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SOME CLASSES OF INTEGRAL EQUATIONS OF CONVOLUTIONS-PAIR GENERATED BY THE KONTOROVICH-LEBEDEV, LAPLACE AND FOURIER TRANSFORMS

Trinh Tuan

Abstract. In this article, we prove the existence of a solution to some classes of integral equations of generalized convolution type generated by the Kontorovich-Lebedev (K) transform, the Laplace ($\mathcal{L}$) transform and the Fourier (F) transform in some appropriate function spaces and represent it in a closed form.

1. Introduction

We investigate the integral equation of second type of the form, see [6].

$$f(x) + \lambda \int_0^T K(x,\tau) f(\tau) d\tau = g(x), \ \ x > 0, \tag{1.1}$$

where λ is an arbitrary real number, $T > 0$, $g(x)$ is a given function, $K(x,\tau)$ is the kernel and f is the unknown function. For general kernels $K(x,\tau)$, an explicit solution to (1.1) is not known, and approximate solutions have been sought instead. Nevertheless, some authors tried to get explicit analytic solutions to particular cases of (1.1), for example, in [6] of H. M. Srivastava and R. G. Buschman have found analytic solutions to (1.1) for the kernels $K(x,\tau) = K(x-\tau) = (x-\tau)^{\alpha}; e^{-a|x-\tau|}; \sinh(a(x-\tau))$, and $aJ_1(a(x-\tau))$ with J_1 being the Bessel functions. Also, in the fifties of the last century, there were some results for Toeplitz-Hankel equations, e.g., for (1.1) with

$$K(x,\tau) = k_1(x-\tau) + k_2(x+\tau),$$

where k_1 is a Toeplitz kernel, k_2 is a Hankel kernel, see [2,4,13]. However, the explicit form of solution is not known for general Toeplitz-Hankel equations. To this end, during the last years some authors used the notion of convolutions and generalized convolution, for example those of the Kontorovich-Lebedev (K),

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Hartley (H), Laplace ($\mathcal{L}$), Fourier sine (F_s) and Fourier cosine (F_c) transforms, see [8–11, 14, 16, 18] to obtain explicit solutions to (1.1). In addition, it is also possible to study the problem in the form of Fredholm integral equations using the generalized convolution technique in [17].

In this paper, we will investigate the equation (3.1) on $(0, +\infty)$ with separable kernels of the form

$$K(x, u) = k_1(x, u) + k_2(x, u),$$

where k_1 and k_2 are kernels some generalized convolutions generated by the Kontorovich-Lebedev transform, the Laplace transform and the Fourier transform. To deal with this equation, we transform it into integral equations of convolutions-pair and then use the properties of generalized convolutions to prove the existence of a solution as well as represent it in a closed form, see Theorems 3.1, 3.2 and 3.3 bellow. By choosing k_1, k_2 are the kernel of generalized convolutions then the equation (3.1) to become a kind of integral equation of convolution-pair and apply the results in [3,7,11,14,18] to provide solutions.

We note that we prove these results without using the Wiener-Lévy theorem if comparing with the results in [8–11, 14–16, 18]. Our results are the first ones which combine the Kontorovich-Lebedev transform, Laplace's transform and the Fourier transform.

2. Preliminaries

The Fourier cosine transform (F_c) and its inverse (F_c^{-1}) formula is of the form (see [12])

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cos(xy) dx, \ y > 0, \tag{2.1}$$

and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (F_c f)(y) \cos(xy) dy, \ x > 0. \tag{2.2}$$

The Fourier sine transform (F_s) and its inverse (F_s^{-1}) formula is of the form (see [12])

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \sin(xy) dx, \ y > 0, \tag{2.3}$$

and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (F_s f)(y) \sin(xy) dy, \ x > 0. \tag{2.4}$$

The Laplace transform ($\mathcal{L}$) is of the form (see [5])

$$(\mathcal{L} f)(y) = \int_0^{\infty} f(x) e^{-xy} dx, \ y > 0. \tag{2.5}$$

The Kontorovichh-Lebedev transform (K) is of the form (see [19])

$$(Kf)(x) = \int_0^\infty K_{ix}(t)f(t)dt, \tag{2.6}$$

here $K_{ix}(t)$ is the Macdonald function (see [19])

$$K_{ix}(t) = \int_0^\infty e^{-t\cosh u}\cos xu\, du, \ \ x \geqslant 0, \ t > 0.$$

The inverse Kontorovichh-Lebedev transform (K^{-1}) is defined as follow (see [19])

$$(K^{-1}f)(x) = \frac{2}{\pi^2}x\sinh \pi x \int_0^\infty K_{ix}(y)y^{-1}f(y)dy, \ x > 0. \tag{2.7}$$

The following function spaces will be used in the next sections. First, the space $L_p(\mathbb{R}_+, \gamma)$ is defined as follow (see [1])

$$L_p(\mathbb{R}_+, \gamma) = \left\{ f : \left(\int_{\mathbb{R}+} \gamma(x)\cdot |f(x)|^p dx \right)^{\frac{1}{p}} < +\infty \right\}, \ \ 1 \leqslant p < \infty.$$

One can easily see that $L_p(\mathbb{R}_+) \subset L_p(\mathbb{R}_+, \gamma)$.

The space $\mathcal{A}_c, \mathcal{A}_s$ and $H(\mathbb{R})$ is defined as follow (see [11])

$$\mathcal{A}_c = \{f = F_c k, k \in L_1(\mathbb{R}_+)\}, \ \ \mathcal{A}_s = \{f = F_s k, k \in L_1(\mathbb{R}_+)\}$$

with norm $\|f\|_{\mathcal{A}_c} := \|k\|_{L_1(\mathbb{R}_+)}; \|f\|_{\mathcal{A}_s} := \|k\|_{L_1(\mathbb{R}_+)}$ and

$$H(\mathbb{R}) = \{f : \mathcal{L}f \in L_2(\mathbb{R}_+)\}.$$

One can easily see that $L_2(\mathbb{R}_+) \subset H(\mathbb{R}_+)$.

3. Integral equations of convolution-pair

In this section, we consider solve in a closed form of the integral equations (3.1) by using the kernel pair k_1 and k_2 are the kernel of differently generalized convolution which were predefined. After that, we will use the generalized convolutions of techniques for integral transforms (K), ($\mathcal{L}$), (F_s) and (F_c) in [3, 7, 11, 14, 18] to convert equation (3.1) becomes a kind of integral equation of convolution-pair.

$$f(x) + \int_0^\infty K(x,u)f(u)du = g(x), \ x > 0. \tag{3.1}$$

Here g is the given functions, assume that kernel is $K(x,u) = k_1(x,u) + k_2(x,u)$ and f is an unknown function.

First, we solve the equation (3.1) in the case k_1, k_2 as follows:

$$\begin{cases} k_1(x,u) = \dfrac{1}{\pi^2}\displaystyle\int_0^\infty \frac{1}{u}\left[e^{-u\cosh(x+v)} + e^{-u\ \cosh(x-v)}\right]\varphi_1(u)du, \ x, v > 0, \\ k_2(x,u) = \dfrac{1}{\pi}\displaystyle\int_0^\infty \left[\frac{v}{v^2+(x-u)^2} + \frac{v}{v^2+(x+u)^2}\right]\Psi_1(u)du, \ x, v > 0. \end{cases} \tag{3.2}$$

Theorem 3.1. *Let* φ_1, Ψ_1, g *be known functions and suppose that*

$$\varphi_1 \in L_1\left(\mathbb{R}_+, \frac{1+\sqrt{x^3}}{\sqrt{x^3}}\right), \Psi_1 \in L_2\left(\mathbb{R}_+\right), g \in L_1(\mathbb{R}_+) \cap \mathcal{A}_c$$

such that

$$\frac{(F_c g)(y)}{1 + \dfrac{(K^{-1}\varphi_1)(y)}{y \sinh \pi y} + (\mathcal{L}\Psi_1)(y)} \in L_1(\mathbb{R}_+) \cap \mathcal{A}_c, \forall y > 0.$$

Then the integral equation (3.1) *has a unique solution in* $L_1(\mathbb{R}_+) \cap \mathcal{A}_c$ *which is of the form*

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(F_c g)(y)}{1 + \dfrac{(K^{-1}\varphi)(y)}{y \sinh \pi y} + (\mathcal{L}\psi)(y)} \cos(xy) dy, \; x > 0,$$

where the $F_c, K^{-1}, \mathcal{L}$ *are the respectively defined by* (2.1)*,* (2.7) *and* (2.5)*.*

Proof. The main tool for our proof will be the following generalized convolution. Let $\varphi_1 \in L_1\left(\mathbb{R}_+, \frac{1+\sqrt{x}}{\sqrt{x^3}}\right)$, $f \in L_1(\mathbb{R}_+)$, $\gamma_1 = \frac{1}{y \sinh \pi y}$, the generalized convolution with weight function γ_1 for the (F_c, K^{-1}) transforms is defined as follow (see [14])

$$\begin{aligned}(3.3) \qquad & (\varphi_1 \overset{\gamma_1}{\underset{1}{*}} f)(x) \\ &= \frac{1}{\pi^2} \int_{\mathbb{R}_+^2} \frac{1}{u} \left[e^{-u \cosh(x+v)} + e^{-u \cosh(x-v)} \right] \varphi_1(u) f(v) du dv, \; \forall x > 0.\end{aligned}$$

Generalized convolution (3.3) belongs to $L_1(\mathbb{R}_+)$, and the following factorization equality holds

$$(3.4) \qquad F_c(\varphi_1 \overset{\gamma_1}{\underset{1}{*}} f)(y) = \frac{1}{y \sinh \pi y} (K^{-1}\varphi_1)(y)(F_c f)(y), \; y > 0.$$

Let $\Psi_1, f \in L_2(\mathbb{R}_+)$, the generalized convolution for the $(F_c, \mathcal{L})$ transforms is defined as follow (see [11])

$$\begin{aligned}(3.5) \qquad & (f \underset{\{ {2 \atop 3} \}}{*} \Psi_1)(x) \\ &= \frac{1}{\pi} \int_{\mathbb{R}_+^2} \left[\frac{v}{v^2 + (x-u)^2} \pm \frac{v}{v^2 + (x+u)^2} \right] f(u) \Psi_1(v) du dv, \; \forall x > 0.\end{aligned}$$

Moreover, $(f \underset{\{ {2 \atop 3} \}}{*} \Psi_1) \in \mathcal{A}_{\{ {c \atop s} \}}$, and the following factorization equality holds

$$(3.6) \qquad F_{\{ {c \atop s} \}}(f \underset{\{ {2 \atop 3} \}}{*} \Psi_1)(y) = (F_{\{ {c \atop s} \}} f)(y)(\mathcal{L}\Psi_1)(y), \; y > 0.$$

With the kernel pair k_1, k_2 were determined in (3.2), combined with the formula (3.3) and (3.5) then the equation (3.1) becomes integral equations of convolution-pair can be rewritten in the form

$$f(x) + (\varphi_1 \overset{\gamma_1}{\underset{1}{*}} f)(x) + (f \underset{2}{*} \Psi_1)(x) = g(x),\ x > 0, \tag{3.7}$$

where, $\varphi_1 \in L_1\left(\mathbb{R}_+, \frac{1+\sqrt{x^3}}{\sqrt{x^3}}\right)$, $\Psi_1 \in L_2(\mathbb{R}_+)$, $g \in L_1(\mathbb{R}_+) \cap \mathcal{A}_c$.

Applying the Fourier cosine F_c on both sides of (3.7), basing on the factorizations (3.4), (3.6) and inverse formula (2.2), we have

$$(F_c f)(y) + F_c(\varphi_1 \overset{\gamma_1}{\underset{1}{*}} f)(y) + F_c(f \underset{2}{*} \Psi_1)(y) = (F_c g)(y),\ y > 0.$$

$$(F_c f)(y) + \frac{1}{y \sinh \pi y}(K^{-1}\varphi_1)(y)(F_c f)(y) + (F_c f)(y)(\mathcal{L}\Psi_1)(y) = (F_c g)(y),\ \forall y > 0.$$

Then

$$(F_c f)(y)\left[1 + \frac{(K^{-1}\varphi_1)(y)}{y \sinh \pi y} + (\mathcal{L}\Psi_1)(y)\right] = F_c g(y),\ \forall y > 0.$$

Under the hypothesis, $\frac{(F_c g)(y)}{1+\frac{(K^{-1}\varphi_1)(y)}{y \sinh \pi y}+(\mathcal{L}\Psi_1)(y)} \in L_1(\mathbb{R}_+) \cap \mathcal{A}_c$, by the inverse for Fourier cosine transform, we obtain

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(F_c g)(y)}{1 + \dfrac{(K^{-1}\varphi_1)(y)}{y \sinh \pi y} + (\mathcal{L}\Psi_1)(y)} \cos(xy) dy,\ x > 0,$$

and f belongs to $L_1(\mathbb{R}_+) \cap \mathcal{A}_c$. The proof is complete. □

Next, we solve the equation (3.1) in the case k_3, k_4 as follows:

$$\begin{cases} k_3(x,u) = \dfrac{1}{\pi^2} \displaystyle\int_0^\infty \dfrac{1}{u}\left[e^{-u \cosh(x-v)} - e^{-u \cosh(x+v)}\right] \varphi_2(u) du,\ x, v > 0, \\ k_4(x,u) = \dfrac{1}{\pi} \displaystyle\int_0^\infty \left[\dfrac{v}{v^2 + (x-u)^2} - \dfrac{v}{v^2 + (x+u)^2}\right] \Psi_2(u) du,\ x, v > 0. \end{cases} \tag{3.8}$$

Theorem 3.2. *Let φ_2, ψ_2, g known functions and suppose that*

$$\varphi_2 \in L_1\left(\mathbb{R}_+, \frac{1}{\sqrt{x^3}}\right), \Psi_2 \in L_2(\mathbb{R}_+),\ g \in L_1(\mathbb{R}_+) \cap \mathcal{A}_s$$

such that

$$\frac{(F_s g)(y)}{1 + (K\,\varphi_2)(y) + (\mathcal{L}\Psi_2)(y)]} \in L_1(\mathbb{R}_+) \cap \mathcal{A}_c,\ \forall y > 0.$$

Then the equation (3.1) *has a unique solution $f \in L_1(\mathbb{R}_+) \cap \mathcal{A}_s$ which is of the form*

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(F_s g)(y)}{1 + (K\,\varphi_2)(y) + (\mathcal{L}\Psi_2)(y)} \sin(xy) dy,\ x > 0,$$

where the $F_s, K, \mathcal{L}$ are the respectively defined by (2.3), (2.6) *and* (2.5).

Proof. In order to prove the theorem, we recall the following generalized convolution for the (F_s, K) transform as follow:

Let $\Psi_2 \in L_1\left(\mathbb{R}_+, \frac{1}{\sqrt{x^3}}\right)$, $f \in L_1(\mathbb{R}_+)$, the generalized convolution $(\Psi_2 \underset{4}{*} f)(x)$ for the Fourier sine and Kontorovichh-Lebedev transform is of the form (see [18])

$$(\Psi_2 \underset{4}{*} f)(x) = \frac{1}{\pi^2}\int_{\mathbb{R}_+^2} \frac{1}{u}\left[e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}\right]\Psi_2(u)f(v)dudv,\ \forall x>0. \tag{3.9}$$

We known that $(\Psi_2 \underset{4}{*} f) \in L_1(\mathbb{R}_+)$ and satisfies the following factorization equality

$$F_s(\Psi_2 \underset{4}{*} f)(y) = (K\,\Psi_2)(y)(F_s\, f)(y),\ y>0. \tag{3.10}$$

With the kernel pair k_3, k_4 were determined in (3.8) and using the generalized convolutions (3.5), (3.9) then the equation (3.1) becomes integral equations of convolution-pair can be rewritten in the form

$$f(x) + (\varphi_2 \underset{4}{*} f)(x) + (f \underset{3}{*} \Psi_2)(x) = g(x),\ x>0. \tag{3.11}$$

Applying the Fourier sine transform F_s on both sides of equation (3.11), with the help of equalities (3.10), (3.6), and inverse formula (2.4), we obtain

$$(F_s f)(y) + F_s(\varphi_2 \underset{4}{*} f)(y) + F_s(f \underset{3}{*} \Psi_3)(y) = (F_s g)(y),\ \ y>0,$$

$$(F_s f)(y)\left[1 + (K\,\varphi_2)(y) + (\mathcal{L}\Psi_2)(y)\right] = (F_s g)(y),\ y>0.$$

Under the hypothesis, $\frac{(F_s g)(y)}{1+(K\,\varphi_2)(y)+(\mathcal{L}\Psi_2)(y)} \in L_1(\mathbb{R}_+)\cap\mathcal{A}_s, \forall y>0$, thanks to the inverse formula of Fourier sine transform (2.4), we have a solution in $L_1(\mathbb{R}_+)\cap\mathcal{A}_s$ as follow:

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(F_s g)(y)}{1+(K\,\varphi_2)(y)+(\mathcal{L}\Psi_2)(y)}\sin(xy)dy,\ x>0.$$

The proof of theorem is complete. □

Next, we will choose kernel k_5, k_6 as follows:

$$\begin{cases} k_5(x,y) = \dfrac{1}{\pi^2}\displaystyle\int_0^\infty \left[\sinh(x+y)e^{-u\cosh(x+y)} \pm \sinh(x-y)e^{-u\cosh(x-y)}\right]\varphi_3(u)du, \\ \qquad x,y>0, \\ k_6(x,y) = \dfrac{1}{2\pi}\displaystyle\int_0^\infty \left[\Theta_{\{\frac{1}{2}\}}(x-1,y,v) - \Theta_{\{\frac{1}{2}\}}(x+1,y,v)\right]\Psi_3(v)dv,\ x,y>0. \end{cases} \tag{3.12}$$

Here $\varphi_3 \in L_1\left(\mathbb{R}_+, \frac{1}{x}\right)$, $\Psi_3 \in H(\mathbb{R}_+)$, $g \in L_1(\mathbb{R}_+)\cap L_2(\mathbb{R}_+)$, $\gamma_2 = \frac{1}{\sinh(\pi y)}$, $\gamma_{3,4} = \mp\sin y$ are given functions, $\Theta_{\{\frac{1}{2}\}}$ are defined by

$$\Theta_{\{\frac{1}{2}\}}(x,u,v) = \frac{v}{v^2+(x-u)^2} \mp \frac{v}{v^2+(x+u)^2}.$$

Theorem 3.3. *Suppose that* $\varphi_3 \in L_1\left(\mathbb{R}_+, \frac{1}{x}\right)$, $\Psi_3 \in H(\mathbb{R}_+)$, $g \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$ *such that*

$$\frac{\left(F_{\left\{ s \atop c \right\}} g\right)(y)}{1 + \dfrac{(K^{-1}\varphi_3)(y)}{\sinh \pi y} \mp \sin y (\mathcal{L}\Psi_3)(y)} \in L_1\left(\mathbb{R}_+, \frac{1}{x}\right) \cap L_2(\mathbb{R}_+), \ \forall y > 0.$$

Then the equation (3.1) *has a unique solution* $f \in L_1\left(\mathbb{R}_+, \frac{1}{x}\right) \cap L_2(\mathbb{R}_+)$ *which is defined by formula*

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\left(F_{\left\{ s \atop c \right\}} g\right)(y)}{1 + \dfrac{(K^{-1}\varphi_3)(y)}{\sinh \pi y} \mp \sin y (\mathcal{L}\Psi_3)(y)} \begin{Bmatrix} \cos(xy) \\ \sin(xy) \end{Bmatrix} dy, \ x > 0.$$

where the F_s, F_c, K^{-1}, L *are the respectively defined by* (2.3), (2.1), (2.7), (2.5).

Proof. First, we recall the following generalized convolutions for the Kontorovichh-Lebedev, Laplace, and Fourier transforms which are useful for our proof. Let $\varphi_3 \in L_1\left(\mathbb{R}_+, \frac{1}{x}\right)$, $f \in L(\mathbb{R}_+)$, $\gamma_2 = \frac{1}{\sinh(\pi y)}$. The generalized convolutions with the weight function γ_2 for the integral transforms F_s, F_c, K^{-1} are defined as follow (see [7]):

$$(3.13) \qquad (\varphi_3 \overset{\gamma_2}{\underset{\left\{ 5 \atop 6 \right\}}{*}} f)(x) = \frac{1}{\pi^2} \int_{\mathbb{R}_+^2} \left[\sinh(x+v)\, e^{-u \cosh(x+v)} \pm \sinh(x-v)\, e^{-u \cosh(x-v)} \right] \varphi_3(u) f(v) du dv, \ \forall x > 0.$$

Moreover, $(\varphi_3 \overset{\gamma_2}{\underset{\left\{ 5 \atop 6 \right\}}{*}} f) \in L_1(\mathbb{R}_+)$, and

$$(3.14) \qquad F_{\left\{ s \atop c \right\}}(\varphi_3 \overset{\gamma_2}{\underset{\left\{ 5 \atop 6 \right\}}{*}} f)(y) = \gamma_2(y)(K^{-1}\varphi_3)(y)(F_{\left\{ c \atop s \right\}} f)(y), \ \forall y > 0.$$

The generalized convolutions for the Fourier cosine, Fourier sine and the Laplace transforms with the weight function $\gamma_{3,4} = \mp \sin y$ of two functions $h \in L_2(\mathbb{R}_+)$, $k \in H(\mathbb{R}_+)$ are defined as follow (see [3]):

$$(3.15) \quad (f \overset{\gamma_{3,4}}{\underset{\left\{ 7 \atop 8 \right\}}{*}} \Psi_3)(x) = \frac{1}{2\pi} \int_{\mathbb{R}_+^2} \left[\Theta_{\left\{ 1 \atop 2 \right\}}(x-1, u, v) - \Theta_{\left\{ 1 \atop 2 \right\}}(x+1, u, v) \right] f(u) \Psi_3(v) du dv, \ \forall x > 0.$$

Here $\Theta_{\left\{ 1 \atop 2 \right\}}(x, u, v) = \frac{v}{v^2 + (x-u)^2} \mp \frac{v}{v^2 + (x+u)^2}$.

Then, $(f \overset{\gamma_{3,4}}{\underset{\left\{\substack{7\\8}\right\}}{*}} \Psi_3) \in L_2(\mathbb{R}_+)$ and satisfy the following factorization equalities

$$F_{\left\{\substack{c\\s}\right\}}(f \overset{\gamma_{3,4}}{\underset{\left\{\substack{7\\8}\right\}}{*}} \Psi_3)(y) = \mp \sin y (F_{\left\{\substack{s\\c}\right\}} f)(y)(\mathcal{L}\, \Psi_3)(y),\ \forall y > 0. \tag{3.16}$$

With how to choose the kernel pair k_5, k_6 as in (3.12), combined with the generalized convolutions (3.13), (3.15), then the equation (3.1) becomes integral equations of convolution-pair can be rewritten in the form

$$f(x) + (\varphi_3 \overset{\gamma_2}{\underset{\left\{\substack{5\\6}\right\}}{*}} f)(x) + (f \overset{\gamma_{3,4}}{\underset{\left\{\substack{7\\8}\right\}}{*}} \Psi_3)(x) = g(x),\ x > 0, \tag{3.17}$$

where $\varphi_3 \in L_1\left(\mathbb{R}_+, \frac{1}{x}\right)$, $\Psi_3 \in H\left(\mathbb{R}_+\right)$, $g \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$, $\gamma_2 = \frac{1}{\sinh(\pi y)}$, $\gamma_{3,4} = \mp \sin y$.

Applying the $F_{\left\{\substack{s\\c}\right\}}$ transform on both sides of (3.17), we have

$$(F_{\left\{\substack{s\\c}\right\}} f)(y) + F_{\left\{\substack{s\\c}\right\}}(\varphi_3 \overset{\gamma_2}{\underset{\left\{\substack{5\\6}\right\}}{*}} f)(y) + F_{\left\{\substack{s\\c}\right\}}(f \overset{\gamma_{3,4}}{\underset{\left\{\substack{7\\8}\right\}}{*}} \Psi_3)(y) = (F_{\left\{\substack{s\\c}\right\}} g)(y),\ \forall y > 0.$$

With the help of factorization equalities (3.14), (3.16), and inverse formula (2.2), (2.4), this equality becomes

$$\begin{aligned}&(F_{\left\{\substack{s\\c}\right\}} f)(x) + \frac{1}{\sinh \pi y}(K^{-1}\varphi_3)(y)(F_{\left\{\substack{c\\s}\right\}} f)(y) \mp \sin y (F_{\left\{\substack{c\\s}\right\}} f)(y)(\mathcal{L}\Psi_3)(y)\\ &= (F_{\left\{\substack{s\\c}\right\}} g)(y),\ \forall y > 0.\end{aligned}$$

Then, we have

$$(F_{\left\{\substack{c\\s}\right\}} f)(y) = \frac{(F_{\left\{\substack{s\\c}\right\}} g)(y)}{1 + \dfrac{(K^{-1}\varphi_3)(y)}{\sinh \pi y} \mp \sin y (\mathcal{L}\Psi_3)(y)},\ \forall y > 0.$$

Since the hypothesis

$$\frac{\left(F_{\left\{\substack{s\\c}\right\}} g\right)(y)}{1 + \dfrac{(K^{-1}\varphi_3)(y)}{\sinh \pi y} \mp \sin y (\mathcal{L}\Psi_3)(y)} \in L_1\left(\mathbb{R}_+, \frac{1}{x}\right) \cap L_2\left(\mathbb{R}_+\right),\ \forall y > 0.$$

We have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(F_{\left\{\substack{s\\c}\right\}} g)(y)}{1 + \dfrac{(K^{-1}\varphi_3)(y)}{\sinh \pi y} \mp \sin y (\mathcal{L}\Psi_3)(y)} \begin{Bmatrix}\cos(xy)\\ \sin(xy)\end{Bmatrix} dy,\ x > 0.$$

Moreover, we can see that $f(x) \in L_1\left(\mathbb{R}_+, \frac{1}{x}\right) \cap L_2\left(\mathbb{R}_+\right)$.

The proof of theorem is complete. □

In the following studies, we will evaluate the estimates of solutions and thereby study the properties of boundedness of solution of these problems.

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Trinh Tuan
Department of Mathematics
Electric Power University
235 Hoang Quoc Viet, Hanoi, Vietnam
Email address: tuantrinhpsac@yahoo.com