in \mathbb{N}$. So we have

$$
\begin{aligned}
&\|J'_n \tilde{f}(x) - F'(x)\| = \|J'_n \tilde{f}(x) - J'_n F'(x)\| \\
&\quad \le \frac{2}{4^n}\left\|(\tilde{f}_e - F'_e)(2^n x)\right\| + \frac{1}{4^{n+1}}\left\|(\tilde{f}_e - F'_e)(2^{n+1} x)\right\| + \frac{1}{4^{n+2}}\left\|(\tilde{f}_e - F'_e)(2^{n+2} x)\right\| \\
&\quad + \frac{2}{2^n}\left\|(\tilde{f}_o - F'_o)(2^n x)\right\| + \frac{1}{2^{n+2}}\left\|(\tilde{f}_o - F'_o)(2^{n+1} x)\right\| + \frac{1}{2^{n+2}}\left\|(\tilde{f}_o - F'_o)(2^{n+2} x)\right\| \\
&\quad \le 4\sum_{i=0}^{\infty} \frac{\Phi(2^{n+i}x) + \Phi'(2^{n+i}x)}{2^{n+i}} + 2\sum_{i=0}^{\infty} \frac{\Phi(2^{n+i+1}x) + \Phi'(2^{n+i+1}x)}{2^{n+i+1}} \\
&\quad + 2\sum_{i=0}^{\infty} \frac{\Phi(2^{n+i+2}x) + \Phi'(2^{n+i+2}x)}{2^{n+i+2}} \\
&\quad \le 8\sum_{i=n}^{\infty} \frac{\Phi(2^i x) + \Phi'(2^i x)}{2^i}
\end{aligned}
$$

for all $x \in V$ and all positive integer $n$. Taking the limit in the above inequality as $n \to \infty$, we obtain the equality $F'(x) = \lim_{n\to\infty} J'_n \tilde{f}(x)$ for all $x \in V$, which means that $F(x) = F'(x)$ for all $x \in V$. □

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