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A RELATIONSHIP BETWEEN CAYLEY-DICKSON PROCESS AND THE GENERALIZED STUDY DETERMINANT

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Abstract. The Study determinant is known as one of replacements for the determinant of matrices with entries in a noncommutative ring. In this paper, we give a generalization of the Study determinant and show its relationship with the Cayley-Dickson process. We also give some properties of a non-associative ring obtained by the Cayley-Dickson process with a not necessarily commutative, but associative ring as the initial ring.

1. Introduction

The Cayley-Dickson process is known as an important tool to construct a sequence of rings with involution. The well-known example is to produce complex numbers, quaternions, and octonions from real numbers. Yamaguchi [7] investigated the ring obtained by applying the Cayley-Dickson process from a commutative ring, and Flaut [3] used the process from an algebra over field, but not from any arbitrary ring. The Cayley-Dickson process has provided fundamental role for the study of some topics in mathematics, especially the functional analysis, composition algebras, and also in the algebraic geometry. Actually, some algebras which constructed by using the Cayley-Dickson process give contributions in physics, especially on particle physics [5]. For example, the class of supersymmetric oscillators with dimension $N \leq 7$ associated with the algebras obtained by the Cayley-Dickson process was introduced in [2].

The Study determinant was first introduced by Study [6]. His idea was as follow. Let A be a square matrix with size n and entries in $\mathbb{H}$. Define a map Ψ to obtain a square matrix B with entries in $\mathbb{C}$ and size $2n$. Study defined the Study determinant by taking a determinant of matrix B. The properties of the Study determinant have been further studied in [1] and [7].

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In this paper we will first construct a non-associative ring by applying the Cayley-Dickson process to any arbitrary (not necessarily commutative, but associative) ring inductively, and give its matrix representation. At the next step, we will give a generalization of the Study determinant. Lastly, the connection between the Cayley-Dickson process and the generalized Study determinant will be presented in the last section.

2. Cayley-Dickson process

In this section, we will apply the Cayley-Dickson process to an arbitrary associative ring R to obtain a ring R_1. Further, by similar process, we will construct a ring R_2 from R_1. We will define the norms in R_1 and R_2, which play important roles to our result. We will also give the properties of the rings obtained by the Cayley-Dickson process.

Let R be a (not necessarily commutative but associative) ring, and we write $\bar{\cdot} : R \to R$ to mean an involutive anti-automorphism of R. Assume

$$ab - ba + \overline{ab - ba} = 0 \quad (a, b \in R), \tag{2.1}$$

$$\bar{a}a = a\bar{a} \quad (a \in R), \tag{2.2}$$

$$\bar{a}ab = b\bar{a}a \quad (a, b \in R). \tag{2.3}$$

Example 2.1. Let

$$\mathbb{H} = \{x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k} : x, y, z, w \in \mathbb{R}\}$$

be a quaternion ring over real numbers, where

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}, \quad \mathbf{ji} = -\mathbf{k}, \quad \mathbf{kj} = -\mathbf{i}, \quad \mathbf{ik} = -\mathbf{j},$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

Define $\overline{x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}} = x - y\mathbf{i} - z\mathbf{j} - w\mathbf{k}$ for every $x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k} \in \mathbb{H}$. Then $\bar{\cdot} : \mathbb{H} \to \mathbb{H}$ is an involutive anti-automorphism of $\mathbb{H}$. It can be checked that the conditions of (2.1), (2.2), and (2.3) are also satisfied.

Now, by using the Cayley-Dickson process, we will produce a non-associative ring from R. Define

$$R_1 = R \oplus R$$

with the following multiplication:

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}). \tag{2.4}$$

Define $\tau : R_1 \to R_1$ by

$$(a, b)^\tau = (\bar{a}, -b). \tag{2.5}$$

Since $((a, b)^\tau)^\tau = (\bar{a}, -b)^\tau = (a, b)$, τ is involutive. We state properties of τ in several lemmas.

Lemma 2.2. *τ is a ring anti-automorphism.*

Proof. Let (a,b), $(c,d) \in R_1$. Then

$$\begin{aligned}((a,b)+(c,d))^\tau &= (a+c,b+d)^\tau = (\overline{a+c}, -b-d) = (\bar{a},-b)+(\bar{c},-d)\\ &= (a,b)^\tau + (c,d)^\tau.\end{aligned}$$

Also,

$$\begin{aligned}((a,b)(c,d))^\tau &= (ac-\bar{d}b, da+b\bar{c})^\tau = (\overline{ac-\bar{d}b}, -da-b\bar{c}),\\ (c,d)^\tau(a,b)^\tau &= (\bar{c},-d)(\bar{a},-b) = (\bar{c}\bar{a}-\bar{b}d, (-b)\bar{c}+(-d)a),\end{aligned}$$

and hence $((a,b)(c,d))^\tau = (c,d)^\tau(a,b)^\tau$. Thus, the result follows. □

We define *the norm in* R_1 by

$$n((a,b)) = \bar{a}a + \bar{b}b \quad (a,b \in R). \tag{2.6}$$

Then

$$(a,b)^\tau(a,b) = (n(a,b),0) \quad (a,b \in R). \tag{2.7}$$

We have the following property.

Lemma 2.3. *Let* $\alpha,\beta,\gamma \in R_1$. *Then*

(i) $\alpha\beta - \beta\alpha + (\alpha\beta-\beta\alpha)^\tau = 0$,
(ii) $\alpha^\tau\alpha = \alpha\alpha^\tau$,
(iii) $\alpha\alpha^\tau\beta = \beta\alpha\alpha^\tau$,
(iv) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ *if* R *is commutative.*

Proof. Let $\alpha=(a,b)$, $\beta=(c,d)$, $\gamma=(e,f)$ $(a,b,c,d,e,f \in R)$.

(i) Since

$$\begin{aligned}\alpha\beta-\beta\alpha &= (a,b)(c,d)-(c,d)(a,b)\\ &= (ac-\bar{d}b, da+b\bar{c}) - (ca-\bar{b}d, bc+d\bar{a})\\ &= (ac-ca+\bar{b}d-\bar{d}b, da-d\bar{a}+b\bar{c}-bc),\end{aligned}$$

and

$$(\alpha\beta-\beta\alpha)^\tau = (\overline{ac-ca}+\overline{\bar{b}d-\bar{d}b}, -(da-d\bar{a}+b\bar{c}-bc)),$$

we have

$$\begin{aligned}\alpha\beta-\beta\alpha+(\alpha\beta-\beta\alpha)^\tau &= (ac-ca+\bar{b}d-\bar{d}b+\overline{ac-ca}+\overline{\bar{b}d-\bar{d}b}, 0)\\ &= 0 \qquad\qquad \text{(by (2.1))}.\end{aligned}$$

(ii)

$$\begin{aligned}\alpha\alpha^\tau &= (n(\alpha),0) && \text{(by (2.7))}\\ &= (\bar{a}a+\bar{b}b,0) && \text{(by (2.6))}\\ &= (\bar{a}a+\bar{b}b, b\bar{a}+(-b)\bar{a})\\ &= (\bar{a},-b)(a,b)\\ &= \alpha^\tau\alpha.\end{aligned}$$

(iii)

$$\begin{aligned}\alpha\alpha^{\tau}\beta &= (\bar{a}a+\bar{b}b,0)(c,d) && \text{(by (2.7))}\\ &= ((\bar{a}a+\bar{b}b)c, d(\bar{a}a+\bar{b}b))\\ &= (c(\bar{a}a+\bar{b}b), d(\bar{a}a+\bar{b}b)) && \text{(by (2.3))}\\ &= (c,d)(\bar{a}a+\bar{b}b,0)\\ &= \beta\alpha\alpha^{\tau}.\end{aligned}$$

(iv) Since R is commutative and

$$\begin{aligned}(\alpha\beta)\gamma &= ((a,b)(c,d))(e,f)\\ &= (ac-\bar{d}b, da+b\bar{c})(e,f)\\ &= (ace-\bar{d}be-\bar{f}da-\bar{f}b\bar{c}, fac-f\bar{d}b+da\bar{e}-b\bar{c}\bar{e}),\end{aligned}$$

$$\begin{aligned}\alpha(\beta\gamma) &= (a,b)((c,d)(e,f))\\ &= (a,b)(ce-f\bar{f}d, fc-d\bar{e})\\ &= (ace-a\bar{f}d-\bar{f}\bar{c}b-\bar{d}eb, fca+d\bar{e}a+b\bar{c}\bar{e}-bf\bar{d}),\end{aligned}$$

the result follows. □

Clearly, if R is commutative, then R_1 is associative.

Lemma 2.4. *The norm in R_1 has the multiplicative property.*

Proof. Let $\alpha=(a,b)$, $\beta=(c,d)$, where $a,b,c,d\in R$. By (2.1), we have

$$c(\bar{b}da)-(\bar{b}da)c+\overline{c(\bar{b}da)-(\bar{b}da)c}=0. \tag{2.8}$$

Thus the result follows from (2.1) and (2.3). □

Next, we extend R_1 to a non-associative ring R_2 by using the Cayley-Dickson process. Similar with the previous procedure, we define $R_2=R_1\oplus R_1$ with the following multiplication

$$(\alpha,\beta)(\gamma,\delta)=(\alpha\gamma-\delta^{\tau}\beta, \delta\alpha+\beta\gamma^{\tau}) \quad (\alpha,\beta,\gamma,\delta\in R_1). \tag{2.9}$$

Hence, if

$$\alpha=(a,b),\quad \beta=(c,d),\quad \gamma=(e,f),\quad \delta=(g,h),\quad (a,b,c,d,e,f,g,h\in R),$$

then (2.9) becomes

$$\begin{aligned}(\alpha,\beta)(\gamma,\delta) = ((&ae-\bar{f}b-\bar{g}c-\bar{d}h, fa+b\bar{e}-d\bar{g}+h\bar{c}),\\ &(ga-\bar{b}h+c\bar{e}+\bar{f}d, bg+h\bar{a}-fc+de)).\end{aligned} \tag{2.10}$$

We remark here that the equation below has a relation with the special case of Lagrange's identity (see [8] for more information related to the particular case of Lagrange's identity). That is, let

$$\begin{aligned} q&=ae-\bar{f}b-\bar{g}c-\bar{d}h, \quad & r&=fa+b\bar{e}-d\bar{g}+h\bar{c},\\ s&=ga-\bar{b}h+c\bar{e}+\bar{f}d, \quad & t&=bg+h\bar{a}-fc+de.\end{aligned}$$

Then $q\bar{q}+r\bar{r}+s\bar{s}+t\bar{t}=(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d})(e\bar{e}+f\bar{f}+g\bar{g}+h\bar{h})$. Furthermore, Flaut and Shpakivskyi [4] discovered some identities that can be also obtained from the Cayley-Dickson process.

Now, define $*: R_2 \to R_2$ by $(\alpha,\beta)\mapsto(\alpha^\tau,-\beta)$ for every $\alpha,\beta\in R_1$. It is easy to see that $*$ is an involutive anti-automorphism of R_2 by Lemma 2.2. We note here that similar property of Lemma 2.3(i)–(iv) also holds in R_2. Finally, define $N: R_2 \to R$, *the norm in* R_2, by $N((a,b),(c,d)) = a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}$. This will help us to see the connection between the Cayley-Dickson process and the generalized Study determinant in the next section.

3. A generalization of the Study determinant

In this section, we will give a matrix representation of the Cayley-Dickson process of previous section and a generalization of the Study determinant.

Firstly, define

$$\widetilde{R}=\left\{\begin{bmatrix}\alpha & \beta\\ -\beta^\tau & \alpha^\tau\end{bmatrix} \mid \alpha,\beta\in R_1\right\}.$$

The addition in $\widetilde{R}$ is the standard matrix addition and the multiplication in $\widetilde{R}$ is defined by

$$\begin{aligned}\begin{bmatrix}\alpha & \beta\\ -\beta^\tau & \alpha^\tau\end{bmatrix}\star\begin{bmatrix}\gamma & \delta\\ -\delta^\tau & \gamma^\tau\end{bmatrix} &= \begin{bmatrix}\alpha\gamma-\delta^\tau\beta & \delta\alpha+\beta\gamma^\tau\\ -\gamma\beta^\tau-\alpha^\tau\delta^\tau & -\beta^\tau\delta+\gamma^\tau\alpha^\tau\end{bmatrix}\\ &= \begin{bmatrix}\alpha\gamma-\delta^\tau\beta & \delta\alpha+\beta\gamma^\tau\\ -(\delta\alpha+\beta\gamma^\tau)^\tau & (\alpha\gamma-\delta^\tau\beta)^\tau\end{bmatrix}\end{aligned}\tag{3.1}$$

for $\alpha,\beta,\gamma,\delta\in R_1$. The operation $\star$ gives the multiplication operation from the Cayley-Dickson process in the first row of R_1 and the form of the conjugate product in the second row. Define $\sigma:\widetilde{R}\to\widetilde{R}$ by

$$\left(\begin{bmatrix}\alpha & \beta\\ -\beta^\tau & \alpha^\tau\end{bmatrix}\right)^\sigma = \begin{bmatrix}\alpha^\tau & -\beta\\ \beta^\tau & \alpha\end{bmatrix}$$

for every $\alpha,\beta\in R_1$. It is easy to see that σ is an involutive anti-automorphism of $\widetilde{R}$. Indeed, let $\alpha,\beta,\gamma,\delta\in R_1$. Then

$$\left(\begin{bmatrix}\alpha & \beta\\ -\beta^\tau & \alpha^\tau\end{bmatrix}\star\begin{bmatrix}\gamma & \delta\\ -\delta^\tau & \gamma^\tau\end{bmatrix}\right)^\sigma = \left(\begin{bmatrix}\alpha\gamma-\delta^\tau\beta & \delta\alpha+\beta\gamma^\tau\\ -(\delta\alpha+\beta\gamma^\tau)^\tau & (\alpha\gamma-\delta^\tau\beta)^\tau\end{bmatrix}\right)^\sigma.$$

On the other hand,

$$\begin{aligned}\left(\begin{bmatrix}\gamma & \delta\\ -\delta^\tau & \gamma^\tau\end{bmatrix}\right)^\sigma\star\left(\begin{bmatrix}\alpha & \beta\\ -\beta^\tau & \alpha^\tau\end{bmatrix}\right)^\sigma &= \begin{bmatrix}\gamma^\tau & -\delta\\ \delta^\tau & \gamma\end{bmatrix}\star\begin{bmatrix}\alpha^\tau & -\beta\\ \beta^\tau & \alpha\end{bmatrix}\\ &= \begin{bmatrix}\gamma^\tau\alpha^\tau-\beta^\tau\delta & -\beta\gamma^\tau-\delta\alpha\\ \alpha^\tau\delta^\tau+\gamma\beta^\tau & -\delta^\tau\beta+\alpha\gamma\end{bmatrix}\\ &= \begin{bmatrix}(-\delta^\tau\beta+\alpha\gamma)^\tau & -(\beta\gamma^\tau+\delta\alpha)\\ (\beta\gamma^\tau+\delta\alpha)^\tau & -\delta^\tau\beta+\alpha\gamma\end{bmatrix}\end{aligned}$$

$$= \left(\begin{bmatrix} -\delta^\tau \beta + \alpha\gamma & \beta\gamma^\tau + \delta\alpha \\ -(\beta\gamma^\tau + \delta\alpha)^\tau & (-\delta^\tau\beta + \alpha\gamma)^\tau \end{bmatrix} \right)^\sigma .$$

So, σ is an anti-automorphism of $\widetilde{R}$. Also,

$$\left(\begin{bmatrix} \alpha^\tau & -\beta \\ \beta^\tau & \alpha \end{bmatrix} \right)^\sigma = \begin{bmatrix} \alpha & \beta \\ -\beta^\tau & \alpha^\tau \end{bmatrix}.$$

Hence, σ is involutive.

Note that if we denote $1 = (1,0)$ and $j = (0,1)$ in R_1, then we use the identification $R = R \oplus 0 \subset R_1$. Also, for every $a, b \in R$, we have $a = (a,0), b = (b,0) \in R_1$ and

$$a + b = (a,0) + (b,0) = (a+b,0) \in R \oplus 0 = R,$$
$$ab = (a,0)(b,0) = (ab,0) \in R \oplus 0 = R.$$

Therefore, the fact that R is a subring of R_1 implies that $M_n(R)$ is a subring of $M_n(R_1)$, where $M_n(A)$ denotes as usual the ring of matrices of size $n \times n$ over a ring A.

Lemma 3.1. *For every $z \in R$,*

$$zj = j\bar{z}. \tag{3.2}$$

Proof. Since $z, \bar{z} \in R$ and R is a subring of R_1, z ($\bar{z}$ resp.) can be written as $(z,0)$ ($(\bar{z},0)$ resp.) $\in R_1$. Therefore,

$$zj = (z,0)(0,1) = (0,z) = (0,1)(\bar{z},0) = j\bar{z}.$$ □

Note that R is a subring of R_1. If $z \in R$, then $z = (z,0) \in R_1$ and $z^\tau = (\bar{z},0) \in R_1$. Therefore

$$zj = (z,0)(0,1) = (0,z) = (0,1)(\bar{z},0) = jz^\tau.$$

Let $z = (a,b) \in R_1$. Then we have $zj = (a,b)(0,1) = (-b,a)$ and $jz^\tau = (0,1)(\bar{a},-b) = (b,a)$. So, if $zj = jz^\tau$, then $b = -b = 0$. Thus, $z = (a,0) = a \in R$. Hence, if $z \in R_1$, then

$$zj = jz^\tau \tag{3.3}$$

if and only if $z \in R \subset R_1$. In other word, Lemma 3.1 shows that $zj = j\bar{z}$ for every $z \in R$. Now, denote $\overline{A} = (\overline{a_{ij}}) \in M_n(R)$ for every $A = (a_{ij}) \in M_n(R)$. Then

$$jA = (ja_{ij}) = (\overline{a_{ij}}j) = \overline{A}j.$$

Also, for every $(a,b) \in R_1$,

$$(a,b) = (a,0) + (0,b) = (a,0) + j(\bar{b},0).$$

Therefore, for every $z \in R_1$, there exist $z_1, z_2 \in R$ such that $z = z_1 + jz_2$. Consequently, for every $Z \in M_n(R_1)$, there exist $Z_1, Z_2 \in M_n(R)$ such that $Z = Z_1 + jZ_2$.

Next, we define

$$\mathcal{J} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Proposition 3.2. *Define* $\Psi : M_n(R_1) \to M_{2n}(R)$ *by*

$$\Psi(A + jB) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}, \qquad (A, B \in M_n(R)).$$

Then Ψ *is an injective homomorphism and*

$$\Psi(M_n(R_1)) = \{N \in M_{2n}(R) \mid \mathcal{J}N = \overline{N}\mathcal{J}\}.$$

Proof. Since for every $A_1, A_2, B_1, B_2 \in M_n(R)$,

$$\begin{aligned}
&\Psi((A_1 + jB_1)(A_2 + jB_2)) \\
&= \Psi(A_1A_2 + A_1jB_2 + jB_1A_2 + jB_1jB_2) \\
&= \Psi((A_1A_2 + j\overline{A_1}B_2 + jB_1A_2 - \overline{B_1}B_2)) \quad \text{(by Lemma 3.1)} \\
&= \Psi(A_1A_2 - \overline{B_1}B_2 + j(\overline{A_1}B_2 + B_1A_2)) \\
&= \begin{pmatrix} A_1A_2 - \overline{B_1}B_2 & -A_1\overline{B_2} + \overline{B_1A_2} \\ B_1A_2 + \overline{A_1}B_2 & -B_1\overline{B_2} + \overline{A_1A_2} \end{pmatrix} \\
&= \begin{pmatrix} A_1 & -\overline{B_1} \\ B_1 & \overline{A_1} \end{pmatrix} \begin{pmatrix} A_2 & -\overline{B_2} \\ B_2 & \overline{A_2} \end{pmatrix} \\
&= \Psi(A_1 + jB_1)\Psi(A_2 + jB_2),
\end{aligned}$$

Ψ is a homomorphism. Furthermore, $Ker(\Psi) = \{0\}$, which shows that Ψ is injective.

Now, suppose

$$N = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in M_{2n}(R_1),$$

where $A, B, C, D \in M_n(R)$. Then we have

$$\mathcal{J}N = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} -B & -D \\ A & C \end{pmatrix}$$

and

$$\overline{N}\mathcal{J} = \begin{pmatrix} \overline{A} & \overline{C} \\ \overline{B} & \overline{D} \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} \overline{C} & -\overline{A} \\ \overline{D} & -\overline{B} \end{pmatrix}.$$

Therefore, $JN = \overline{N}J$ if and only if $C = -\overline{B}$ and $D = \overline{A}$. Hence

$$\Psi(M_n(R_1) = \left\{ \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \mid A, B \in M_n(R) \right\} = \{N \in M_{2n}(R) \mid \mathcal{J}N = \overline{N}\mathcal{J}\}.$$

□

Definition 3.3. Define $\det_R : M_n(R) \to R$ by

$$\det_R((a_{ij})) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

with $\mathrm{sgn}(\sigma)$ is a sign function of permutation and $\det_{R_1} : M_n(R_1) \to R$ by

$$\det_{R_1} = \det_R \Psi.$$

Lemma 3.4. *The multiplicative property holds for* $\det_{R_1}$.

Proof. Since Ψ is a homomorphism by Lemma 3.2, the result follows. □

Lemma 3.5. *Extend* $\tau : R_1 \to R_1$ *to* $\tau : M_n(R_1) \to M_n(R_1)$ *by* $A + jB \mapsto \overline{A^t} - jB^t$ *for every* $A, B \in M_n(R))$. *Then*

(i) τ *is an anti-automorphism of* $M_n(R_1)$,
(ii) $\Psi(M^\tau) = \overline{\Psi(M)}^t \quad (M \in M_n(R_1))$,
(iii) $\det_{R_1}(M) = \overline{\det_{R_1}(M)} \quad (M \in M_n(R_1))$.

Proof. (i) For every $A_1, A_2, B_1, B_2 \in M_n(R)$,

$$\begin{aligned} &((A_1 + jB_1)(A_2 + jB_2))^\tau \\ &= (A_1A_2 + A_1jB_2 + jB_1A_2 + jB_1jB_2)^\tau \\ &= (A_1A_2 + j\overline{A_1}B_2 + jB_1A_2 - \overline{B_1}B_2)^\tau && \text{(by Lemma 3.1)} \\ &= (A_1A_2 - \overline{B_1}B_2 + j(\overline{A_1}B_2 + B_1A_2))^\tau \\ &= (\overline{A_1A_2 - \overline{B_1}B_2})^t - j(\overline{A_1}B_2 + B_1A_2)^t, \end{aligned}$$

$$\begin{aligned} (A_1 + jB_1)^\tau(A_2 + jB_2)^\tau &= (\overline{A_2}^t - jB_2^t)(\overline{A_1} - jB_1^t) \\ &= \overline{A_2}^t\overline{A_1}^t - \overline{A_2}^t jB_1^t - jB_2^t\overline{A_1}^t + jB_2^t jB_1^t \\ &= \overline{A_2}^t\overline{A_2}^t - \overline{B_2}^t B_1^t - j(A_2^t B_1^t + B_2^t\overline{A_1}^t). \end{aligned}$$

(ii) Let $M = A + jB \in M_n(R_1)$ $(A, B \in M_n(R))$. Then

$$\Psi(M^\tau) = \Psi(\overline{A^t} - jB^t) = \begin{bmatrix} \overline{A^t} & \overline{B}^t \\ -B^t & A^t \end{bmatrix} = \overline{\Psi(M)}^t.$$

(iii) Let $M = A + jB \in M_n(R_1)$ $(A, B \in M_n(R))$. Notice that

$$\Psi(M^\tau) = \Psi(\overline{A}^t - jB^t) = \begin{bmatrix} \overline{A}^t & \overline{B}^t \\ -B^t & A^t \end{bmatrix}$$

and

$$(\Psi(M^\tau))^t = \begin{bmatrix} \overline{A} & -B \\ \overline{B} & A \end{bmatrix}.$$

Therefore, we have

$$\begin{aligned} \det_R\Psi(M) = \det_R\left(\begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix}\right) &= \det_R\left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\begin{bmatrix} \overline{A} & -B \\ \overline{B} & A \end{bmatrix}\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}\right) \\ = \det_R\left(\begin{bmatrix} \overline{A} & -B \\ \overline{B} & A \end{bmatrix}\right) &= \det_R(\Psi(M^\tau)^t) \end{aligned}$$

$$= \det_R \left(\overline{\Psi(M)}\right) = \overline{\det_R(\Psi(M))}. \qquad \square$$

Now, when $R = \mathbb{R}$, we will obtain $R_1 = \mathbb{C}$ by the Cayley-Dickson process. Therefore, every element in $\mathbb{C}$ can be expressed as $a + ib$ where $a, b \in \mathbb{R}$ and $i = (0, 1)$. Next, we will apply the Cayley-Dickson process to obtain $R_2 = \mathbb{H}$ from $\mathbb{C}$. Similarly, every element in $\mathbb{H}$ can be expressed as $a+jb$ where $a, b \in \mathbb{C}$ and $j = (0, 1)$. Note that we have $zj = j\bar{z}$ for every $z \in \mathbb{C}$. By applying Proposition 3.2, we have the following.

Corollary 3.6. *Define* $\Psi_1 : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ *by*

$$\Psi_1(C + iD) = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}, \qquad (C, D \in M_n(\mathbb{R})).$$

Then Ψ_1 *is an injective homomorphism and*

$$\Psi_1(M_n(\mathbb{C})) = \{P \in M_{2n}(\mathbb{R}) \mid \mathcal{J}P = P\mathcal{J}\}.$$

Corollary 3.7. *Define* $\Psi_2 : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ *by*

$$\Psi_2(A + jB) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}, \qquad (A, B \in M_n(\mathbb{C})),$$

where $\overline{A}$ *denote the conjugate of* $A \in M_n(\mathbb{C})$*. Then* Ψ_2 *is an injective homomorphism and*

$$\Psi_2(M_n(\mathbb{H})) = \{N \in M_{2n}(\mathbb{C}) \mid \mathcal{J}N = \overline{N}\mathcal{J}\}.$$

Based on our notation, the Study determinant is well-known as $\det_{\mathbb{C}}$. Corollary 3.6 and Corollary 3.7 are crucial for the concept of Study determinant. We refer [1] for further information of Study determinant.

We now give a relationship between the Cayley-Dickson process and a generalization of the Study determinant. It seems that the relation is natural but we need an additional bijection $f : R_2 \to R_2$ to show the connection between the multiplication defined in the Cayley-Dickson process (or the multiplication operation defined in 3.1) and the standard matrix multiplication.

Proposition 3.8. *Let* $\Omega = (\alpha, \beta) \in R_2$*, where* $\alpha, \beta \in R_1$*. Then* $f : R_2 \to R_2$ *with* $f(\alpha, \beta) = (\beta, \alpha)$ *satisfies* $N^2 = \det \Psi\pi f$*.*

Proof. Let $\Omega = (\alpha, \beta) = (((a, b), (c, d)), ((e, f), (g, h)))$ where $a, b, c, d, e, f, g, h \in R$. By applying f to Ω and using (2.1), (2.2), and (2.3) to each term of the result of determinant of $\Psi\pi f(\alpha)$, for example $a\bar{a}\bar{a}a = a\bar{a}a\bar{a}$, we have

$$\det_R \Psi\pi f(\Omega) = \det_R \begin{bmatrix} c & a & -\bar{d} & -\bar{b} \\ -\bar{a} & \bar{c} & -\bar{b} & \bar{d} \\ d & b & \bar{c} & \bar{a} \\ b & -d & -a & c \end{bmatrix} = (a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d})^2 = N^2(\alpha). \qquad \square$$

Finally, we give a commutative diagram to show the relation between our generalization of the Study determinant and the Cayley-Dickson process. The Proposition 3.8 shows that the following diagram is commutative.

$$\begin{tikzcd}
R_2 \times R_2 \arrow[r, dashed, "\pi f \times \pi f"] \arrow[d, "\mathsf{CD}"] & \widetilde{R} \times \widetilde{R} \arrow[r, hook, dashed] \arrow[d, "\star"] & M_2(R_1) \times M_2(R_1) \arrow[r, "\Psi \times \Psi"] \arrow[d, "{[\cdot][\cdot]}"] & M_4(R) \times M_4(R) \arrow[r, "\det \times \det"] \arrow[d, "{[\cdot][\cdot]}"] & R \times R \arrow[d, "{[\cdot][\cdot]}"] \\
R_2 \arrow[r, dashed, "\pi f"] \arrow[rrrr, bend right, "N^2"'] & \widetilde{R} \arrow[r, hook, dashed] & M_2(R_1) \arrow[r, "\Psi"] & M_4(R) \arrow[r, "\det"] & R
\end{tikzcd}$$

Here, "CD" denote the Cayley-Dickson multiplication, $[\cdot][\cdot]$ denote the standard matrix multiplication and $N^2(\Omega)$ means $(N(\Omega))^2$ for every $\Omega \in R_2$.

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