Commun. Korean Math. Soc. **36** (2021), No. 3, pp. 447–463
https://doi.org/10.4134/CKMS.c190332
pISSN: 1225-1763 / eISSN: 2234-3024

GLOBAL ATTRACTOR FOR A CLASS OF QUASILINEAR DEGENERATE PARABOLIC EQUATIONS WITH NONLINEARITY OF ARBITRARY ORDER

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Abstract. In this paper we study the existence and long-time behavior of weak solutions to a class of quasilinear degenerate parabolic equations involving weighted p-Laplacian operators with a new class of nonlinearities. First, we prove the existence and uniqueness of weak solutions by combining the compactness and monotone methods and the weak convergence techniques in Orlicz spaces. Then, we prove the existence of global attractors by using the asymptotic *a priori* estimates method.

1. Introduction

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for a system having some dissipativity properties is to analyze the existence and structure of its global attractor. The existence of the global attractor has been derived for a large class of nondegenerate PDEs (see e.g. [5, 21] and references therein). In recent years, the existence and long-time behavior of solutions to degenerate parabolic equations have attracted the attention of many mathematicians.

In this paper we consider the following problem

$$\begin{cases} u_t - \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + f(u) = g(x), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, $2 \leq p \leq N$, $u_0 \in L^2(\Omega)$ given, the coefficient $a(\cdot)$, the nonlinearity f and the external force g satisfy the following conditions:

Received September 23, 2019; Revised February 2, 2021; Accepted February 9, 2021.
2010 *Mathematics Subject Classification.* 35B41, 35K65, 35D05.
Key words and phrases. Quasilinear degenerate parabolic equation, weighted p-Laplacian operator, weak solution, global attractor, compactness method, monotonicity method, weak convergence techniques, Orlicz spaces, asymptotic a priori estimate method.

(H1) The function $a : \Omega \to \mathbb{R}$ satisfies the following assumptions: $a \in L^1_{\text{loc}}(\Omega)$ and $a(x) = 0$ for $x \in \Sigma$, and $a(x) > 0$ for $x \in \overline{\Omega} \setminus \Sigma$, where Σ is a closed subset of $\overline{\Omega}$ with $\text{meas}(\Sigma) = 0$. Furthermore, we assume that

$$\int_\Omega \frac{1}{[a(x)]^{\frac{N}{\alpha}}} dx < \infty \text{ for some } \alpha \in (0, p);$$

(H2) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$f(u)u \geq -\mu u^2 - c_1, \tag{1.2}$$

$$f'(u) \geq -\ell, \tag{1.3}$$

where c_1, ℓ, μ are positive constants, and if $p = 2$, then we assume furthermore that $0 < \mu < c_0$ with c_0 is determined in (2.1).

(H3) $g \in L^2(\Omega)$.

The degeneracy of problem (1.1) is considered in the sense that the measurable, nonnegative diffusion coefficient $a(x)$ is allowed to vanish somewhere. The physical motivation of the assumption (H1) is related to the modeling of reaction diffusion processes in composite materials, occupying a bounded domain Ω, in which at some points they behave as *perfect insulator*. Following [7, p. 79], when at some points the medium is perfectly insulating, it is natural to assume that $a(x)$ vanishes at these points. As mentioned in [13, 18], the assumption (H1) implies that the degenerate set may consist of an infinite many number of points, which is different from the weight of Caldiroli-Musina type in [3, 4, 6] that is only allowed to have at most a finite number of zeroes. A typical example of the weight $a(\cdot)$ is $\text{dist}(x, \partial\Omega)$.

Problem (1.1) contains some important classes of parabolic equations, such as the semilinear heat equation (when $a = 1, p = 2$), semilinear degenerate parabolic equations (when $p = 2$), the p-Laplacian equations (when $a = 1, p \neq 2$), etc. It is noticed that the existence and long-time behavior of weak solutions to problem (1.1) in a particular case, namely when $p = 2$ and the nonlinearity is growth and dissipative of polynomial type, were studied by Li, Ma and Zhong in [13]. For this kind of nonlinearities, the existence of entropy solutions and the existence of a global attractor in $L^1(\Omega)$ of this problem have been studied very recently in [18]. We also refer the interested reader to [1–4, 9–12, 14, 15, 17, 20] for related results on degenerate parabolic equations.

To study problem (1.1) we first introduce the energy space $W_0^{1,p}(\Omega, a)$ defined as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{W_0^{1,p}(\Omega,a)} := \Big(\int_\Omega a(x)|\nabla u|^p dx \Big)^{1/p},$$

and prove some compact embedding results related to this space (see Section 2 for details). Then, under assumptions (H1)-(H3), we prove the existence of global weak solutions and the existence of global attractors for the semigroup generated by problem (1.1) in $L^2(\Omega)$ and $W_0^{1,p}(\Omega, a)$. Thus, in some sense, we improve previous results about the p-Laplacian parabolic equations in bounded domains.

Let us explain the methods used in the paper. First, using the compactness and monotonicity methods [16, Chapters 1-2] and weak convergence techniques in Orlicz spaces [9] we prove the existence and uniqueness of a global weak solution to problem (1.1). Then we study the existence of global attractors in some function spaces for the semigroup associated to problem (1.1). Thanks to *a priori* estimates of the solutions in $W_0^{1,p}(\Omega,a)$ and $D(L_{p,a})$ and the compactness of the embeddings $W_0^{1,p}(\Omega,a) \hookrightarrow L^2(\Omega)$ and $D(L_{p,a}) \hookrightarrow W_0^{1,p}(\Omega,a)$, we get the existence of a global attractor in $L^2(\Omega)$ and $W_0^{1,p}(\Omega,a)$.

The rest of the paper is organized as follows. In Section 2, we introduce some function spaces and prove some compactness results, which are frequently used later. Section 3 is devoted to the proof of global existence of a weak solution to problem (1.1) by using compactness and monotonicity methods and weak convergence techniques in Orlicz spaces. In Section 4, we prove the existence of global attractors in $L^2(\Omega)$ and $W_0^{1,p}(\Omega,a)$ for the semigroup associated to problem (1.1).

2. Preliminaries

To study problem (1.1), we introduce the weighted Sobolev space $W_0^{1,p}(\Omega,a)$, defined as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{W_0^{1,p}(\Omega,a)} := \left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{1}{p}},$$

and denote by $W^{-1,q}(\Omega,a)$ its dual space, with $\frac{1}{p}+\frac{1}{q}=1$.

It is noticed that the assumption (H1) has been particularly made in [18], where the authors use the following expression

$$\int_\Omega [a(x)]^{-(1/\gamma)} dx < \infty \quad \text{for some } \gamma \in (0, p-1),$$

which gives (H1) by taking $\gamma = \alpha/N$. Therefore, from the corresponding results in [18], we have the following embeddings, which are generalizations of the corresponding results in the case $p=2$ of Li et al. [13].

Proposition 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ and $a(\cdot)$ satisfies* (H1). *Then the following embeddings hold*:

(i) $W_0^{1,p}(\Omega,a) \hookrightarrow W_0^{1,\beta}(\Omega)$ *continuously if* $1 \leq \beta \leq \frac{pN}{N+\alpha}$;

(ii) $W_0^{1,p}(\Omega,a) \hookrightarrow L^r(\Omega)$ *continuously if* $1 \leq r \leq p_\alpha^*$, *where* $p_\alpha^* = \frac{pN}{N-p+\alpha}$;

(iii) $W_0^{1,p}(\Omega,a) \hookrightarrow L^r(\Omega)$ *compactly if* $1 \leq r < p_\alpha^*$.

Thanks to Proposition 2.1, there exists a best constant c_0 such that

$$c_0\|u\|^2_{L^2(\Omega)} \leq \|u\|^2_{W_0^{1,p}(\Omega,a)}. \tag{2.1}$$

Putting

$$L_{p,a}u = -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u), \quad u \in W_0^{1,p}(\Omega, a).$$

The following proposition, its proof is straightforward, gives some important properties of the operator $L_{p,a}$.

Proposition 2.2. *The operator $L_{p,a}$ maps $W_0^{1,p}(\Omega, a)$ into its dual $W^{-1,q}(\Omega, a)$. Moreover,*

- (i) *$L_{p,a}$ is hemicontinuous, i.e., for all $u, v, w \in W_0^{1,p}(\Omega, a)$, the map $\lambda \mapsto \langle L_{p,a}(u + \lambda v), w\rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$;*
- (ii) *$L_{p,a}$ is strongly monotone when $p \geq 2$, i.e.,*

$$\langle L_{p,a}u - L_{p,a}v, u - v\rangle \geq \delta\|u - v\|^p_{W_0^{1,p}(\Omega,a)} \text{ for all } u, v \in W_0^{1,p}(\Omega, a).$$

We introduce the Banach space $D(L_{p,a})$, defined as the domain of the operator $L_{p,a}$ with the homogeneous Dirichlet boundary condition

$$D(L_{p,a}) := \{u \in W_0^{1,p}(\Omega, a) \,|\, L_{p,a}u \in L^2(\Omega)\},$$

endowed with the norm

$$\|u\|_{D(L_{p,a})} := \Big(\int_\Omega |\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)|^2 dx\Big)^{\frac{1}{2(p-1)}}.$$

Proposition 2.3. *Assume that Ω is bounded domain in $\mathbb{R}^N (N \geq 2)$ and $a(\cdot)$ satisfies* (H1)*. Then the embedding $D(L_{p,a}) \hookrightarrow W_0^{1,p}(\Omega, a)$ is compact.*

Proof. For any function $u \in D(L_{p,a})$, we have

$$\begin{aligned}
\|u\|^p_{W_0^{1,p}(\Omega,a)} &= \int_\Omega a(x)|\nabla u|^p dx \\
&= -\int_\Omega \mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)u dx \\
&\leq \Big(\int_\Omega |\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)|^2 dx\Big)^{1/2}\Big(\int_\Omega |u|^2 dx\Big)^{1/2} \\
&\leq \|u\|^{p-1}_{D(L_{p,a})} \cdot \|u\|_{L^2(\Omega)}. \qquad (2.2)
\end{aligned}$$

Noting that $\|u\|_{L^2(\Omega)} \leq C\|u\|_{W_0^{1,p}(\Omega,a)}$ by Proposition 2.1. From (2.2), we obtain

$$\|u\|^{p-1}_{W_0^{1,p}(\Omega,a)} \leq C\|u\|^{p-1}_{D(L_{p,a})}.$$

It implies that $D(L_{p,a}) \hookrightarrow W_0^{1,p}(\Omega, a)$. Next, we will prove that for any $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that

$$\|u\|^p_{W_0^{1,p}(\Omega,a)} \leq \varepsilon\|u\|^p_{D(L_{p,a})} + C(\varepsilon)\|u\|^p_{L^1(\Omega)} \qquad (2.3)$$

for all $u \in D(L_{p,a})$. Indeed, since $W_0^{1,p}(\Omega, a) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega)$, by the Ehrling lemma (see [19, p. 215]), we have for any $\eta > 0$, there exists a constant $C(\eta)$ such that

$$\|u\|_{L^2(\Omega)} \leq \eta\|u\|_{W_0^{1,p}(\Omega,a)} + C(\eta)\|u\|_{L^1(\Omega)}.$$

Combining this inequality into (2.2) and using the Cauchy inequality, we obtain

$$\begin{aligned}\|u\|^p_{W^{1,p}_0(\Omega,a)} &\le \|u\|^{p-1}_{D(L_{p,a})}(\eta\|u\|_{W^{1,p}_0(\Omega,a)} + C(\eta)\|u\|_{L^1(\Omega)})\\ &\le C_1(\eta,p)\|u\|^p_{W^{1,p}_0(\Omega,a)} + C_2(\eta,p)\|u\|^p_{D(L_{p,a})} + C_3(\eta,p)\|u\|^p_{L^1(\Omega)}.\end{aligned}$$

Hence, we obtain (2.3) for suitable choosing of η. Let $\{u_n\}$ be a bounded sequence in $D(L_{p,a})$. Since $D(L_{p,a}) \hookrightarrow W^{1,p}_0(\Omega,a) \hookrightarrow\hookrightarrow L^1(\Omega)$, there exists a subsequence $\{u_{nk}\}$ such that $u_{nk} \to u$ in $L^1(\Omega)$. Using (2.3), we have

$$\|u_{nk} - u\|^p_{W^{1,p}_0(\Omega,a)} \le \varepsilon\|u_{nk} - u\|^p_{D(L_{p,a})} + C(\varepsilon)\|u_{nk} - u\|^p_{L^1(\Omega)}.$$

By the boundedness of this subsequence in $D(L_{p,a})$, we conclude that $u_{nk} \to u$ in $W^{1,p}_0(\Omega,a)$, up to a subsequence if necessary. This completes the proof. □

Proposition 2.4. *Let $\{u_n\}$ be a bounded sequence in $L^p(0,T;W^{1,p}_0(\Omega,a))$ such that $\{u'_n\}$ is bounded in $L^q(0,T;W^{-1,q}(\Omega,a))$ where $q = p/(p-1)$. If* (H1) *holds, then $\{u_n\}$ converges almost everywhere in $\Omega_T := \Omega \times (0,T)$ up to a subsequence.*

Proof. By Proposition 2.1, one can take a number $r \in [2, p^*_\alpha)$ such that

$$W^{1,p}_0(\Omega,a) \hookrightarrow\hookrightarrow L^r(\Omega). \tag{2.4}$$

Since $r' = r/(r-1) \le 2$, we have

$$L^p(\Omega) \hookrightarrow L^{r'}(\Omega),$$

and therefore,

$$L^r(\Omega) \hookrightarrow L^q(\Omega). \tag{2.5}$$

Using Proposition 2.1 once again and noticing that $p < p^*_\alpha$ since $\alpha \in (0,p)$, we see that

$$W^{1,p}_0(\Omega,a) \hookrightarrow L^p(\Omega).$$

This and (2.5) follow that

$$L^r(\Omega) \hookrightarrow W^{-1,q}(\Omega,a).$$

Now with (2.4), we have an evolution triple

$$W^{1,p}_0(\Omega,a) \hookrightarrow\hookrightarrow L^r(\Omega) \hookrightarrow W^{-1,q}(\Omega,a).$$

The assumption of $\{u'_n\}$ in $L^q(0,T;W^{-1,q}(\Omega,a))$ implies that

$$\{u'_n\} \text{ is also bounded in } L^q(0,T;W^{-1,q}(\Omega,a)).$$

Thanks to the well-known Aubin-Lions compactness lemma (see [16, p. 58]), $\{u_n\}$ is precompact in $L^p(0,T;L^r(\Omega))$ and therefore in $L^s(0,T;L^s(\Omega))$, $s = \min(p,r)$, so it has an a.e. convergent subsequence. □

3. Existence and uniqueness of global weak solutions

Denote $\Omega_T = \Omega \times (0,T)$ and let (p,q) be conjugate, i.e., $\frac{1}{p}+\frac{1}{q}=1$. We give the definition of weak solutions to problem (1.1).

Definition. A function u is called a weak solution of problem (1.1) on the interval $(0,T)$ if

$$\begin{aligned} &u \in L^p(0,T;W_0^{1,p}(\Omega,a)) \cap C([0,T];L^2(\Omega)) \\ &\frac{du}{dt} \in L^q(0,T;W^{-1,q}(\Omega,a)) + L^1(\Omega_T), \\ &u|_{t=0} = u_0 \ \text{ a.e. in } \ \Omega, f(u) \in L^1(\Omega_T), \end{aligned}$$

and

$$\int_{\Omega_T}\left(\frac{\partial u}{\partial t}\eta + a(x)|\nabla u|^{p-2}\nabla u \nabla \eta + f(u)\eta - g\eta\right)dxdt = 0$$

for all test functions $\eta \in L^p(0,T;W_0^{1,p}(\Omega,a)\cap L^\infty(\Omega))$.

Theorem 3.1. *Under assumptions* (H1)-(H3)*, for each* $u_0 \in L^2(\Omega)$ *and* $T>0$ *given, the problem* (1.1) *has a unique weak solution on* $(0,T)$*. Moreover, the mapping* $u_0 \mapsto u(t)$ *is continuous on* $L^2(\Omega)$.

Proof. (i) *Existence.* Consider the approximating solution $u_n(t)$ in the form

$$u_n(t) = \sum_{k=1}^n u_{nk}(t)e_k,$$

where $\{e_j\}_{j=1}^\infty$ is dense in $W_0^{1,p}(\Omega,a)\cap L^\infty(\Omega)$ and orthogonal in $L^2(\Omega)$. We get u_n from solving the problem

$$\begin{cases} \langle \frac{du_n}{dt}, e_k\rangle + \langle L_{p,a}u_n, e_k\rangle + \langle f(u_n), e_k\rangle = \langle g, e_k\rangle, \\ (u_n(0), e_k) = (u_0, e_k), \ k = 1,\dots,n. \end{cases}$$

By the Peano theorem, we obtain the local existence of u_n. We now establish some *a priori* estimates for u_n. Since

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \|u_n\|^p_{W_0^{1,p}(\Omega,a)} + \int_\Omega f(u_n)u_n dx = \int_\Omega g u_n dx. \tag{3.1}$$

In the case $p=2$, it follows from (1.2) and (2.1) that

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + (c_0-\mu)\|u_n\|^2_{L^2(\Omega)} \le c_1|\Omega| + \int_\Omega g u_n dx.$$

Since $c_0 - \mu > 0$, by the Young inequality, we obtain

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + (c_0-\mu)\|u_n\|^2_{L^2(\Omega)} \le 2c_1|\Omega| + \frac{1}{c_0-\mu}\|g\|^2_{L^2(\Omega)}. \tag{3.2}$$

In the case $p > 2$, noting that $c_2\|u\|^p_{L^2(\Omega)} \le \|u\|^p_{W_0^{1,p}(\Omega,a)}$ due to Proposition 2.1, from (3.1) and (1.2) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} + c_2\|u_n\|^p_{L^2(\Omega)} \\
&\le \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} - \int_\Omega f(u_n)u_n dx + \int_\Omega g u_n dx \\
&\le (\mu+1)\|u_n\|^2_{L^2(\Omega)} + c_1|\Omega| + \frac{1}{2}\|g\|^2_{L^2(\Omega)}.
\end{aligned}$$

Moreover, there exists a positive constant C_1 such that

$$-c_2|s|^p + (\mu+1)|s|^2 \le C_1.$$

Thus,

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \|u_n\|^2_{L^2(\Omega)} \le 2C_1 + 2c_1|\Omega| + \|g\|^2_{L^2(\Omega)}. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} \le C,$$

where $C = C(c_0, c_1, c_2, \mu, C_1, |\Omega|, \|g\|_{L^2(\Omega)})$. Integrating from 0 to t, $0 \le t \le T$ and using the fact that $\|u_n(0)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)}$, we obtain

$$\|u_n(t)\|^2_{L^2(\Omega)} \le \|u_0\|^2_{L^2(\Omega)} + CT. \tag{3.4}$$

On the other hand, from (3.1) and using (1.2), (3.4) we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \|u_n\|^p_{W_0^{1,p}(\Omega,a)} &\le c_1|\Omega| + \left(\mu + \frac{1}{2}\right)\|u_n\|^2_{L^2(\Omega)} + \frac{1}{2}\|g\|^2_{L^2(\Omega)} \\
&\le C.
\end{aligned}$$

Integrating from 0 to t, $0 \le t \le T$ and using the fact that $\|u_n(0)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)}$, we obtain

$$\|u_n(t)\|^2_{L^2(\Omega)} + 2\int_0^t \|u_n\|^p_{W_0^{1,p}(\Omega,a)} dt \le C.$$

It follows that

- $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$;
- $\{u_n\}$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega,a))$.

On the other hand, by the Hölder inequality we have

$$\begin{aligned}
\Big|\int_0^T \langle L_{p,a}u_n, v\rangle dt\Big| &= \Big|\int_0^T\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n \nabla v dx dt\Big| \\
&\le \int_0^T\int_\Omega \big(a(x)^{\frac{p-1}{p}}|\nabla u_n|^{p-1}\big)\big(a(x)^{\frac{1}{p}}|\nabla v|\big) dx dt \\
&\le \|u_n\|^{\frac{p}{q}}_{L^p(0,T;W_0^{1,p}(\Omega,a))}\|v\|_{L^p(0,T;W_0^{1,p}(\Omega,a))}
\end{aligned}$$

for any $v \in L^p(0,T;W_0^{1,p}(\Omega,a))$.

Using the boundedness of $\{u_n\}$ in $L^p(0,T;W_0^{1,p}(\Omega,a))$, we infer that $\{L_{p,a}u_n\}$ is bounded in $L^q(0,T;W^{-1,q}(\Omega,a))$.

We now prove that $\{f(u_n)\}$ is bounded in $L^1(\Omega_T)$. It follows from (3.1) and (3.4) that

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \int_\Omega f(u_n)u_n dx \le C.$$

Integrating from 0 to T, we obtain

$$\frac{1}{2}\|u_n(T)\|^2_{L^2(\Omega)} + \int_{\Omega_T} f(u_n)u_n dxdt \le \frac{1}{2}\|u_n(0)\|^2_{L^2(\Omega)} + TC.$$

Hence

$$\int_{\Omega_T} f(u_n)u_n dxdt \le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + TC. \tag{3.5}$$

Setting $h(u_n) = f(u_n) - f(0) + \nu u_n$ with $\nu > \ell$. It follows from (1.2) that $h(s)s \ge 0$ for all $s \in \mathbb{R}$. Therefore, we deduce from (3.5) and the boundedness of $\{u_n\}$ in $L^\infty(0,T;L^2(\Omega))$ that

$$\begin{aligned}\int_{\Omega_T} |h(u_n)|dxdt &\le \int_{\Omega_T\cap\{|u_n|>1\}} |h(u_n)u_n|dxdt + \int_{\Omega_T\cap\{|u_n|\le 1\}} |h(u_n)|dxdt \\ &\le \int_{\Omega_T} h(u_n)u_n dxdt + \sup_{|s|\le 1}|h(s)|\,|\Omega_T| \\ &= \int_{\Omega_T} f(u_n)u_n dxdt + \nu\int_{\Omega_T}|u_n|^2 dxdt + |f(0)|\int_{\Omega_T}|u_n|dxdt \\ &\quad + \sup_{|s|\le 1}|h(s)|\,|\Omega_T| \\ &\le C.\end{aligned}$$

This means that $\{h(u_n)\}$ is bounded in $L^1(\Omega_T)$, and so is $\{f(u_n)\}$. Rewriting (1.1) in $L^q(0,T;W_0^{-1,q}(\Omega,a)) + L^1(\Omega_T)$ as

$$u_{nt} = g - L_{p,a}u_n - f(u_n). \tag{3.6}$$

Therefore, by Proposition 2.4, there is an a.e. convergent subsequence in Ω_T and $\{u_n\}$ is compact in $L^2(0,T;L^2(\Omega))$. Applying a diagonalization procedure and using Lemma 1.3 in [16, p. 12], we obtain (up to a subsequence) that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } L^p(0,T;W_0^{1,p}(\Omega,a)), \\ u_n &\to u \text{ in } L^2(0,T;L^2(\Omega)), \\ u_{nt} &\rightharpoonup u_t \text{ in } L^q(0,T;W^{-1,q}(\Omega,a)) + L^1(\Omega_T), \\ L_{p,a}u_n &\rightharpoonup \psi \text{ in } L^q(0,T;W^{-1,q}(\Omega,a)), \\ u_n(T) &\to u(T) \text{ in } L^2(\Omega). \end{aligned}$$

We now pass to the limit in the nonlinear term. From (1.3) we see that $h(\cdot)$ is a strictly increasing function. Moreover, using (3.5) we have

$$\int_{\Omega_T} h(u_n(t))u_n(t)dxdt \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + TC\|g\|^2_{L^2(\Omega)} + \frac{|f(0)|^2}{2}|\Omega|T + \big(\frac{1}{2}+\nu\big)\|u_n\|^2_{L^2(0,T,L^2(\Omega))}.$$

Since $u_n \to u$ strongly in $L^2(0,T;L^2(\Omega))$, then up to a subsequence, we have $u_n \to u$ a.e. in Ω_T. Applying Lemma 6.1 in [8], we obtain that $h(u) \in L^1(\Omega_T)$ and for all test functions $\varphi \in C_0^\infty([0,T];W_0^{1,p}(\Omega,a)\cap L^\infty(\Omega))$,

$$\int_{\Omega_T} h(u_n)\varphi\, dxdt \to \int_{\Omega_T} h(u)\varphi\, dxdt \text{ as } n\to\infty.$$

Hence, $f(u) \in L^1(\Omega_T)$ and for all $\varphi \in C_0^\infty([0,T];W_0^{1,p}(\Omega,a)\cap L^\infty(\Omega))$,

$$\int_{\Omega_T} f(u_n)\varphi\, dxdt \to \int_{\Omega_T} f(u)\varphi\, dxdt \text{ as } n\to\infty.$$

Now, passing to the limit in (3.6), one has in the distribution sense

$$u_t = g - \psi - f(u). \tag{3.7}$$

We now show that $\psi = L_{p,a}u$. To do this, integrating (3.1) from 0 to T we obtain

$$\int_0^T\int_\Omega a(x)|\nabla u_n|^p dxdt = \int_{\Omega_T} g u_n dxdt - \int_{\Omega_T} f(u_n)u_n dxdt + \frac{1}{2}\|u_n(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_n(T)\|^2_{L^2(\Omega)}.$$

Since $\lim\limits_{n\to\infty}\|u_n(T)\|^2_{L^2(\Omega)} = \|u(T)\|^2_{L^2(\Omega)}$ and $\lim\limits_{n\to\infty}\|u_n(0)\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega)}$, we deduce that

$$\lim_{n\to\infty}\int_0^T\int_\Omega a(x)|\nabla u_n|^p dxdt = \int_{\Omega_T} gudxdt - \int_{\Omega_T} f(u)udxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)}. \tag{3.8}$$

Using Proposition 2.2, we have

$$\int_{\Omega_T}\Big(a(x)|\nabla u_n|^{p-2}\nabla u_n - a(x)|\nabla v|^{p-2}\nabla v\Big)\cdot\nabla(u_n - v)dxdt \geq 0$$

for all $v \in L^p(0,T;W_0^{1,p}(\Omega,a))$. Thus, taking the limits leads to

$$\lim_{n\to\infty}\int_0^T\int_\Omega a(x)|\nabla u_n|^p dxdt - \int_0^T\langle\psi,v\rangle dt - \int_{\Omega_T} a(x)|\nabla v|^{p-2}\nabla v\cdot\nabla(u-v)dxdt \geq 0.$$

Putting this with (3.8), we have

$$\int_{\Omega_T} gudxdt - \int_{\Omega_T} f(u)udxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)}$$

$$- \int_0^T \langle \psi, v\rangle dt - \int_{\Omega_T} a(x)|\nabla v|^{p-2}\nabla v\cdot\nabla(u-v)dxdt \geq 0. \tag{3.9}$$

We see that $f(u) \in L^1(\Omega_T)$ and u does not belong to $W_0^{1,p}(\Omega, a)\cap L^\infty(\Omega)$. Therefore, u cannot be chosen as a test function in (3.7). We will use some ideas in [9]. Let $B_k : \mathbb{R} \to \mathbb{R}$ be the truncated function defined by

$$B_k(s) = \begin{cases} k & \text{if} \quad s > k, \\ s & \text{if} \quad |s| \leq k, \\ -k & \text{if} \quad s < -k. \end{cases}$$

We construct the following Nemytskii mapping

$$\widehat{B}_k : W_0^{1,p}(\Omega, a)\cap L^\infty(\Omega) \to W_0^{1,p}(\Omega, a)\cap L^\infty(\Omega)$$
$$v \mapsto \widehat{B}_k(v)(x) = B_k(v(x)).$$

It follows from Lemma 2.3 in [9] that $\|\widehat{B}_k(v) - v\|_{W_0^{1,p}(\Omega,a)\cap L^\infty(\Omega)} \to 0$ as $k\to\infty$. We now can test (3.7) by $\widehat{B}_k(u)$. Multiplying (3.7) by $\widehat{B}_k(u)$, then integrating from ε to T, we have

$$\int_\varepsilon^T\int_\Omega \frac{d}{dt}\big(u(t)\widehat{B}_k(u)(t)\big)dxdt - \int_\varepsilon^T\int_\Omega u\frac{d}{dt}\big(\widehat{B}_k(u)(t)\big)dxdt + \int_\varepsilon^T \langle\psi, \widehat{B}_k(u)\rangle dt$$
$$= \int_\varepsilon^T\int_\Omega g\widehat{B}_k(u)dxdt - \int_\varepsilon^T\int_\Omega f(u)\widehat{B}_k(u)dxdt.$$

Noting that $u\frac{d}{dt}\big(\widehat{B}_k(u)\big) = \frac{1}{2}\frac{d}{dt}\big(\big(\widehat{B}_k(u)\big)^2\big)$, we have

$$\begin{aligned}\int_\varepsilon^T \langle\psi, \widehat{B}_k(u)\rangle dt = &\int_\varepsilon^T\int_\Omega g\widehat{B}_k(u)dxdt - \int_\varepsilon^T\int_\Omega h(u)\widehat{B}_k(u)dxdt \\ &- \int_\varepsilon^T\int_\Omega (f(0)-\nu u)\widehat{B}_k(u)dxdt + \int_\Omega u(\varepsilon)\widehat{B}_k(u)(\varepsilon)dx \\ &- \int_\Omega u(T)\widehat{B}_k(u)(T)dx \\ &+ \frac{1}{2}\|\widehat{B}_k(u)(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\widehat{B}_k(u)(\varepsilon)\|^2_{L^2(\Omega)}.\end{aligned}$$

Passing to the limit as $k\to\infty$ we have

$$\int_\varepsilon^T \langle\psi, u\rangle dt = \int_\varepsilon^T\int_\Omega gudxdt - \lim_{k\to\infty}\int_\varepsilon^T\int_\Omega h(u)\widehat{B}_k(u)dxdt$$

$$- \int_\varepsilon^T\int_\Omega (f(0)-\nu u)udxdt + \frac{1}{2}\|u(\varepsilon)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)}, \tag{3.10}$$

where due to the nondecreasing of $\{h(u)\widehat{B}_k(u)\}_{k=1}^{\infty}$ and $\widehat{B}_k(u) \to u$ in $C([0,T]; L^2(\Omega))$, it follows from the monotone convergence theorem that

$$\lim_{k\to\infty} \int_\varepsilon^T \int_\Omega h(u)\widehat{B}_k(u)dxdt = \int_\varepsilon^T \int_\Omega h(u)udxdt.$$

We deduce from (3.10) by passing to the limit as $\varepsilon \to 0$ that

$$\int_0^T \langle \psi, u\rangle dt = \int_{\Omega_T} gudxdt - \int_{\Omega_T} f(u)udxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)}. \tag{3.11}$$

In view of (3.9) and (3.11), we have

$$\int_0^T \langle \psi + \operatorname{div}(a(x)|\nabla v|^{p-2}\nabla v), u - v\rangle dt \geq 0,\ \forall v \in L^p(0,T; W_0^{1,p}(\Omega, a)).$$

Choosing $v = u - \delta\varphi$, we deduce that

$$\int_0^T \langle \psi + \operatorname{div}(a(x)|\nabla(u-\delta\varphi)|^{p-2}\nabla(u-\delta\varphi)), \varphi\rangle dt \geq 0, \quad \text{if } \delta > 0,$$
$$\int_0^T \langle \psi + \operatorname{div}(a(x)|\nabla(u-\delta\varphi)|^{p-2}\nabla(u-\delta\varphi)), \varphi\rangle dt \leq 0, \quad \text{if } \delta < 0,$$

for all $\varphi \in L^p(0,T; W_0^{1,p}(\Omega, a))$. Letting $\delta \to 0$, we get

$$\int_0^T \langle \psi + \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u), \varphi\rangle dt = 0,\ \forall \varphi \in L^p(0,T; W_0^{1,p}(\Omega, a)).$$

This implies that $\psi = -\operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big)$ in $L^q(0,T; W^{-1,q}(\Omega, a))$. We now prove $u(0) = u_0$. Choosing some test function $\varphi \in C^1([0,T]; W_0^{1,p}(\Omega, a) \cap L^\infty(\Omega))$ with $\varphi(T) = 0$ and integrating by parts in t in the approximate equations, we have

$$\int_0^T -\langle u_n, \varphi'\rangle dt + \int_0^T \langle L_{p,a}u_n, \varphi\rangle dt + \int_{\Omega_T} (f(u_n)\varphi - g\varphi)dxdt = (u_n(0), \varphi(0)).$$

Taking limits as $n \to \infty$, we obtain

$$\int_0^T -\langle u, \varphi'\rangle dt + \int_0^T \langle L_{p,a}u, \varphi\rangle dt + \int_{\Omega_T} (f(u)\varphi - g\varphi)dxdt = (u_0, \varphi(0)), \tag{3.12}$$

since $u_n(0) \to u_0$. On the other hand, for the "limiting equation", we have

$$\int_0^T -\langle u, \varphi'\rangle dt + \int_0^T \langle L_{p,a}u, \varphi\rangle dt + \int_{\Omega_T} (f(u)\varphi - g\varphi)dxdt = (u(0), \varphi(0)). \tag{3.13}$$

Comparing (3.12) and (3.13), we get $u(0) = u_0$, which completes the proof of existence.

(ii) *Uniqueness and continuous dependence.* Let u, v be two weak solutions of problem (1.1) with initial data u_0, v_0 in $L^2(\Omega)$, respectively. Then $w := u - v$

satisfies

$$\begin{cases} \dfrac{dw}{dt} + (L_{p,a}u - L_{p,a}v) + (f(u) - f(v)) = 0, \\ w(0) = u_0 - v_0. \end{cases} \tag{3.14}$$

Multiplying the first equation in (3.14) by $\widehat{B}_k(w)$, then integrating from ε to t, we obtain

$$\begin{aligned} &\int_\varepsilon^t \int_\Omega \frac{d}{ds}\big(w(s)\widehat{B}_k(w)(s)\big)dxds - \int_\varepsilon^t \int_\Omega w \frac{d}{ds}\widehat{B}_k(w(s))dxds \\ &\quad + \int_\varepsilon^t \int_\Omega (a(x)|\nabla u|^{p-2}\nabla u - a(x)|\nabla v|^{p-2}\nabla v)\nabla(\widehat{B}_k(w)(s))dxds \\ &\quad + \int_\varepsilon^t \int_\Omega (f(u) - f(v))\widehat{B}_k(w)(s)dxds = 0. \end{aligned} \tag{3.15}$$

Since $w\frac{d}{dt}\big(\widehat{B}_k(w)\big) = \frac{1}{2}\frac{d}{dt}\big((\widehat{B}_k(w))^2\big)$, we introduce from Proposition 2.2 and (1.3) by passing (3.15) to the limit as $k \to \infty$ and $\varepsilon \to 0$ that

$$\|w\|^2_{L^2(\Omega)} \le \|w(0)\|^2_{L^2(\Omega)} + 2\ell \int_0^t \|w(s)\|^2_{L^2(\Omega)}ds.$$

Applying the Gronwall inequality, we obtain

$$\|w(t)\|^2_{L^2(\Omega)} \le \|w(0)\|^2_{L^2(\Omega)}e^{2\ell t} \quad \text{for all } t \in [0,T].$$

This completes the proof. □

4. Existence of global attractors

By Theorem 3.1, we can define a continuous nonlinear semigroup

$$S(t) : L^2(\Omega) \to L^2(\Omega),\ u_0 \mapsto S(t)u_0 := u(t),$$

where $u(t)$ is the unique weak solution to problem (1.1) with initial datum u_0. The aim of this section is to prove the existence of global attractors in various spaces for the semigroup $S(t)$.

For the sake of brevity, in the following lemmas we give some formal calculations, the rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [19].

Lemma 4.1. *The semigroup $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set in $L^2(\Omega)$.*

Proof. Multiplying (1.1) by u and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^p_{W_0^{1,p}(\Omega,a)} + \int_\Omega f(u)udx = \int_\Omega gudx. \tag{4.1}$$

In the case $p = 2$, analogously to (3.2) we have

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + (c_0 - \mu)\|u\|^2_{L^2(\Omega)} \le 2c_1|\Omega| + \frac{1}{c_0 - \mu}\|g\|^2_{L^2(\Omega)}.$$

Applying the Gronwall lemma, we get

$$\|u(t)\|^2_{L^2(\Omega)} \le \|u_0\|^2_{L^2(\Omega)} e^{-(c_0-\mu)t} + C(\|g\|_{L^2(\Omega)}, |\Omega|, \mu, c_0, c_1). \tag{4.2}$$

In the case $p > 2$, analogously to (3.3) we have

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \le 2C_1 + 2c_1|\Omega| + \|g\|^2_{L^2(\Omega)}.$$

Applying the Gronwall lemma, we get

$$\|u(t)\|^2_{L^2(\Omega)} \le \|u_0\|^2_{L^2(\Omega)} e^{-t} + C(\|g\|_{L^2(\Omega)}, |\Omega|, C_1, c_1). \tag{4.3}$$

From (4.2) and (4.3), we see that $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set in $L^2(\Omega)$, i.e., there is a positive constant ρ_0 such that for any bounded subset B in $L^2(\Omega)$, there exists $T_1 = T_1(B)$ which depends only on the L^2-norm of B such that

$$\|S(t)u_0\|^2_{L^2(\Omega)} \le \rho_0 \quad \text{for all } t \ge T_1, u_0 \in B. \tag{4.4}$$

This completes the proof. □

Lemma 4.2. *The semigroup $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set in $W^{1,p}_0(\Omega, a)$.*

Proof. Multiplying the first equation in (1.1) by $L_{p,a}u$ and integrating by parts, we obtain

$$\frac{1}{p}\frac{d}{dt}\|u\|^p_{W^{1,p}_0(\Omega,a)} + \|L_{p,a}u\|^2_{L^2(\Omega)} = -\int_\Omega f'(u)a(x)|\nabla u|^p dx + \int_\Omega g L_{p,a}u dx.$$

Using (H1), (1.3) and the Cauchy inequality, we deduce that

$$\frac{d}{dt}\|u\|^p_{W^{1,p}_0(\Omega,a)} \le \ell p\|u\|^p_{W^{1,p}_0(\Omega,a)} + C\|g\|^2_{L^2(\Omega)}. \tag{4.5}$$

On the other hand, integrating (4.1) from t to $t+1$ and using (1.2) together with the Cauchy inequality, we have

$$\begin{aligned} &\frac{1}{2}\|u(t+1)\|^2_{L^2(\Omega)} + \int_t^{t+1} \|u(s)\|^p_{W^{1,p}_0(\Omega,a)} ds \\ \le\ &\left(\mu + \frac{1}{2}\right)\int_t^{t+1} \|u(s)\|^2_{L^2(\Omega)} ds + \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + c_1|\Omega| + \frac{1}{2}\|g\|^2_{L^2(\Omega)}. \end{aligned}$$

In view of (4.4), we get the following estimate

$$\int_t^{t+1} \|u(s)\|^p_{W^{1,p}_0(\Omega,a)} ds \le \left((\mu + \frac{1}{2})\rho_0 + c_1|\Omega| + \frac{1}{2}\|g\|^2_{L^2(\Omega)}\right) \tag{4.6}$$

for all $t \ge T_0$. As an application of the uniform Gronwall inequality, we deduce from (4.5) and (4.6) that

$$\|u(t)\|^p_{W^{1,p}_0(\Omega,a)} \le \rho_1 \tag{4.7}$$

for all $t \ge T_1 = T_0 + 1$. □

From Lemma 4.1 and the compactness of the embedding $W_0^{1,p}(\Omega, a) \hookrightarrow L^2(\Omega)$, we immediately obtain the following result.

Theorem 4.3. *Assume that assumptions* (H1)-(H3) *are satisfied. Then the semigroup* $\{S(t)\}_{t\geq 0}$ *associated to* (1.1) *has a global attractor* $\mathcal{A}_{L^2}$ *in* $L^2(\Omega)$.

Lemma 4.4. *Assume that assumptions* (H1)-(H3) *hold. Then there exists a bounded absorbing set for semigroup* $\{S(t)\}_{t\geq 0}$ *in* $D(L_{p,a})$.

Proof. By differentiating (1.1) in time and denoting $v = u_t$, we get

$$v_t - \operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla v\big) - (p-2)\operatorname{div}\big(a(x)|\nabla u|^{p-4}(\nabla u \cdot \nabla v)\nabla u\big) + f'(u)v = 0.$$

Multiplying the above equality by v, integrating over Ω and using (1.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Omega)} + \int_\Omega a(x)|\nabla u|^{p-2}|\nabla v|^2 + (p-2)\int_\Omega a(x)|\nabla u|^{p-4}(\nabla u \cdot \nabla v)^2$$
$$\leq \ell\|v\|^2_{L^2(\Omega)},$$

and hence,

$$\frac{d}{dt}\|v\|^2_{L^2(\Omega)} \leq 2\ell\|v\|^2_{L^2(\Omega)}. \tag{4.8}$$

On the other hand, multiplying the first equation in (1.1) by u_t, we get

$$\|u_t\|^2_{L^2(\Omega)} + \frac{1}{p}\frac{d}{dt}\|u\|^p_{W_0^{1,p}(\Omega)} + \int_\Omega f(u)u_t dx - \int_\Omega g u_t dx = 0.$$

We can rewrite this equality as follows

$$\frac{d}{dt}\Big[\frac{1}{p}\int_\Omega a(x)|\nabla u|^p dx + \int_\Omega F(u)dx - \int_\Omega g u dx\Big] = -\|u_t\|^2_{L^2(\Omega)} \leq 0, \tag{4.9}$$

where $F(s) = \int_0^s f(\tau)d\tau$. On the other hand, integrating (4.1) from t to $t+1$ and using (4.4) we get

$$\int_t^{t+1}\left[\|u\|^p_{W_0^{1,p}(\Omega,a)} + \int_\Omega f(u)u dx - \int_\Omega g u dx\right] ds \leq \frac{\rho_0}{2}$$

for all $t \geq T_0$. It follows from (1.3) that

$$F(u) \leq f(u)u + \frac{\ell}{2}u^2 \text{ for all } u \in \mathbb{R}.$$

Hence, we have

$$\int_t^{t+1}\left[\frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega,a)} + \int_\Omega F(u)dx - \int_\Omega g u dx\right] ds \leq \frac{\ell+1}{2}\rho_0 \tag{4.10}$$

for all $t \geq T_0$. Using the uniform Gronwall inequality, it follows from (4.9) and (4.10) that

$$\frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega,a)} + \int_\Omega F(u)dx - \int_\Omega g u dx \leq \rho_2 \tag{4.11}$$

for all $t \geq T_2 = T_1 + 1$, and $\rho_2 = C(\ell, \rho_0, \rho_1, \|g\|_{L^2(\Omega)})$. Integrating (4.9) from t to $t+1$ and using (4.11), we infer that

$$\int_t^{t+1} \|u_t\|^2_{L^2(\Omega)} ds \leq \rho_2. \tag{4.12}$$

Using the uniform Gronwall inequality once again, from (4.8) and (4.12) we deduce that

$$\|u_t(t)\|^2_{L^2(\Omega)} \leq \rho_3 \quad \text{for all } t \geq T_3 = T_2 + 1. \tag{4.13}$$

On the other hand, multiplying the first equation in (1.1) by $L_{p,a}u$, using (1.3) and the Cauchy inequality, we obtain

$$\begin{aligned} \|L_{p,a}u\|^2_{L^2(\Omega)} &= -\int_\Omega u_t L_{p,a}u dx - \int_\Omega f'(u)a(x)|\nabla u|^p dx + \int_\Omega g L_{p,a}u dx \\ &\leq \ell\|u\|^p_{W_0^{1,p}(\Omega,a)} + \frac{1}{2}\|L_{p,a}u\|^2_{L^2(\Omega)} + \|u_t\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)}. \end{aligned}$$

The following estimate is obtained from (4.7) and (4.13),

$$\|u(t)\|^{2(p-1)}_{D(L_{p,a})} = \|L_{p,a}u(t)\|^2_{L^2(\Omega)} \leq \rho_4 \quad \text{for all } t \geq T_3.$$

This inequality implies the desired result. □

Since the embedding $D(L_{p,a}) \hookrightarrow W_0^{1,p}(\Omega, a)$ is compact (see Proposition 2.3), we immediately get the following result.

Theorem 4.5. *Assume that assumptions* (H1)-(H3) *are satisfied. Then the semigroup* $\{S(t)\}_{t\geq 0}$ *associated to* (1.1) *has a global attractor* $\mathcal{A}_{D(L_{p,a})}$ *in* $W_0^{1,p}(\Omega, a)$.

Remark 4.6. It is interesting to extend the results obtained in the present paper to the case that the domain Ω is unbounded. In this case, the problem turns to be much more complicated due to the lack of the compactness of necessary Sobolev type embeddings. This interesting question, which was proposed by a reviewer, will be the subject for our future work.

Acknowledgements. The authors would like to thank the reviewers for their helpful comments and suggestions, which help us to improve the presentation of the paper. We also thank the anonymous reviewer for suggesting an interesting problem, which will be our future research subject.

The research of the third author is supported by the Hung Vuong University.

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