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APPROXIMATE CONTROLLABILITY FOR SEMILINEAR INTEGRO-DIFFERENTIAL CONTROL EQUATIONS WITH QUASI-HOMOGENEOUS PROPERTIES

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Abstract. In this paper, we consider the approximate controllability for a class of semilinear integro-differential functional control equations in which nonlinear terms of given equations satisfy quasi-homogeneous properties. The main method used is to make use of the surjective theorems that is similar to Fredholm alternative in the nonlinear case under restrictive assumptions. The sufficient conditions for the approximate controllability is obtain which is different from previous results on the system operator, controller and nonlinear terms. Finally, a simple example to which our main result can be applied is given.

1. Introduction

In this paper, we deal with the approximate controllability for semilinear integro-differential functional control equations in the form
(1.1)
$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t) + \int_0^t k(t-s)g(s,x(s))ds + (Bu)(t), \quad 0 < t \le T, \\ x(0) &= x_0, \end{cases}$$

where

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))ds$$

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for a k belonging to $L^2(0,T)$. Let V and H be complex Hilbert spaces forming a Gelfand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

by identifying the antidual of H with H. Here, the principal operator A generates an analytic semigroup $(S(t))_{t\geq 0}$ in both H and V^* and B is a bounded linear operator from another Hilbert space $L^2(0,T;U)(T>0)$ to $L^2(0,T;U)$. k belongs to $L^2(0,T)$ and g is a nonlinear mapping as detailed in Section 2.

The controllability problem is a question of whether is possible to steer a dynamic system from an initial state to an arbitrary final state using the set of admissible controls. There are two main ways to deal with the approximate controllability for semilinear control equations, the first is to use the range condition argument of controller as seen in [27, 28]. The approximate controllability of semilinear systems dominated by linear parts(in case $g \equiv 0$) as matters connected with [19] has been studied by assuming that $S(t)$ is compact operator in [6, 10, 13, 20, 26]. Another approach is to use a fixed point theorem combined with technique of operator transformations by configuring the resolvent as seen in [3]. Recently, the approximate controllability of stochastic equations have been studied by authors [4, 21] as a continuous study. Similar considerations of semilinear stochastic systems have been dealt with in many references [2, 8, 15, 16, 17, 18, 22]. Sukavanam and Tomar [23] studied the approximate controllability for the general retarded initial value problem by assuming that the Lipschitz constant of the nonlinear term is less then one. In particular, Wang [26] established the approximate controllability for the equation (1.1) with conditions the range condition of controller provided

$$\limsup_{||u||_{L^2(0,T;H)}\to\infty} \frac{||f(\cdot,u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;H)}} := \gamma \tag{1.2}$$

is sufficiently small. Moreover, [12] dealt with the approximate controllability for the system (1.1) even if $\gamma \neq 1$ of (1.2) by using so called Fredholm theory: $(\lambda I - F)(u) = f$ is solvable in $L^2(0,T;H)$.

In this paper, authors want to use a different method than the previous one. Our used tool is the surjective theorems similar to the Fredholm alternative for nonlinear operators under restrictive assumption, which is on the solution of nonlinear operator equations $\lambda B(x) - F(x) = y$ in dependence on the real number λ, where B is a controller operator and F is a nonlinear operator. In order to obtain the approximate controllability for a class of semilinear integro-differential functional control equations, it is necessary to suppose that B acts as an odd homeomorphism operator while F related to the nonlinear term of (1.1) is quasi-homogeneous as defined in Sect. 3. By using this method, the approximate controllability of (1.1) without restrictions such as the inequality constraints for Lipschitz constant of f or compactness of $S(t)$ can be given as applicable conditions.

Sect. 2 is devoted to constructing a variation of constant formula of L^2-regularity and properties of the strict solutions of (1.1)(see [7] in the linear case). In Sect. 3, in order to apply the surjective theory in the proof of the main theorem, we assume some compactness of the embedding between intermediate spaces. Then by virtue of [1], we can show that the solution mapping of a control space to the terminal state space is completely continuous by means of regularities results. Moreover, the sufficient conditions on the controller and nonlinear terms for approximate controllability for (1.1) can be obtained while the nonlinear term g of (1.1) is odd quasi-homogeneous. Finally, a simple example to which our main result can be applied is given.

2. Semilinear functional equations

The notations $|\cdot|$, $||\cdot||$ and $||\cdot||_*$ denote the norms of H, V and V^*, respectively as usual. Therefore, for the brevity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad \forall u \in V.$$

Let $a(u, v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\text{Re } a(u,u) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0.$$

Let A be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u, v), \quad u,\ v \in V.$$

Then A is a bounded linear operator from V to V^*. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. Moreover, for each $T > 0$, by using interpolation theory, we have

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

From the following inequalities

$$c_0||u||^2 \le \operatorname{Re} a(u,u) + c_1|u|^2 \le |Au|\,|u| + c_1|u|^2 \le (|Au| + c_1|u|)|u|,$$

it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \le C_0||u||_{D(A)}^{1/2}|u|^{1/2}.$$

Therefore, in terms of the intermediate theory, we can see that

$$(D(A),H)_{1/2,2} = V, \text{ and } (V,V^*)_{1/2,2} = H,$$

where $(V,V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [5], [25]). For the sake of simplicity we assume that $c_1 = 0$ and hence the closed half plane $\{\lambda : \operatorname{Re}\lambda \ge 0\}$ is contained in the resolvent set of A. It is known that A generates an analytic semigroup $S(t)$ in both H and V^*. As seen in Lemma 3.6.2 of [24], there exists a constant $M > 0$ such that

$$|S(t)x| \le M|x| \quad \text{and} \quad ||S(t)x||_* \le M||x||_*, \tag{2.1}$$

moreover, for all $t > 0$ and every $x \in H$ or V^*:

$$|S(t)x| \le Mt^{-1/2}||x||_*, ||S(t)x|| \le Mt^{-1/2}|x|.$$

The following initial value problem for the abstract linear parabolic equation

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + k(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{2.2}$$

By virtue of Theorem 3.3 of [7](or Theorem 3.1 of [10]), we have the following result on the corresponding linear equation (2.2).

Proposition 2.1. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*
1) Noting that $V = (D(A),H)_{1/2,2}$, for $x_0 \in V$ and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (2.2) belonging to

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V)$$

and satisfying

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.3}$$

where C_1 is a constant depending on T.
2) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then, noting that

$(V,V^*)_{1/2,2} = H$, there exists a unique solution x of (2.2) belonging to

$$L^2(0,T;V)\cap W^{1,2}(0,T;V^*)\subset C([0,T];H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)}\le C_1(|x_0|+||k||_{L^2(0,T;V^*)}), \tag{2.4}$$

where C_1 is a constant depending on T.

LEMMA 2.2. *Suppose that $k\in L^2(0,T;H)$ and $x(t)=\int_0^t S(t-s)k(s)ds$ for $0\le t\le T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;H)}\le C_2T||k||_{L^2(0,T;H)}, \tag{2.5}$$

and

$$||x||_{L^2(0,T;V)}\le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.6}$$

Proof. By a consequence of (2.3), it is immediate that

$$||x||_{L^2(0,T;D(A))}\le C_1||k||_{L^2(0,T;H)}, \tag{2.7}$$

Since

$$\begin{aligned}||x||^2_{L^2(0,T;H)} &= \int_0^T|\int_0^t S(t-s)k(s)ds|^2dt\le M\int_0^T(\int_0^t|k(s)|ds)^2dt\\ &\le M\int_0^T t\int_0^t|k(s)|^2dsdt\le M\frac{T^2}{2}\int_0^T|k(s)|^2ds,\end{aligned}$$

where M is the constant of (2.1), it follows that

$$||x||_{L^2(0,T;H)}\le T\sqrt{M/2}||k||_{L^2(0,T;H)}. \tag{2.8}$$

From (2.7), and (2.8) it holds that

$$||x||_{L^2(0,T;V)}\le C_0\sqrt{C_1T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2>0$ such that

$$C_2=\max\{\sqrt{M/2},C_0\sqrt{C_1}(M/2)^{1/4}\},$$

Thus, (2.5) and (2.6) are satisfied. □

Consider the following initial value problem for the abstract semilinear parabolic equation

$$\begin{cases}\frac{d}{dt}x(t) &= Ax(t)+\int_0^t k(t-s)g(s,x(s))ds+(Bu)(t),\\ x(0) &= x_0.\end{cases} \tag{2.9}$$

Let U be a Hilbert space and the controller operator B be a bounded linear operator from $L^2(0,T;U)$ to $L^2(0,T;H)$.

Let $g : \mathbb{R}^+ \times V \to H$ be a nonlinear mapping satisfying the following:

Assumption (F).
(i) For any $x \in V$, the mapping $g(\cdot, x)$ is strongly measurable;
(ii) There exist positive constants L_0, L_1 such that
(a) $x \mapsto g(t, x)$ is odd mapping $(g(\cdot, -x) = -g(\cdot, x))$;
(b) for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$,

$$|g(t,x) - g(t,\hat{x})| \le L_1 ||x - \hat{x}||,$$
$$|g(t,0)| \le L_0.$$

For $x \in L^2(0,T;V)$, we set

$$f(t,x) = \int_0^t k(t-s) g(s, x(s)) ds$$

where k belongs to $L^2(0,T)$.

Lemma 2.3. *Let Assumption (F) be satisfied. Assume that $x \in L^2(0,T;V)$ for any $T > 0$. Then $f(\cdot, x) \in L^2(0,T;H)$ and*
(2.10)
$$||f(\cdot,x)||_{L^2(0,T;H)} \le L_0 ||k||_{L^2(0,T)} T/\sqrt{2} + ||k||_{L^2(0,T)} \sqrt{T} L_1 ||x||_{L^2(0,T;V)}.$$
Moreover if $x,\ \hat{x} \in L^2(0,T;V)$, then

$$||f(\cdot,x) - f(\cdot,\hat{x})||_{L^2(0,T;H)} \le ||k||_{L^2(0,T)} \sqrt{T} L_1 ||x - \hat{x}||_{L^2(0,T;V)}. \tag{2.11}$$

Proof. From Assumption (F) and using the Hölder inequality, it is easily seen that

$$\begin{aligned}
&||f(\cdot,x)||_{L^2(0,T;H)} \le ||f(\cdot,0)||_{L^2(0,T;H)} + ||f(\cdot,x) - f(\cdot,0)||_{L^2(0,T;H)} \\
&\le \Big(\int_0^T | \int_0^t k(t-s) g(s,0) ds|^2 dt \Big)^{1/2} \\
&\quad + \Big(\int_0^T | \int_0^t k(t-s) \{ g(s,x(s)) - g(s,0) \} ds|^2 dt \Big)^{1/2} \\
&\le L_0 ||k||_{L^2(0,T)} T/\sqrt{2} + ||k||_{L^2(0,T)} \sqrt{T} ||g(\cdot,x) - g(\cdot,0)||_{L^2(0,T;H)} \\
&\le L_0 ||k||_{L^2(0,T)} T/\sqrt{2} + ||k||_{L^2(0,T)} \sqrt{T} L_1 ||x||_{L^2(0,T;V)}.
\end{aligned}$$

The proof of (2.11) is similar. □

By virtue of Theorem 2.1 of [11], we have the following result on (2.9).

Proposition 2.4. *Let Assumption (F) be satisfied. Then there exists a unique solution x of (2.9) such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

for any $x_0 \in H$. Moreover, there exists a constant C_3 such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_3(|x_0| + ||u||_{L^2(0,T;U)}). \tag{2.12}$$

Corollary 2.5. *Assume that the embedding $D(A) \subset V$ is completely continuous. Let Assumption (F) be satisfied, and x_u be the solution of equation (2.9) associated with $u \in L^2(0,T;U)$. Then the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$.*

Proof. If $u \in L^2(0,T;U)$, then in view of (2.4) in Proposition 2.1

$$||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_3(|x_0| + ||B||\,||u||_{L^2(0,T;U)}). \tag{2.13}$$

Since $x_u \in L^2(0,T;V)$, we have $f(\cdot, x_u) \in L^2(0,T;H)$. Consequently

$$x_u \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H).$$

Hence, with aid of (2.3) of Proposition 2.1, (2.10) and (2.12),

$$\begin{aligned}
&||x_u||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(||x_0|| + ||f(\cdot,x_u) + Bu||_{L^2(0,T;H)})\\
&\le C_1\big\{||x_0|| + L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}L_1||x||_{L^2(0,T;V)} + ||Bu||_{L^2(0,T;H)}\big\}\\
&\le C_1\big\{||x_0|| + L_0||k||_{L^2(0,T)}T/\sqrt{2}\\
&\quad + ||k||_{L^2(0,T)}\sqrt{T}L_1C_3\big(|x_0| + ||u||_{L^2(0,T;U)}\big) + ||Bu||_{L^2(0,T;H)}\big\}.
\end{aligned}$$

Thus, if u is bounded in $L^2(0,T;H)$, then so is x_u in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$. Since $D(A)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;V) \subset L^2(0,T;V)$$

is completely continuous in view of Theorem 2 of [1], the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$. □

3. Approximate controllability

Throughout this section, we assume that $D(A)$ is compactly embedded in V. Let $x(T;f,u)$ be a state value of the system (2.9) at time T

corresponding to the nonlinear term f and the control u. We define the reachable sets for the system (2.9) as follows:

$$R_T(f) = \{x(T; f, u) : u \in L^2(0, T; U)\},$$
$$R_T(0) = \{x(T; 0, u) : u \in L^2(0, T; U)\}.$$

Definition 3.1. The system (2.9) is said to be approximately controllable in the time interval $[0, T]$ if for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0, T; U)$ such that the solution $x(T; f, u)$ of (2.9) satisfies $|x(T; f, u) - x_1| < \epsilon$, that is, if $\overline{R_T(f)} = H$ where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H, then the system (2.9) is called approximately controllable at time T.

Let us introduce the theory of the degree for completely continuous perturbations of the identity operator, which is the infinite dimensional version of Borsuk's theorem. Let $0 \in D$ be a bounded open set in a Banach space X, $\overline{D}$ its closure and ∂D its boundary. The number $d[I - T; D, 0]$ is the degree of the mapping $I - T$ with respect to the set D and the point 0 (see [9] or [14]).

Theorem 3.2. *(Borsuk's theorem) Let D be a bounded open symmetric set in a Banach space X, $0 \in D$. Suppose that $T : \overline{D} \to X$ be odd completely continuous operator satisfying $T(x) \neq x$ for $x \in \partial D$. Then $d[I - T; D, 0]$ is odd integer. That is, there exists at least one point $x_0 \in D$ such that $(I - T)(x_0) = 0$.*

Definition 3.3. Let T be a mapping defined by on a Banach space X with value in a real Banach space Y. The mapping T is said to be a (K, L, α)-homeomorphism of X onto Y if

(i) T is a homeomorphism of X onto Y;
(ii) there exist real numbers $K > 0$, $L > 0$, and $\alpha > 0$ such that

$$L||x||_X^\alpha \leq ||T(x)||_Y \leq K||x||_X^\alpha, \quad \forall x \in X.$$

Lemma 3.4. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ a continuous operator satisfying*

$$\limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+.$$

Then if $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}] \cup \{0\}$ then

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty.$$

Proof. Suppose that there exist a constant $M > 0$ and a sequence $\{x_n\} \subset X$ such that

$$||\lambda T(x_n) - F(x_n)||_Y \leq M$$

as $x_n \to \infty$. From this it follows that

$$\frac{\lambda T(x_n)}{||x_n||_X^\alpha} - \frac{F(x_n)}{||x_n||_X^\alpha} \to 0.$$

Hence, we have

$$\limsup_{n\to\infty} \frac{|\lambda| ||T(x_n)||_Y}{||x_n||_X^\alpha} = N,$$

and so, $|\lambda|K \geq N \geq |\lambda|L$. It is a contradiction with $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}]$. □

PROPOSITION 3.5. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ an odd completely continuous operator. Suppose that for $\lambda \neq 0$,*

$$\lim_{||x||_X\to\infty} ||\lambda T(x) - F(x)||_Y = \infty. \tag{3.1}$$

Then $\lambda T - F$ maps X onto Y.

Proof. We follow the proof Theorem 1.1 in Chapter II of [9]. Suppose that there exists $y \in Y$ such that $\lambda T(x) = y$. Then from (3.1) it follows that $FT^{-1} : Y \to Y$ is an odd completely continuous operator and

$$\lim_{||y||_Y\to\infty} ||y - FT^{-1}(\frac{y}{\lambda})||_Y = \infty.$$

Let $y_0 \in Y$. There exists $r > 0$ such that

$$||y - FT^{-1}(\frac{y}{\lambda})||_Y > ||y_0||_Y \geq 0$$

for each $y \in Y$ satisfying $||y||_Y = r$. Let $Y_r = \{y \in Y : ||y||_Y < r\}$ be a open ball. Then by view of Theorem 3.1, we have $d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0]$ is an odd number. For each $y \in Y$ satisfying $||y||_Y = r$ and $t \in [0, 1]$, there is

$$||y - FT^{-1}(\frac{y}{\lambda}) - ty_0||_Y \geq ||y - FT^{-1}(\frac{y}{\lambda})||_Y - ||y_0||_Y > 0$$

and hence, by the homotopic property of degree, we have

$$d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, y_0] = d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0] \neq 0.$$

Hence, by the existence theory of the Leray-Schauder degree, there exists a $y_1 \in Y_r$ such that

$$y_1 - FT^{-1}(\frac{y_1}{\lambda}) = y_0.$$

We can choose $x_0 \in X$ satisfying $\lambda T(x_0) = y_1$, and so, $\lambda T(x_0) - F(x_0) = y_0$. Thus, it implies that $\lambda T - F$ is a mapping of X onto Y. $\square$

Combining Lemma 3.4. and Proposition 3.5, we have the following results.

Corollary 3.6. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ an odd completely continuous operator satisfying*

$$\limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+.$$

Then if $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}] \cup \{0\}$ then $\lambda T - F$ maps X onto Y. Therefore, if $N = 0$, then for all $\lambda \neq 0$ the operator $\lambda T - F$ maps X onto Y.

Let X be a Banach space with the norm $\| \cdot \|_X$. Denote by X^* the adjoint space of all bounded linear functionals on X. The pairing between $x^* \in X^*$ and $x \in X$ is denoted by (x^*, x). Unless otherwise stated, we use symbols " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence, respectively, i.e., the sequence $\{x_n\}$, $x_n \in X$ converges strongly (weakly) to the point $x_0 \in X$, denote by $x_n \to x_0$ ($x_n \rightharpoonup x_0$), if

$$\lim_{n\to\infty} \|x_n - x_0\|_X = 0 \ (\lim_{n\to\infty} (x^*, x_n) = (x^*, x_0) \quad \text{for each} \quad x* \in X^*).$$

Let F be mapping (nonlinear, in general) with the domain $M \subset X$ and the range in the Banach space Y. F is said to strongly (weakly) continuous on M if $x_n \to x_0$ ($x_n \rightharpoonup x_0$) in X implies $F(x_n) \to F(x_0)$ in Y for $x_n, \ x_0 \in M$, and F is said to be completely continuous on M if F is continuous on M and for each bounded subset $D \subset M$, $F(D)$ is compact subset in Y.

Definition 3.7. Let F be a mapping defined by on a Banach space X with value in a real Banach space Y and $b > 0$ a real number.

(a) F is said to be b-homogeneous if

$$t^b F(u) = F(tu)$$

holds for each $t \geq 0$ and all $u \in X$.

(b) F is said to be b-quasi-homogeneous if there exist nonlinear operators R and F_0 defined on X with value in Y such that F_0 is b-homogeneous and $F(u) = R(u)F_0(u)$ for every $u \in X$ satisfying

$$\lim_{||u||_X \to \infty} ||R(u)||_Y \in \mathbb{R}^+.$$

EXAMPLE 3.8. *Set $X = Y = \mathbb{R}$ and*

$$F(u) = \frac{u}{1+|u|}u^3.$$

Then F is said to be 3-quasi-homogeneous.

REMARK 3.9. In [9], the relationship between F and F_0 is defined in other words as F is said to be b-strongly quasi-homogeneous with respect to F_0, if

$$t_n > 0 \to 0, u_n \rightharpoonup u_0 \Rightarrow t_n^b F(u_n/t_n) \to F_0(u_0) \in Y.$$

If F is the strong continuous and b-quasi-homogeneous, then F is a b-strongly quasi-homogeneous with respect to F_0, So, our basic results follow theorems of [9].

THEOREM 3.10. *Let X be a reflexive space, and let T be odd (K, L, a)-homeomorphism of X onto Y, $F : X \to Y$ an odd strong continuous and b-quasi-homogeneous operator. If $a > b$, then $\lambda T - F$ maps X onto Y for any $\lambda \neq 0$.*

Proof. Since X is a reflexive space, we know that every strong continuous operator $F : X \to Y$ is also completely continuous. Hence according to Corollary 3.6 it is sufficient to prove that

$$\lim_{x\to\infty} \frac{\|F(x)\|_Y}{\|x\|_X^a} = 0.$$

Since F is b-quasi-homogeneous, there exist R and F_0 be a mappings defined by on a Banach space X with value in $\mathbb{R}^+$ and a real Banach space Y, respectively, such that $F = RF_0$ satisfying

$$\lim_{|u|\to\infty} R(u) = c_0$$

for some a constant $c_0 > 0$ holds and F_0 is b-homogeneous. Suppose that there exist $\epsilon > 0$ and a sequence $\{x_n\}$, $x_n \in X$, $\|x_n\|_X \to \infty$ such that

$$\frac{x_n}{\|x_n\|_X} = v_n \rightharpoonup v_0$$

and

$$\frac{\|F(x_n)\|_Y}{\|x\|_X^a} \geq \epsilon$$

for any positive integer n. Then

$$\frac{F(\|x_n\|v_n)}{\|x_n\|_X^b} = R(\|x_n\|v_n)F_0(v_n) \to c_0F_0(v_0),$$

and

$$\frac{\|x_n\|_X^b}{\|x_n\|_X^a} \to 0$$

implies

$$0 < \epsilon \le \frac{\|F(x_n)\|}{\|x_n\|^a} = \frac{\|x_n\|^b}{\|x_n\|^a} \cdot \frac{\|F(x_n)\|}{\|x_n\|^b} \to 0,$$

which is a contradiction. □

THEOREM 3.11. *Let X be a reflexive space, and let T be odd (K, L, a)-homeomorphism of X onto Y, $F : X \to Y$ an odd strongly continuous and b-quasi-homogeneous operator. If $F_0(v) = 0$ imply $v = 0$, and $a < b$, then $\lambda T - F$ maps X onto Y for any $\lambda \neq 0$.*

Proof. According to Proposition 3.5, we shall prove

$$\lim_{x\to\infty} \|\lambda T(x) - F(x)\|_Y = \infty.$$

Since F is b-quasi-homogeneous, there exist R and F_0 be mappings defined by on a Banach space X with value in $\mathbb{R}^+$ and a real Banach space Y, respectively, such that $F = RF_0$ satisfying

$$\lim_{|u|\to\infty} R(u) = c_0$$

for some a constant $c_0 > 0$ holds and F_0 is b-homogeneous. Suppose that there exist a constant $M > 0$ and a sequence $\{x_n\}$, $x_n \in X$, $\|x_n\|_X \to \infty$ such that

$$\frac{x_n}{\|x_n\|} = v_n \to v_0$$

and

$$\|\lambda T(x_n) - F(x_n)\|_Y \le M$$

for any positive integer n. Then

$$\frac{\lambda T(\|x_n\|\, v_n)}{\|x_n\|^b} - \frac{F\,\|x_n\|\, v_n}{\|x_n\|^b} \to 0,$$

and so

$$\frac{\lambda T(\|x_n\|\, v_n)}{\|x_n\|^b} \to c_0 F_0(v_0).$$

But since T is (K, L, a)- homeomorphism, we have

$$K|\lambda|\frac{\|x_n\|^a}{\|x_n\|^b} \ge \frac{\|\lambda T(x_n)\|}{\|x_n\|^b} \ge L|\lambda|\frac{\|x_n\|^a}{\|x_n\|^b}.$$

Thus, noting that $a < b$, it holds

$$\frac{\|\lambda T(x_n)\|}{\|x_n\|^b} \to 0,$$

and $F_0(v_0) = 0$. From our assumption $v_0 = 0$ and this is a contradiction with $\|v_0\|_X = 1$. □

Now, we consider the approximate controllability for the following semilinear control system

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + f(t, x(t)) + (Bv)(t), \\ x(0) & = x_0. \end{cases} \tag{3.2}$$

We shall make use of the following assumption:

Assumption (A) The embedding $D(A) \subset V$ is completely continuous.

By using the Krasnosel'skii theorem(see [2]), we can define an operator $F : L^2(0, T; U) \to L^2(0, T; H)$ as

$$F(v) = -f(\cdot, x_v). \tag{3.3}$$

Assumption (F1) F is b-quasi-homogeneous.

THEOREM 3.12. *Under Assumptions (A), (F) and (F1), if $1 > b$, then we have*

$$R_T(0) \subset \overline{R_T(f)}.$$

Therefore, if the linear system (3.2) with $f \equiv 0$ is approximately controllable, then so is the nonlinear system (3.2).

Proof. Thanks to Corollary 2.5, F defined by (3.3) is a completely continuous mapping from $L^2(0, T; U)$ to $L^2(0, T; H)$. We shall show that F is strongly continuous. Given a sequence $\{u_n\}$, $u_n \in L^2(0, T; U)$, $u_n \rightharpoonup u$, we claim that $F(u_n) \to F(u)$. By (2.11) and (2.12), we have

$$\begin{aligned} ||F(u_n) - F(u)||_{L^2(0,T;H)} &\le ||k||_{L^2(0,T)} \sqrt{T} L_1 ||x_{u_n} - x_u||_{L^2(0,T;V)} \\ &\le ||k||_{L^2(0,T)} \sqrt{T} L_1 C_3 ||u_n - u||_{L^2(0,T;U)}, \end{aligned}$$

and so, $F(u_n) \rightharpoonup F(u)$. Hence, to prove our claim it suffices to show that every subsequence of $\{F(u_n)\}$ contains another subsequence which converge. However, this is immediate because the sequence $\{u_n\}$ is bounded and F is a completely continuous.

Since $1 > b$ and the identity operator I on $L^2(0, T; H)$ is an odd $(1, 1, 1)$-homeomorphism, from Theorem 3.2, it follows that that $\lambda I - F$

maps $L^2(0,T;H)$ onto itself for any $\lambda \neq 0$. Let

$$\eta = \int_0^T S(T-s)(Bv)(s)ds \in R_T(0).$$

We are going to show that there exists w such that

$$\eta = x(T; g, w).$$

We denote the range of the operator B by H_B, its closure $\overline{H}_B$ in $L^2(0,T;H)$. Let $\overline{H}_B^{\perp}$ be the orthogonal complement of $\overline{H}_B$ in $L^2(0,T;H)$. Let $X = L^2(0,T;H)/\overline{H}_B^{\perp}$ be the quotient space and the norm of a coset $\tilde{y} = y_B + \overline{H}_B^{\perp} \in X$ is defined of $||\tilde{y}|| = ||y_B + \overline{H}_B^{\perp}|| = \inf\{|y_B + g| : y_B \in \overline{H}_B,\ \ g \in \overline{H}_B^{\perp}\}$. We define by Q the isometric isomorphism from X onto $\overline{H}_B$, that is, $Q\tilde{y} = Q(y_B + g : y_B \in \overline{H}_B, \quad g \in \overline{H}_B^{\perp}) = y_B$. Let

$$\mathcal{F}\tilde{y} = F(Q\tilde{y}) + \overline{H}_B^{\perp}$$

for $\tilde{y} \in X$. Then $\mathcal{F}$ is also a completely continuous mapping from X to itself.

Set $z = Bv$. Then $z \in \overline{H}_B$ and $\tilde{z} = z + \overline{H}_B^{\perp} \in X$. Hence, by Theorem 3.10 with $\lambda = 1$, there exists $\tilde{w} \in X$ such that

$$\tilde{z} = \tilde{w} - \mathcal{F}\tilde{w}. \tag{3.4}$$

Put $w_B = Q\tilde{w}$. Then we have that $w - w_B = \in \overline{H}_B^{\perp}$. Hence,

$$\tilde{z} = w - F(w_B) + \overline{H}_B^{\perp} = w_B - F(w_B) + \overline{H}_B^{\perp}. \tag{3.5}$$

Thus, from (3.4) and (3.5) it follows that

$$\begin{aligned} \eta &= \int_0^T S(T-s)(-\mathcal{F}(w_B)(s) + w_B(s))ds \\ &= \int_0^T S(T-s)(f(s, \hat{x}_{w_B}) + w_B(s))ds. \end{aligned}$$

Since $w_B \in \overline{H}_B$, there exists a sequence $\{v_n\} \in L^2(0,T;U)$ such that $Bv_n \mapsto w_B$ in $L^2(0,T;H)$.

Let y_f be the solution of the equation with $B = I$

$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t) + f(t, x(t)) + f(t), \\ x(0) &= x_0. \end{cases}$$

Then

$$y_f = S(T)x_0 + \int_0^T S(T-s)\{f(s, x(s)) + f(s)\}ds.$$

Here, we note $y_{Bv_n}f = x(T; g, v_n)$. Thus, for each $\epsilon > 0$, we can choose a control function v_N such that

$$||Bv_N - w_B||_{L^2(0,T;H)} < \left\{ \left(M\sqrt{T}\left(||k||_{L^2(0,T)}\sqrt{T}L_1C_3 + 1\right) \right\}^{-1} \epsilon,$$

where M is a constant in (2.1). Then we have

$$||y_{v_N} - \eta||_{L^2(0,T;H)} \leq \left(M\sqrt{T}\left(||k||_{L^2(0,T)}\sqrt{T}L_1C_3 + 1\right)||Bv_N - w_B|| < \epsilon.$$

Since ϵ is given arbitrary, we conclude $\eta \in \overline{R_T(f)}$. □

THEOREM 3.13. *Let Assumptions (A), (F) and (F1) hold. If* $F(v) = 0$ *imply* $v = 0$ *and* $1 \neq b$, *then we have*

$$R_T(0) \subset \overline{R_T(f)}.$$

Proof. If $1 > b$, it holds from Theorem 3.10. The case if $1 < b$ is obvious from Theorem 3.11. □

We need to impose following assumption:

Assumption (B). There exist positive constants β, γ such that

$$\beta\|u\| \leq |Bu| \leq \gamma\|u\|, \quad \forall u \in L^2(0,T;U).$$

THEOREM 3.14. *Under Assumptions (A), (F), (F1), and (B), if* $1 < b$ *then the semilinear control system (3.2) is approximately controllable.*

Proof. Since B is odd $(\gamma, \beta, 1)$- homeomorphism of $L^2(0,T;U)$ onto $L^2(0,T;H)$, $F : L^2(0,T;U) \to L^2(0,T;H)$ an odd strong continuous b-homogeneous operator. From Theorem 3.10, it follows that if $1 > b$ then $\lambda B - F$ maps $L^2(0,T;U)$ onto $L^2(0,T;H)$ for any $\lambda \neq 0$. Let $\xi \in D(A)$. Then there exists a function $p \in C^1(0,T;H)$ such that

$$\xi = \int_0^T S(T-s)p(s)ds,$$

for instance, put $p(s) = (\xi + sA\xi)/T$. Hence, there exists a function $u \in L^2(0,T;U)$ such that

$$p = (\lambda B - F)u,$$

that is,

$$\xi = \int_0^T S(T-s)\{f(s, x(s)) + (Bu)(s))ds.$$

Therefore, if $1 > b$, then $D(A) \subset R_T(f)$, which complete the proof. □

THEOREM 3.15. *Let Assumptions (A), (B), (F) and (F1) hold. If* $F(v) = 0$ *imply* $v = 0$ *and* $1 \neq b$, *then the semilinear control system (3.2) is approximately controllable.*

Proof. This theorem is obvious from Theorems 3.13 and 3.14. □

EXAMPLE 3.16. *We consider the semilinear heat equation dealt with by* [19] *and* [27]. *Let*

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi),$$
$$a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx$$

and

$$A = d^2/dx^2 \quad \text{with} \quad D(A) = \{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

We consider the following retarded functional differential equation

$$\frac{d}{dt}x(t) = Ax(t) + f(x(t)) + Bw(t), \tag{3.6}$$

where

$$f(x) = \frac{\sigma x}{1+|x|}x^3, \quad \sigma > 0.$$

For $x,\ y \in H$, set $\max\{|x(\xi)|, |x(\xi)|\}$ *for almost all $\xi \in (0,\pi)$. Then we have*

$$|f(x(\xi)) - f(y(\xi))| \leq 3\sigma m^3(1+m)^{-1}|x(\xi) - y(\xi)|$$

for almost all $\xi \in (0,\pi)$. It is easily seen that Assumption (F) is satisfied and f is 3-quasi-homogeneous.

The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $\phi_n(x) =$ $\sin nx$, *respectively. Let*

$$U = \{\sum_{n=2}^\infty u_n\phi_n : \sum_{n=2}^\infty u_n^2 < \infty\},$$
$$Bu = 2u_2\phi_1 + \sum_{n=2}^\infty u_n\phi_n, \quad \text{for} \quad u = \sum_{n=2}^\infty u_n \in U.$$

Now we can define bounded linear operator $\hat{B}$ from $L^2(0,T;U)$ to $L^2(0,T;H)$ by $(\hat{B}u) = Bu(t)$, $u \in L^2(0,T;U)$. It is easily known that the operator $\hat{B}$ is one to one and the range of $\hat{B}$ is closed. it follows that the operator satisfies Assumption (B). We can see many examples which satisfy Assumption (B) as seen in [27, 28].

The solution of the following equation

$$\frac{d}{dt}x(t) = Ax(t) + Bw(t)$$

with initial datum 0 *is*

$$x(t) = \int_0^t e^{(t-s)A} Bw(s)ds.$$

Let $\xi \in D(A)$ *and*

$$u(s) = B^{-1}(\xi + sA\xi)/T.$$

Then it follows that $x(T) = \xi$*, which says that the reachable set* $R_T(0)$ *for linear system is a dense subspace. Moreover, from Theorem 3.15 with* $\lambda = 1$*, it follows that the system of (3.6) is approximately controllable.*

4. Conclusion

The purpose of this paper is obtained some sufficient conditions for the approximate controllability of a class of semilinear integro-differential functional control equations in which nonlinear terms of given equations satisfy quasi-homogeneous properties. Our used tool is the surjective theorems similar to the Fredholm alternative for nonlinear operators under restrictive assumption, which is on the solution of nonlinear operator equations $\lambda I(x) - F(x) = y$ in dependence on the real number λ, where I is the identity operator and F is a nonlinear operator. To solve this problem, we prove that $\lambda T - F$ maps for any $\lambda \neq 0$ provided that T is an odd (K, L, a)- homeomorphism, F an odd strongly continuous and b-quasi-homogeneous operator satisfying a¿b. Motivated by this consideration, we derive the approximate controllability of semilinear systems provided the approximate controllability of the corresponding linear systems considering T as the identity function. In the finite dimensional case we prove the same assertion under the assumption $a < b$, but it seems to be unsolved up to this time in infinite dimensional space.

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