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SOLVABILITY OF ODES DESCRIBING CURVES ON $\mathbb{S}^2$ OR $\mathbb{H}^2$

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Abstract. We show the solvability of the ordinary differential equations describing curves on sphere or hyperbolic space. Making use of the geometric properties of the equations, we derive explicit solution formulae.

1. Introduction

We are interested in the following ordinary differential equations describing curves on the sphere $\mathbb{S}^2 = \{(x, y, x) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

$$\begin{aligned} &\frac{dv}{dt} = f(t)\big(\mathbf{n} \times v\big)(t) + g(t)\Big(v \times \big(\mathbf{n} \times v\big)\Big)(t), \\ &v(0) = v_0, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &\frac{d^2v}{dt^2} + \gamma\frac{dv}{dt} + \left|\frac{dv}{dt}\right|^2 v = 0, \\ &v(0) = v_0, \quad \frac{dv}{dt}(0) = v_1, \end{aligned} \tag{1.2}$$

satisfying $|v_0|^2 = 1$ and $\langle v_0, v_1\rangle = 0$. Here $v : \mathbb{R}^+ \to \mathbb{S}^2 \subset \mathbb{R}^3$ and $\mathbf{n} = (0, 0, 1)$. The scalar functions f and g are continuous functions of t. We assume that the constant γ is non-negative. A damping effect is represented by $\gamma > 0$. The usual inner product $\langle\cdot, \cdot\rangle$ and cross product $\times$ in $\mathbb{R}^3$ are defined by

$$\begin{aligned} &\langle \mathbf{a}, \mathbf{b}\rangle = a_1b_1 + a_2b_2 + a_3b_3, \\ &\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1), \end{aligned}$$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. We also denote $|\mathbf{a}|^2 = \langle \mathbf{a}, \mathbf{a}\rangle$.

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The equation (1.1) comes from the inertial spin model [2,3].

$$\begin{aligned}\frac{dv_i}{dt} &= f(t)\big(s_i \times v_i\big)(t) + g(t)\Big(v_i \times \big(s_i \times v_i\big)\Big)(t),\\ \frac{ds_i}{dt} &= v_i \times \frac{\kappa}{N}\sum_{j=1}^{N} p_{ij} v_j,\end{aligned} \tag{1.3}$$

where $v_i : \mathbb{R}^+ \to \mathbb{S}^2$ and $s_i : \mathbb{R}^+ \to \mathbb{R}^3$ for $i = 1, 2, \ldots, N$. The coupling constants are p_{ij}. The case of $f = 1$ and $g = 0$ in (1.3) was studied in [2]. A variant of the inertial spin model was proposed in [3] where the alignment control is added to make the feasibility of velocity alignment more applicable than the original inertial spin model. For the trivial coupling constants $p_{ij} = 0$, we have $s_i(t) = s_i(0)$ and let $s_i = \mathbf{n}$ for simplicity. Then we derive the equation (1.1). Note that the equation (1.1) is still a system of ODEs with quadratic nonlinearity.

The equation (1.2) is a simplified version of the damped wave map equation [5,6].

$$\partial_{tt} v - \Delta v + \gamma \partial_t v + v(|\partial_t v|^2 - |\nabla v|^2) = 0.$$

When $v(x,t) = v(t)$, we can derive the equation (1.2). A new method for computing wave maps into sphere was proposed in [4] using angular momentum which is related with inertial spin model in [2].

We study a system (3.1) which is a hyperbolic space version of (1.1). A first-order particle swarm model on the hyperbolic space has been studied in [1].

In Section 2, we find explicit solution formulae for (1.1) and (1.2). The solution formula for (3.1) is studied in Section 3.

2. The solution of ODEs describing curves on $\mathbb{S}^2$

2.1. Solution of (1.1)

For the solution of (1.1), it is easy to check that $\frac{d|v|^2}{dt} = 0$ from which we can derive, for $|v_0|^2 = 1$,

$$|v(t)|^2 = 1. \tag{2.1}$$

Then we have $v : \mathbb{R}^+ \to \mathbb{S}^2$, where $\mathbb{S}^2$ is the unit sphere. Making use of (2.1) and

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \langle \mathbf{a}, \mathbf{c}\rangle \mathbf{b} - \langle \mathbf{a}, \mathbf{b}\rangle \mathbf{c},$$

we can rewrite (1.1) as

$$\frac{dv}{dt} = f \underbrace{\left(\mathbf{n} \times v\right)}_{\text{(I)}} + g \underbrace{\left(\mathbf{n} - \langle v, \mathbf{n}\rangle v\right)}_{\text{(II)}}. \tag{2.2}$$

For $v_0 = \pm\mathbf{n}$, we have a trivial solution $v(t) = \pm\mathbf{n}$. From now on, we assume that $v \neq \pm\mathbf{n}$. It is easy to check

$$\frac{d}{dt}\langle \mathbf{n}, v\rangle = g\left(1 - \langle \mathbf{n}, v\rangle^2\right).$$

Considering $1 - \langle \mathbf{n}, v\rangle^2 > 0$, the sign of the right hand side is determined by the sign of g. When $g > 0$, v moves to the north pole $\mathbf{n}$. The first term (I) in (2.2) represents the rotation around $\mathbf{n}$. The attraction and repulsion with respect to $\mathbf{n}$ are expressed by the second term (II).

To study the behavior of the solution to (2.2) more precisely, we use spherical coordinate

$$v = (x, y, z) = (\sin\phi\cos\theta,\, \sin\phi\sin\theta,\, \cos\phi), \tag{2.3}$$

where ϕ and θ are functions of t. Then the third component of (2.2) leads us to

$$-\sin\phi\frac{d\phi}{dt} = g(1 - \cos^2\phi), \tag{2.4}$$

which can be integrated by

$$\cos\phi(t) = \frac{(1+\cos\phi_0)\exp\left(2\int_0^t g(\tau)d\tau\right) - (1-\cos\phi_0)}{(1+\cos\phi_0)\exp\left(2\int_0^t g(\tau)d\tau\right) + (1-\cos\phi_0)}, \tag{2.5}$$

where $0 < \phi_0 = \phi(0) < \pi$ is an initial value. Making use of (2.4), the first and second components in (2.2) reduce to $\frac{d\theta}{dt} = f$ from which we obtain

$$\theta(t) = \theta_0 + \int_0^t f(\tau)d\tau,$$

where $\theta_0 = \theta(0)$ is an initial value.

The function ϕ is determined by g only and θ by f only. For the function g satisfying $\lim_{t\to\infty} \exp\left(2\int_0^t g(\tau)d\tau\right) = M$, we have, considering the formula (2.5),

$$\lim_{t\to\infty}\cos\phi = \frac{(1+\cos\phi_0)M - (1-\cos\phi_0)}{(1+\cos\phi_0)M + (1-\cos\phi_0)}.$$

In particular, we have $\cos\phi \to \pm 1$ for $\int_0^t g(\tau)d\tau \to \pm\infty$ which implies $v \to \pm\mathbf{n}$ as $t \to \infty$.

2.2. Solution of (1.2)

Here we assume $\gamma > 0$ which is a damping constant. The case of $\gamma = 0$ is solved in Remark 2.1. For the solution of (1.2), we can check

$$\frac{d^2}{dt^2}\left(|v|^2 - 1\right) + \gamma\frac{d}{dt}\left(|v|^2 - 1\right) + 2\left|\frac{dv}{dt}\right|^2\left(|v|^2 - 1\right) = 0. \tag{2.6}$$

For the initial data

$$|v_0|^2 - 1 = 0 \quad \text{and} \quad \frac{d}{dt}(|v|^2 - 1)|_{t=0} = \langle v_0, v_1 \rangle = 0,$$

the solution of (2.6) becomes

$$|v(t)|^2 = 1.$$

Then we have $v : \mathbb{R}^+ \to \mathbb{S}^2$.

From now on, we use the notation $' = \frac{d}{dt}$. Taking inner product (1.2) by v', we can derive

$$(|v'|^2)' + 2\gamma|v'|^2 = 0,$$

where $\langle v, v' \rangle = 0$ is used. Then we obtain

$$|v'(t)|^2 = |v_1|^2 e^{-2\gamma t}. \tag{2.7}$$

When $|v_1| = 0$, we have a trivial solution $v(t) = v_0$. From now on, we assume that $|v_1| > 0$.

In the spherical coordinate (2.3), the equation (2.7) can be rewritten as

$$(\phi')^2 + (\theta')^2 \sin^2 \phi = |v_1|^2 e^{-2\gamma t}. \tag{2.8}$$

The calculation $\cos\theta$ (second component of (1.2)) $-\sin\theta$ (first component of (1.2)) leads to

$$\theta'' \sin\phi + \gamma\theta' \sin\phi + 2\phi'\theta' \cos\phi = 0. \tag{2.9}$$

The equation (2.9) can be rewritten as

$$\frac{\theta''}{\theta'} + \gamma + 2\frac{\cos\phi}{\sin\phi}\phi' = 0$$

which is $\big(\log|\theta'|\big)' + \gamma + \big(\log|\sin^2\phi|\big)' = 0$. Then we have

$$|\theta' \sin^2\phi| = e^{a-\gamma t}, \tag{2.10}$$

where a is an integral constant.

Making use of (2.8) and (2.10), we arrive at

$$(\phi')^2 = e^{-2\gamma t}\left[|v_1|^2 - \frac{e^{2a}}{\sin^2\phi}\right], \tag{2.11}$$

where $e^{2a} = \big(|v_1|^2 - (\phi'(0))^2\big)\sin^2\phi(0)$. Making change of variables

$$\frac{\sqrt{|v_1|^2 - e^{2a}}}{|v_1|}h(t) = \cos\phi(t),$$

the equation (2.11) can be rewritten as

$$\frac{1}{\sqrt{1-h^2}}\frac{dh}{dt} = |v_1|e^{-\gamma t}, \tag{2.12}$$

which can be solved by

$$h(t) = \cos\left(\frac{|v_1|}{\gamma}e^{-\gamma t} - \frac{|v_1|}{\gamma} + \arccos h(0)\right).$$

Considering that $z(t) = \frac{\sqrt{|v_1|^2 - e^{2a}}}{|v_1|}h(t)$ and $|v_1|^2 - e^{2a} = |v_1|^2 z^2(0) + (z'(0))^2$, we have

$$z(t) = \frac{\sqrt{|v_1|^2 z^2(0) + (z'(0))^2}}{|v_1|}\cos\left(\frac{|v_1|}{\gamma}e^{-\gamma t} - \frac{|v_1|}{\gamma} + \arccos\left(\frac{|v_1|z(0)}{\sqrt{|v_1|^2 z^2(0) + (z'(0))^2}}\right)\right).$$

Making use of (2.7), the equation (1.2) can be written as

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \gamma\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + |v_1|^2 e^{-2\gamma t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

By the symmetry of the above equations, we can derive the following formulae for $x(t)$ and $y(t)$.

$$x(t) = \frac{\sqrt{|v_1|^2 x^2(0) + (x'(0))^2}}{|v_1|}\cos\left(\frac{|v_1|}{\gamma}e^{-\gamma t} - \frac{|v_1|}{\gamma} + \arccos\left(\frac{|v_1|x(0)}{\sqrt{|v_1|^2 x^2(0) + (x'(0))^2}}\right)\right),$$

$$y(t) = \frac{\sqrt{|v_1|^2 y^2(0) + (y'(0))^2}}{|v_1|}\cos\left(\frac{|v_1|}{\gamma}e^{-\gamma t} - \frac{|v_1|}{\gamma} + \arccos\left(\frac{|v_1|y(0)}{\sqrt{|v_1|^2 y^2(0) + (y'(0))^2}}\right)\right).$$

Remark 2.1. For the case of $\gamma = 0$, the equation (2.12) implies

$$h(t) = \cos\Big(\arccos h(0) - |v_1|t\Big),$$

from which we can derive

$$z(t) = \frac{\sqrt{|v_1|^2 z^2(0) + (z'(0))^2}}{|v_1|}\cos\left(\arccos\left(\frac{|v_1|z(0)}{\sqrt{|v_1|^2 z^2(0) + (z'(0))^2}}\right) - |v_1|t\right).$$

3. The solution of ODE describing curves on $\mathbb{H}^2$

We study the following ordinary differential equation describing curves on the hyperbolic space $\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^{2+1} \mid x_1^2 + x_2^2 - x_3^2 = -1 \text{ and } x_3 > 0\}$, which is a non-compact target.

$$\begin{aligned} \frac{du}{dt} &= f(t)\big(\mathbf{n}\dot{\times}u\big)(t) + g(t)\big(\mathbf{n} + \langle u, \mathbf{n}\rangle_h u\big)(t), \\ u(0) &= u_0 \end{aligned} \tag{3.1}$$

satisfying $|u_0|_h^2 = -1$. Here $u : \mathbb{R}^+ \to \mathbb{R}^{2+1}$ and $\mathbf{n} = (0,0,1)$. The scalar functions f and g are continuous functions of t. The pseudo inner product $\langle\cdot,\cdot\rangle_h$ and the pseudo cross product $\dot{\times}$ in $\mathbb{R}^{2+1}$ are defined by

$$\begin{aligned} \langle\mathbf{a},\mathbf{b}\rangle_h &= a_1b_1 + a_2b_2 - a_3b_3, \\ \mathbf{a}\dot{\times}\mathbf{b} &= (a_2b_3 - a_3b_2,\, a_3b_1 - a_1b_3,\, a_2b_1 - a_1b_2), \end{aligned}$$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. We also denote $|\mathbf{a}|_h^2 = \langle\mathbf{a},\mathbf{a}\rangle_h$.

It is easy to check that

$$\frac{1}{2}\frac{d}{dt}\big(1+|u|_h^2\big) = g\langle u,\mathbf{n}\rangle_h\big(1+|u|_h^2\big),$$

from which we can derive, for $1+|u_0|_h^2 = 0$,

$$|u(t)|_h^2 = -1.$$

Then we have $u:\mathbb{R}^+\to\mathbb{H}^2$.

For $u_0=\mathbf{n}$, we have a trivial solution $u(t)=\mathbf{n}$. From now on, we assume that $u\neq\mathbf{n}$. Then we can assume that

$$u=(x_1,x_2,x_3)=(\sinh\psi\cos\theta,\ \sinh\psi\sin\theta,\ \cosh\psi),$$

where $0<\psi<\infty$ and θ are functions of t. Then the third component of (3.1) leads us to

$$\sinh\psi\frac{d\psi}{dt} = g(1-\cosh^2\psi), \tag{3.2}$$

from which we derive

$$\cosh\psi(t) = \frac{(\cosh\psi_0+1)\exp\big(2\int_0^t g(\tau)d\tau\big)+\cosh\psi_0-1}{(\cosh\psi_0+1)\exp\big(2\int_0^t g(\tau)d\tau\big)+1-\cosh\psi_0},$$

where $0<\psi_0=\psi(0)<\infty$ is an initial value. Making use of (3.2), the first and second components in (3.1) reduce to $\frac{d\theta}{dt}=f$ from which we derive $\theta(t)=\theta_0+\int_0^t f(\tau)d\tau$ for the initial value $\theta_0=\theta(0)$.

The function ψ is determined by g only and θ by f only. For the positive function g satisfying $\int_0^t g(\tau)d\tau\to\infty$, we have $\cosh\psi\to 1$ which implies $u\to\mathbf{n}$ as $t\to\infty$. The equation (3.2) may have a finite time blow-up. For instance, we have, for $g=-1$,

$$\frac{d\psi}{dt} = \sinh\psi,$$

which has a blow-up solution ψ for the initial value $\psi_0>0$.

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