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UNIQUENESS OF MEROMORPHIC FUNCTION WITH ITS LINEAR DIFFERENTIAL POLYNOMIAL SHARING TWO VALUES

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Abstract. The paper has been devoted to study the uniqueness problem of meromorphic function and its linear differential polynomial sharing two values. We have pointed out gaps in one of the theorem due to [1]. We have further extended the corrected form of Chen-Li-Li's result which in turn extend the an earlier result of [8] in a large extent. In fact, we have subtly use the notion of weighted sharing of values in this particular section of literature which was unexplored till now. A handful number of examples have been provided by us pertinent to different discussions. Specially we have given an example to show that one condition in a theorem can not be dropped.

1. Introduction

Throughout the paper by $\mathbb{C}$ and $\mathbb{N}$ we respectively mean the set of all complex numbers and natural numbers. We denote $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. By any meromorphic function f we always mean that it is defined on $\mathbb{C}$. For any non-constant meromorphic function $h(z)$ we define $S(r,h) = o(T(r,h))$, $(r \to \infty, r \notin E)$, where E denotes any set of positive real numbers having finite linear measure. We follow the standard notation of Nevanlinna theory as given in [4]. According to the same theory, for a non-constant meromorphic function, $T(r,f)$ denotes the Nevanlinna characteristic function, $N(r,\frac{1}{f-a}) = N(r,a;f)$ $(\overline{N}(r,\frac{1}{f-a}) = \overline{N}(r,a;f))$ denotes the counting function (reduced counting function) of a-points of f. On the other hand, we use $N(r,f) = N(r,\infty;f)$ $\big(\overline{N}(r,f) = \overline{N}(r,\infty;f)\big)$ to denote counting (reduced counting) function of poles of f. Let us define $\chi_{_k} = \begin{cases} 1, & \text{if } k = 1 \\ k+1, & \text{if } k \geq 2 \end{cases}$: The following definitions are used in the paper.

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Definition 1.1 ([5])**.** For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a-points of f whose multiplicities are not greater (less) than m where each a-point is counted according to its multiplicity. $\overline{N}(r, a; f \mid\leq m)(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also $N(r, a; f \mid< m), N(r, a; f \mid> m), \overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined similarly.

The notation $E(a, f) \subseteq E(a, g)$ means if z_0 is a zero of $f-a$ with multiplicity p the z_0 is also a zero of $g - a$ with multiplicity at least p.

In the middle of 2001 the gradation of sharing known as weighted sharing was introduced by Lahiri in [6] as follows.

Definition 1.2 ([6])**.** Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity p is counted p times if $p \leq k$ and $k + 1$ times if $p > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, l) for any integer l such that $0 \leq l < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.3 ([7])**.** For a set $S \subset \mathbb{C} \cup \{\infty\}$ we define $E_f(S, k) = \cup_{a \in S} E_k(a; f)$, where k is a nonnegative integer or infinity. Clearly $E_f(S) = E_f(S, \infty)$. If $E_f(S, k) = E_g(S, k)$, we say that f, g share the set S with weight k.

Inspired by the famous Five value and Four value theorems of R. Nevanlinna, in [12], Rubel-Yang first showed that two shared values are enough to make a non constant entire function f and its first derivative $f^{(1)}$ identical. Their results were as follows:

Theorem A ([12])**.** *Let f be a non-constant entire function. If f and $f^{(1)}$ share two distinct values (a, ∞), (b, ∞), then $f \equiv f^{(1)}$.*

In 1979, Mues-Steinmetz [10] reduced the sharing condition in Theorem A from CM to IM. Their result is the following.

Theorem B ([10])**.** *Let f be a non-constant entire function. If f and $f^{(1)}$ share two distinct values $(a, 0)$, $(b, 0)$, then $f \equiv f^{(1)}$.*

In 2000, Li-Yang [9] obtained the following result which settled the conjecture of Frank et al. [2] and extended the result Theorem B.

Theorem C ([9])**.** *Let f be a non-constant entire function. If f and $f^{(k)}$ share two distinct values $(a, 0)$, $(b, 0)$, then $f \equiv f^{(k)}$.*

In the mean time Mues-Steinmetz [11] and Gundersen [3] independently investigated about the uniqueness problem of non constant meromorphic function, f with its derivative $f^{(1)}$ corresponding to two CM shared values.

Theorem D ([3, 11]). *Let f be a non-constant meromorphic function. If f and $f^{(1)}$ share values (a,∞), (b,∞), then $f \equiv f^{(1)}$.*

In 2006, Tanaiadchawoot [13] tackle Theorem D under one CM and one IM shared value and proved the following result.

Theorem E ([13]). *Let f be a non-constant meromorphic function, a, b be nonzero distinct finite complex constant. If f and $f^{(1)}$ share (a,∞), $(b,0)$ and $\overline{N}(r,f) = S(r,f)$, then $f \equiv f^{(1)}$.*

For two IM shared values, in 2013, Li [8] obtained the following result.

Theorem F ([8]). *Let f be a non-constant meromorphic function such that $\overline{N}(r,f) < \lambda T(r,f)$, where $\lambda \in [0, \frac{1}{9})$, and a, b be two distinct finite values. If f and $f^{(1)}$ share $(a,0)$, $(b,0)$, then $f \equiv f^{(1)}$.*

However, Frank et al. [2] (see [9]) investigated the uniqueness of a meromorphic function f and its k-th derivative $f^{(k)}$ sharing two values CM without any additional suppositions. Below we recall the result of Frank et al.

Theorem G ([2]). *Let f be a non-constant meromorphic function. If f and $f^{(k)}$ share distinct finite values (a,∞), (b,∞), then $f \equiv f^{(k)}$.*

In 2018, Chen et al. [1] made a major breakthrough by investigating Theorem G under IM shared values and presented the following two results.

Theorem H ([1]). *Let f be a non-constant meromorphic function and k be a positive integer. If $\overline{N}(r,f) < T(r,f)/(3k+1)$, f and $f^{(k)}$ share two different non-zero values $(a,0), (b,0)$, then $f \equiv f^{(k)}$.*

Theorem I ([1]). *Let f be a non-constant meromorphic function, and k be a positive integer. If $\overline{N}(r,f) < T(r,f)/(3k^2+4k+2)$, f and $f^{(k)}$ share $(0,0)$, $(a,0)$, where $a \neq 0$ and $E(0,f) \subseteq E(0,f^{(k)})$, $E(1,f) \subseteq E(1,f^{(k)})$, then $f \equiv f^{(k)}$.*

Remark 1.1. The results obtained in Theorem H and Theorem I are really praiseworthy in the context of Li's [8] own result. But unfortunately the statement as well as in the proof of Theorem H are not flawless. Actually, we would like to point out that in the proof of Theorem H (p. 380), to deduce the inequality

$$T(r,f) \leq m\left(r, \frac{1}{f^{(k)}}\right) + k\overline{N}(r,f) + S(r,f),$$

Chen et al. [1] used the following two inequalities

$$\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) \leq T(r,f) + k\overline{N}(r,f) + S(r,f),$$
$$m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f-b}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r,f).$$

But this fact is true under the hypothesis that f and $f^{(k)}$ have only simple a, b points. Consequently for general meromorphic functions Theorem H cease to be hold.

Remark 1.2. In Theorem I ([1, p. 379]) the hypothesis that f and $f^{(k)}$ share $(0,0)$, $(1,0)$ and $E(0,f) \subseteq E(0,f^{(k)})$, $E(1,f) \subseteq E(1,f^{(k)})$. Let z_0 be a zero of f with multiplicity $p+k$ $(p \geq 1)$ then z_0 is a zero of $f^{(k)}$ with multiplicity p but this can not happen as $E(0,f) \subseteq E(0,f^{(k)})$. If z_0 is a zero of f with multiplicity k, then by Taylor series expansion we see that z_0 is not a zero of $f^{(k)}$, but this again contradicts $E(0,f) \subseteq E(0,f^{(k)})$. So f will not have any zero of multiplicity exactly k. Thus we can see that $E(0,f) \subseteq E(0,f^{(k)})$ implies that the multiplicity of zeros of f is always $\leq k-1$. When $k=1$, then as f and $f^{'}$ share $(0,0)$, we see that 0 is a Picard exceptional value of both f and $f^{'}$, and so f and $f^{'}$ actually share $(0,\infty)$. So for the case $k=1$, the sharing conditions as imposed in Theorem I means that f and $f^{'}$ share $(0,\infty)$, $(1,0)$ together with $E(1,f) \subseteq E(1,f^{'})$. However, it is to be noted that Theorem F is obtained under much weaker sharing hypothesis and hence for $k=1$ Theorem I is of no importance.

Next we introduce linear differential polynomial of a meromorphic function f denoted by $L(f)$ and defined as follows:

$$L(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_1 f^{(1)} + a_0 f,$$

where $a_i, i = 0,1,\ldots,k$ are the complex constants and $a_k \neq 0$.

The above two remarks will definitely motivate the researchers to explore the relation between f and its linear differential polynomial under relaxed sharing hypothesis. The purpose of writing the paper is to provide the corrected and extended forms of Theorem H and to present Theorem I in a systematic manner under the assumption of some different sharing conditions, which can also accommodate all the previous results presented so far in some sense.

Notice that the uniqueness of f and $f^{(k)}$ is actually the following differential equation

$$f^{(k)}(z) - f(z) = 0. \tag{1.1}$$

It is easy to see that one of the solutions of the above differential equations given in Theorems A-G are of the form

$$f(z) = \sum_{i=1}^{n} A_i e^{\theta_i z}, \tag{1.2}$$

where A_i are complex constants and θ_i are the k-th roots of unity. One immediate question comes out from the above discussion is that if we replace $f^{(k)}$ by its most general form $L(f)$, whether the same conclusion occurs and naturally this query demands further attention.

Note 1.1: Consider $L(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_1 f^{(1)} + a_0 f$. When $a_0 = 1$, then one of the solutions of $f = L(f)$ may be $f = P(z)$, where $P(z)$ is a polynomial of degree $\leq k-1$. On the other hand, if $a_0 \neq 1$, then one of the solutions of $f = L(f)$ is of the form $f = \sum_{j=1}^{k} A_j e^{\theta_j z}$, where A_j are complex constants, not all zero and θ_j are the roots of the equation $a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0 = 1$.

To find under which sufficient condition $L(f)$ becomes identical to f or in other words f takes the form as mentioned in Note 1.1, is the main motivation of this paper. To this end, we present three theorems which investigate the effect of weighted sharing on Theorem G under $L(f)$.

Theorem 1.1. *Let f be a non-constant meromorphic function such that f, $L(f)$ share the value $(0,k)$ and (b,k) and satisfies $\frac{(k+1)^2}{k}\overline{N}(r,f) < T(r,f)$. Then $f \equiv L(f)$.*

The following examples show that in Theorem 1.1, two value sharing can not be relaxed to one value sharing.

Example 1.1. Consider $f = e^{\lambda z}$ and $L(f) = \sum_{j=0}^{k} \lambda^{k-j} f^{(j)} = k e^{\lambda z}$, where $\lambda^k = 1$. Then f and $L(f)$ share $(0,\infty)$ but $f \not\equiv L(f)$.

Example 1.2. Consider $f = Ae^{\lambda z} + \frac{B}{2}$ and $L(f) = f^{(k)} + \lambda^{k-j} f^{(j)} = 2Ae^{\lambda z}$, where A, B be two non zero complex numbers and $j = 1, 2, \ldots, k-1$; $\lambda^k = 1$. Then f and $L(f)$ share (B,∞) but $f \not\equiv L(f)$.

Example 1.3. Let $f = e^{\lambda z} + \lambda - 1$, where $\lambda^{k-1} = 1$ and $L(f) = \sum_{j=1}^{k} f^{(j)} = \lambda e^{\lambda z}$. Then f and $L(f)$ share (λ,∞) but $f \not\equiv L(f)$.

Example 1.4. Consider $f = e^{\lambda z} + A$ and $L(f) = \lambda^{k-1} f^{(1)} + f = 2e^{\lambda z} + A$, where A is a non zero complex number and λ is the k-th root of unity. Then f and $L(f)$ share (A,∞) but $f \not\equiv L(f)$.

The following examples show that the conclusion of Theorem 1.1 cease to hold for one CM and one IM shared values.

Example 1.5. Consider $f = a + a(1 \pm e^{\lambda z})^2$ and $L(f) = \frac{1}{2} f^{(k)} - \lambda f^{(k-1)} = \mp a e^{\lambda z}$, where a is a non zero complex number and λ is the k-th root of unity; $k \geq 2$. Then f and $L(f)$ share $(2a,\infty)$ and $(a,0)$ but $f \not\equiv L(f)$.

Example 1.6. Consider $f = (1-e^z)^2$ and $L(f) = \frac{1}{2} f^{(1)} - f = e^z - 1$. Then it is clear that f and $L(f)$ share $(1,\infty)$ and $(0,0)$ but $f \not\equiv L(f)$.

Example 1.7. Consider $f = (1-e^z)^2$ and $L(f) = -\frac{1}{4}\left(f^{(3)} - 3f^{(2)} + 4f\right) = e^z - 1$. Then it is clear that f and $L(f)$ share $(1,\infty)$ and $(0,0)$ but $f \not\equiv L(f)$.

Example 1.8. Consider $f = 3e^{2z} - 6e^z + 1$ and $L(f) = -2f^{(2)} + \frac{13}{2} f^{(1)} - 5f = 3e^z - 5$. Then f and $L(f)$ share $(1,\infty)$ and $(-2,0)$ but $f \not\equiv L(f)$.

From the next three examples we show that the conclusion of Theorem 1.1 does not hold even if two shared values is replaced by a set with two elements.

Example 1.9. Consider $f = e^{\lambda z} + \lambda$, where $\lambda^k = 1$. Then choosing $L(f) = \sum_{j=1}^{k-1} f^{(j)} = -e^{\lambda z}$, it is clear that f and $L(f)$ share the set $\{0, \lambda\}$ CM but $f \not\equiv L(f)$.

Example 1.10. Consider $f = e^{\lambda z} + a + b$ and $L(f) = \frac{1}{k}\sum_{j=0}^{k-1} \lambda^j f^{(k-j)} = -e^{\lambda z}$, where $\lambda^k = -1$. Then it is easy to verify f and $L(f)$ share the set $\{a, b\}$ CM but $f \not\equiv L(f)$.

Example 1.11. Consider $f = \cos z$. Then for $m, n \in \mathbb{N}$, choosing $L(f) = -\frac{1}{2}f^{(4m-1)} + \frac{1}{2}f^{(4n-3)} = -\sin z$, it is easy to verify that f and $L(f)$ share the set $\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$ but $f \not\equiv L(f)$.

The next theorem improve Theorem F for two distinct complex numbers a and b with $ab \neq 0$.

Theorem 1.2. *Let f be a non-constant meromorphic function with $\overline{N}(r, f) < \lambda T(r, f)$; $0 \leq \lambda < \frac{1}{3k^2+4k+2}$, such that f and $L(f)$ share $(a, k-1)$ and $(b, k-1)$ with $a \cdot b \neq 0$ and $a_0 = 0$, then $f \equiv L(f)$.*

The next example shows that in Theorem 1.2, the condition $a_0 = 0$ is essential.

Example 1.12. Consider $f = \cos z$. Then choosing $L(f) = -if^{(1)} + f = i \sin z + \cos z$, it is easy to verify that f and $L(f)$ share $(-1, 0)$ and $(1, 0)$ but $f \not\equiv L(f)$.

In the following theorem we consider the case for $ab = 0$, in Theorem 1.2.

Theorem 1.3. *Let f be a non-constant meromorphic function with $\overline{N}(r, f) < \lambda T(r, f)$; $0 \leq \lambda < \frac{1}{3k^2+4k+2}$, such that f and $L(f)$ share $(0, \chi_k - 1)$ and $(b, k-1)$ with $a_0 = 0$, then $f \equiv L(f)$.*

Remark 1.3. For any two distinct complex numbers a, b combining Theorems 1.2 and 1.3 one can get improved and extended version of Li's [8] result.

The following examples show that when $k \geq 2$, the conclusion of Theorem 1.2 does not hold good for two IM shared values.

Example 1.13. Consider $f = \frac{a^2}{2}e^{\lambda z} + \frac{1}{2}e^{-\lambda z}$ and $L(f) = f^{(k)} + \lambda^{k-j}f^{(j)} = a^2 e^{\lambda z} + \frac{(-1)^k+(-1)^j}{2}e^{-\lambda z}$, where a is non zero complex constant and $j \in \{k-1, k-3, k-5, \ldots, 2\}$ or $\{k-1, k-3, k-5, \ldots, 1\}$ according as k is odd or even and $\lambda^k = 1$. Then f and $L(f)$ share $(-a, 0)$ and $(a, 0)$ but $f \not\equiv L(f)$.

Example 1.14. Consider $f = \frac{1}{2}e^{\lambda z} + \frac{a^2}{2}e^{-\lambda z}$ and $L(f) = f^{(k)} + \lambda^{k-j}f^{(j)} = e^{\lambda z} + \frac{a^2\left[(-1)^k+(-1)^j\right]}{2}e^{-\lambda z}$, where a is non zero complex constant and $j \in \{k-1, k-3, k-5, \ldots, 2\}$ or $\{k-1, k-3, k-5, \ldots, 1\}$ according as k is odd or even and $\lambda^k = 1$. Then it is easy to verify that f and $L(f)$ share $(-a, 0)$ and $(a, 0)$ but $f \not\equiv L(f)$.

The following example shows that in Theorem 1.2 the weight $k-1$ of sharing can not be further reduced at least for the case $k=2$.

Example 1.15. Consider $f=\cos z$. Then for $m,n\in\mathbb{N}$, choosing $L(f)=if^{(4m-1)}-\frac{1}{2}f^{(4n-2)}-\frac{1}{2}f''=i\sin z+\cos z$, it is easy to verify that f and $L(f)$ share $(-1,0)$ and $(1,0)$ but $f\not\equiv L(f)$.

2. Lemma

Lemma 2.1. *Let f be a non-constant meromorphic function such that f and $L(f)$ with $a_0=0$ share $(b,0)$, where $b\neq 0$ and satisfies $\overline{N}(r,f)<\lambda T(r,f)+S(r,f)$ and $\lambda\in[0,1), k\in\mathbb{Z}^+$. Then*

$$\overline{N}\left(r,\frac{1}{f-b}\right)>\frac{1-\lambda}{k+1}T(r,f)+S(r,f).$$

Proof. We note that as f and $L(f)$ share $(b,0)$, then multiplicity of b-points of f is at most k. Now

$$\begin{aligned}(2.1)\qquad & m\left(r,\frac{1}{f}\right)+m\left(r,\frac{1}{L(f)-b}\right)\\ &\leq m\left(r,\frac{1}{L(f)}\right)+m\left(r,\frac{L(f)}{f}\right)+m\left(r,\frac{1}{L(f)-b}\right)+S(r,f)\\ &\leq m\left(r,\frac{1}{L(f)}\right)+m\left(r,\frac{1}{L(f)-b}\right)+S(r,f)\\ &\leq m\left(r,\frac{1}{(L(f))^{(1)}}\right)+S(r,f).\end{aligned}$$

So by (2.1) we get

$$\begin{aligned}(2.2)\qquad & T(r,f)+T\left(r,L(f)\right)\\ &\leq N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{L(f)-b}\right)+T\left(r,\frac{1}{(L(f))^{(1)}}\right)\\ &\quad -N\left(r,\frac{1}{(L(f))^{(1)}}\right)+S(r,f)\\ &\leq N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{L(f)-b}\right)+T\left(r,(L(f))^{(1)}\right)\\ &\quad -N\left(r,\frac{1}{(L(f))^{(1)}}\right)+S(r,f).\end{aligned}$$

From Milloux Inequality we have

$$(2.3)\qquad T\left(r,(L(f))^{(1)}\right)\leq T\left(r,L(f)\right)+\overline{N}(r,f)+S(r,f).$$

Note that

$$(2.4)\qquad N\left(r,\frac{1}{L(f)-b}\right)-N\left(r,\frac{1}{(L(f))^{(1)}}\right)$$

$$\leq \overline{N}\left(r, \frac{1}{L(f)-b}\right) - N_0\left(r, \frac{1}{(L(f))^{(1)}}\right),$$

where $N_0\left(r, \frac{1}{(L(f))^{(1)}}\right)$ is the counting function of those zeros of $(L(f))^{(1)}$ which are not b-point of $L(f)$. Now combining (2.2), (2.3), (2.4) we get

$$\begin{aligned}T(r,f) \leq \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{L(f)-b}\right)\\ - N_0\left(r, \frac{1}{(L(f))^{(1)}}\right) + S(r,f).\end{aligned} \tag{2.5}$$

Now we set $g = f - b$. Clearly

$$\overline{N}\left(r, \frac{1}{g}\right) = \overline{N}\left(r, \frac{1}{f-b}\right) = \overline{N}\left(r, \frac{1}{L(f)-b}\right) = \overline{N}\left(r, \frac{1}{L(g)-b}\right). \tag{2.6}$$

If z_1 is a zero of $L(g) - b$ with order t, then z_1 is counted $t-1$ times in $N\left(r, \frac{1}{(L(g))^{(1)}}\right)$. Note that each zero of g is of order at most k. Now using (2.5) for the function g and (2.6) we get

$$\begin{aligned}T(r,f) &\leq T(r,g) + O(1)\\ &\leq \overline{N}(r,g) + N(r, \frac{1}{g}) + \overline{N}\left(r, \frac{1}{L(g)-b}\right) - N_0\left(r, \frac{1}{(L(g))^{(1)}}\right) + S(r,g)\\ &\leq \overline{N}(r,f) + N(r, \frac{1}{g}) + \overline{N}\left(r, \frac{1}{L(g)-b}\right) + S(r,f)\\ &< \lambda T(r,f) + k\overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S(r,f)\\ &= \lambda T(r,f) + (k+1)\overline{N}\left(r, \frac{1}{g}\right) + S(r,f).\end{aligned}$$

Therefore

$$\overline{N}\left(r, \frac{1}{f-b}\right) > \frac{1-\lambda}{k+1} T(r,f) + S(r,f). \qquad \square$$

3. Proofs of the theorems

Proof of Theorem 1.1. Let us assume $f \not\equiv L(f)$. The condition that f and $L(f)$ share the values $(0,k)$ and (b,k), implies the zeros of f are of multiplicity $\geq 2k+1$. Let us consider the function

$$\phi = \frac{f^{(1)}(f - L(f))}{f(f-b)}. \tag{3.1}$$

We see that

$$m(r,\phi) \leq m\left(r, \frac{f^{(1)}}{f-b}\right) + m\left(r, 1 - \frac{L(f)}{f}\right) = S(r,f).$$

Now

$$\begin{aligned} m\left(r,\frac{1}{f}\right) &\leq m\left(r,\frac{1}{\phi}\right)+m\left(r,\left(\frac{f^{(1)}(f-L(f))}{f^2(f-b)}\right)\right)+S(r,f)\\ &\leq m\left(r,\frac{1}{\phi}\right)+m\left(r,\frac{f^{(1)}}{b}\left\{\frac{1}{f-b}-\frac{1}{f}\right\}\right)+m\left(r,\left(1-\frac{L(f)}{f}\right)\right)\\ &\quad+S(r,f)\\ &\leq m\left(r,\frac{1}{\phi}\right)+S(r,f). \end{aligned}$$

Let z_0 be a zero of f with multiplicity $p \geq 2k+1$ then z_0 is also a zero of ϕ with multiplicity $p-(k+1)$. It is easy to see that the b-point of f will not contribute to any zero or pole of ϕ. On the other hand, if z_1 is a pole of f with multiplicity q, then z_1 is also a pole of ϕ with multiplicity $(k+1)$.

Now

$$\begin{aligned} &N\left(r,\frac{1}{f}\right)-(k+1)\overline{N}\left(r,\frac{1}{f}\right) \\ &\leq N\left(r,\frac{1}{\phi}\right)+S(r,f)\\ &\leq T\left(r,\frac{1}{\phi}\right)-m\left(r,\frac{1}{\phi}\right)+S(r,f)\\ &\leq N(r,\phi)+m(r,\phi)-m\left(r,\frac{1}{\phi}\right)+S(r,f)\\ &\leq (k+1)\overline{N}(r,f)+m(r,\phi)-m\left(r,\frac{1}{\phi}\right)+S(r,f).\\ &\leq (k+1)\overline{N}(r,f)-m\left(r,\frac{1}{f}\right)+S(r,f). \end{aligned} \tag{3.2}$$

Therefore by (3.2)

$$\begin{aligned} T(r,f) &= T\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq (k+1)\overline{N}\left(r,\frac{1}{f}\right)+(k+1)\overline{N}(r,f)+S(r,f). \end{aligned} \tag{3.3}$$

Next we consider $V=\frac{(L(f))^{(1)}}{L(f)-b}-\frac{f^{(1)}}{f-b}$.

It is clear that $V \not\equiv 0$, since otherwise $f \equiv L(f)$. As f and $L(f)$ share $(0,k)$, it follows that f has no zeros of order $\leq 2k$ and $L(f)$ has no zeros of order $\leq k$. Also f and $L(f)$ share (b,k), it follows that the b-points of f and $L(f)$ are equal (counting multiplicity) up to multiplicity k. Now it is easy to check that f and $L(f)$ will not have any b-points of multiplicity $\geq k+1$. In other words, f and $L(f)$ share the value (b,∞) and this implies $N(r,V) \leq \overline{N}(r,f)+S(r,f)$.

Thus we get

$$k\overline{N}\left(r,0;f|\geq 2k+1\right)\leq N\left(r,\frac{1}{V}\right)\leq T(r,V)+O(1) \tag{3.4}$$
$$\leq N(r,V)+S(r,f)\leq \overline{N}(r,f)+S(r,f).$$

Combining (3.3) and (3.4) we get

$$T(r,f)\leq \frac{k+1}{k}\overline{N}(r,f)+(k+1)\overline{N}(r,f)+S(r,f) \tag{3.5}$$
$$=\frac{(k+1)^2}{k}\overline{N}(r,f)+S(r,f).$$

Clearly (3.5) contradicts the given condition. Therefore $f\equiv L(f)$. □

Proof of Theorem 1.2. Let us assume $f\not\equiv L(f)$. Consider the function

$$\psi=\frac{L(f)\left(L(f)-f\right)}{(f-a)(f-b)}. \tag{3.6}$$

Note that if z_0 is a pole of f with multiplicity p, then z_0 is also a pole of ψ with multiplicity $2k$. As f and $L(f)$ share $(a,k-1)$, $(b,k-1)$ it is clear that the a (b)-points will not contribute any pole of ψ. Now

$$\psi=\frac{L(f)}{f-a}\sum_{j=1}^{k}\frac{a_jf^{(j)}}{f-b}-\frac{1-a_0}{a-b}\left(\frac{aL(f)}{f-a}-\frac{bL(f)}{f-b}\right).$$

It is easy to see that

$$m(r,\psi)=S(r,f). \tag{3.7}$$

From (3.6) we get that $\psi(f^2-(a+b)f+ab)=L(f)\left(L(f)-f\right)$. Differentiating both sides we get

$$\psi^{(1)}(f^2-(a+b)f+ab)+\psi(2ff^{(1)}-(a+b)f^{(1)}) \tag{3.8}$$
$$=(L(f))^{(1)}\left(L(f)-f\right)+L(f)\left((L(f))^{(1)}-f^{(1)}\right).$$

Let z_1 be a b-point of f. As f and $L(f)$ share $(b,k-1)$ then $f(z_1)=L(f)(z_1)=b$. Using it in (3.8) we get

$$(L(f))^{(1)}(z_1)=\left(1+\left(1-\frac{a}{b}\right)\psi(z_1)\right)f^{(1)}(z_1).$$

Now we consider a function

$$\sigma=\frac{\left((L(f))^{(1)}-\left(1+\left(1-\frac{a}{b}\right)\psi\right)f^{(1)}\right)\left(L(f)-f\right)}{(f-a)(f-b)}.$$

Let $g=(L(f))^{(1)}-\left(1+(1-\frac{a}{b})\psi\right)f^{(1)}$. Note that $m(r,\frac{g}{f-a})=S(r,f)$ and $m(r,\frac{g}{f-b})=S(r,f)$. We can write

$$\sigma=\frac{g}{f-a}\sum_{j=1}^{k}\frac{a_jf^{(j)}}{f-b}-\frac{1-a_0}{a-b}\left(\frac{ag}{f-a}-\frac{bg}{f-b}\right).$$

Thus $m(r,\sigma) = S(r,f)$. Notice that poles of f is also poles of σ with multiplicity $3k+1$. Thus

$$T(r,\sigma) = N(r,\sigma) + S(r,f) \leq (3k+1)\overline{N}(r,f) + S(r,f).$$

As f and $L(f)$ share $(a,k-1)$ and $(b,k-1)$, the multiplicity of zeros of $f-b$ is always $\leq k$. If z_1 is a zero of $f-b$ with multiplicity $p \leq k-1$, then z_1 is also zero of $L(f)-f$ with multiplicity exactly p. If z_2 is a zero of $f-b$ with multiplicity k, then z_2 is also zero of $L(f)-b$ with multiplicity $k+j$, where $j \geq 0$. So z_2 be zero of $L(f)-f$ with multiplicity equal to $\min\{k, k+j\} = k$. Thus in both the cases zeros of $f-b$ will not contribute to the zeros or poles of $\frac{L(f)-f}{f-b}$. Note that the zeros of $f-b$ must be a zero of $\left(L(f)\right)^{(1)} - \left(1 + (1-\frac{a}{b})\psi\right) f^{(1)}\big)$ of multiplicity at least one. Thus b-points of f must be the zeros of σ. So

$$\begin{aligned}(3.9) \qquad \overline{N}\left(r, \frac{1}{f-b}\right) \leq N\left(r, \frac{1}{\sigma}\right) &\leq T(r,\sigma) \leq (3k+1)\overline{N}(r,f) + S(r,f) \\ &< \lambda(3k+1)T(r,f) + S(r,f).\end{aligned}$$

By Lemma 2.1 we have

$$(3.10) \qquad \overline{N}\left(r, \frac{1}{f-b}\right) > \frac{1-\lambda}{k+1}T(r,f) + S(r,f).$$

Combining (3.9) and (3.10) we get

$$(3.11) \qquad \frac{1-\lambda}{k+1} < \lambda(3k+1) \implies \lambda > \frac{1}{3k^2+4k+2}.$$

Clearly (3.11) contradicts the given condition. Therefore $f \equiv L(f)$. ☐

Proof of Theorem 1.3. For the case $k=1$, we refer to [8]. So we consider the case $k \geq 2$. Clearly f, $L(f)$ share the value $(0,k)$ and a non-zero value $(b,k-1)$. Let us assume $f \not\equiv L(f)$. We follow the similar procedure as adopted in the proof of Theorem 1.2. Here only the functions ψ and σ will be constructed as follows:

$$\psi = \frac{L(f)\left(L(f)-f\right)}{f(f-b)},$$

$$\sigma = \frac{\left(\left(L(f)\right)^{(1)} - (1+\psi) f^{(1)}\right)\left(L(f)-f\right)}{f(f-b)}.$$

As f and $L(f)$ share $(0,k)$ so multiplicity of zeros always $\geq 2k+1$. It is easy to see that zeros of f will not contribute any poles of σ. Remaining part of the proof is same as proof of Theorem 1.2. So we omit the details. ☐

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