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BERGER TYPE DEFORMED SASAKI METRIC ON THE COTANGENT BUNDLE

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Abstract. In this paper, we introduce the Berger type deformed Sasaki metric on the cotangent bundle T^*M over an anti-paraKähler manifold (M, φ, g) as a new natural metric with respect to g non-rigid on T^*M. Firstly, we investigate the Levi-Civita connection of this metric. Secondly, we study the curvature tensor and also we characterize the scalar curvature.

1. Introduction

In the field, one of the first works which deal with the cotangent bundles of a manifold as a Riemannian manifold is that of E. M. Patterson and A. G. Walker [7], who constructed from an affine symmetric connection on a manifold a Riemannian metric on the cotangent bundle, which they call the Riemann extension of the connection. A generalization of this metric had been given by M. Sekizawa [11] in his classification of natural transformations of affine connections on manifolds to metrics on their cotangent bundles, obtaining the class of natural Riemann extensions which is a 2-parameter family of metrics, and which had been intensively studied by many authors. On the other hand, inspired by the concept of g-natural metrics on tangent bundles of Riemannian manifolds, F. Ağca considered another class of metrics on cotangent bundles of Riemannian manifolds, that he called g-natural metrics [1]. Also, there are studies by other authors, A. A. Salimov and F. Ağca [2, 8], K. Yano and S. Ishihara[12], F. Ocak and S. Kazimova [6], F. Ocak [5], A. Gezer and M. Altunbas [4] etc..

The main idea in this note consists of the modification of the Sasaki metric [8, 10]. First, we introduce the Berger type deformed Sasaki metric on the cotangent bundle T^*M over an anti-paraKähler manifold (M, φ, g). Then, we establish the Levi-Civita connection (Theorem 3.4 and Proposition 3.6) and the curvature tensor (Theorem 4.1) and we characterize the sectional curvature

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(Theorem 4.3 and Proposition 4.4) and the scalar curvature (Theorem 4.7 and Proposition 4.9).

2. Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)_{i=\overline{1,m}, \bar{i}=m+i}$ on T^*M, where p_i is the component of covector p in each cotangent space $T^*_x M$, $x \in U$ with respect to the natural coframe dx^i. Let $C^\infty(M)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M (resp. T^*M) and $\Im^r_s(M)$ (resp. $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r, s). Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on T^*M, the vertical distribution $VT^*M = \mathrm{Ker}(d\pi)$ and the horizontal distribution HT^*M that define a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M. \tag{1}$$

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be local expressions in $(U, x^i)_{i=\overline{1,m}}$, $U \subset M$ of a vector and covector (1-form) field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the complete and horizontal lifts X^C, $X^H \in \Im^1_0(T^*M)$ of $X \in \Im^1_0(M)$ and the vertical lift $\omega^V \in \Im^1_0(T^*M)$ of $\omega \in \Im^0_1(M)$ are defined, respectively by

$$X^C = X^i \frac{\partial}{\partial x^i} - p_h \frac{\partial X^h}{\partial x^i} \frac{\partial}{\partial x^{\bar{i}}}, \tag{2}$$

$$X^H = X^i \frac{\partial}{\partial x^i} + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{3}$$

$$\omega^V = \omega_i \frac{\partial}{\partial x^{\bar{i}}}, \tag{4}$$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, (see [12] for more details).

From (3) and (4) we see that $(\frac{\partial}{\partial x^i})^H$ and $(dx^i)^V$ have respectively local expressions of the form

$$\tilde{e}_i = (\frac{\partial}{\partial x^i})^H = \frac{\partial}{\partial x^i} + p_a \Gamma^a_{hi} \frac{\partial}{\partial x^{\bar{h}}}, \tag{5}$$

$$\tilde{e}_{\bar{i}} = (dx^i)^V = \frac{\partial}{\partial x^{\bar{i}}}. \tag{6}$$

The set of vector fields $\{\tilde{e}_i\}$ on $\pi^{-1}(U)$ defines a local frame for HT^*M over $\pi^{-1}(U)$ and the set of vector fields $\{\tilde{e}_{\bar{i}}\}$ on $\pi^{-1}(U)$ defines a local frame for VT^*M over $\pi^{-1}(U)$. The set $\{\tilde{e}_\alpha\} = \{\tilde{e}_i, \tilde{e}_{\bar{i}}\}$ defines a local frame on T^*M, adapted to the direct sum decomposition (1). The indices $\alpha, \beta, \ldots = \overline{1, 2m}$ indicate the indices with respect to the adapted frame.

Using (3), (4) we have

$$X^H = X^i \tilde{e}_i \,, \quad X^H \; = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{7}$$

$$\omega^V = \omega_i \tilde{e}_{\bar{i}} \,, \quad \omega^V \; = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}, \tag{8}$$

with respect to the adapted frame $\{\tilde{e}_\alpha\}_{\alpha=\overline{1,2m}}$, (see [12] for more details).

Lemma 2.1 ([12])**.** *Let (M, g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M of M satisfies the following*:

(1) $[\omega^V, \theta^V] = 0$,
(2) $[X^H, \theta^V] = (\nabla_X \theta)^V$,
(3) $[X^H, Y^H] = [X, Y]^H + (pR(X, Y))^V$,

for all vector fields $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

For a Riemannian manifold (M, g), we define the map

$$\begin{array}{ccc} \Im^0_1(M) & \to & \Im^1_0(M) \\ \omega & \mapsto & \tilde{\omega} \end{array}$$

$g(\tilde{\omega}, X) = \omega(X)$ for all $X \in \Im^1_0(M)$.

Locally for all $\omega = \omega_i dx^i \in \Im^0_1(M)$, we have $\tilde{\omega} = g^{ij}\omega_i \frac{\partial}{\partial x^j}$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

For each $x \in M$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $T^*_x M$ by $g^{-1}(\omega, \theta) = g(\tilde{\omega}, \tilde{\theta}) = g^{ij}\omega_i\theta_j$. In this case we have $\tilde{\omega} = g^{-1} \circ \omega$.

We define the map

$$\begin{array}{ccc} \Im^1_0(M) & \to & \Im^0_1(M) \\ X & \mapsto & \tilde{X} \end{array}$$

by $\tilde{X}(Y) = g(X, Y)$ for all $Y \in \Im^1_0(M)$.

Locally for all $X = X^i \frac{\partial}{\partial x^i} \in \Im^1_0(M)$, we have $\tilde{X} = g_{ij} X^i dx^j = g \circ X$.

Lemma 2.2. *For a Riemannian manifold (M, g), we have the followings*:

$$\tilde{\tilde{\omega}} = \omega, \quad \tilde{\tilde{X}} = X, \tag{9}$$

$$g^{-1}(\omega, \theta\varphi) = g(\varphi\tilde{\omega}, \tilde{\theta}), \tag{10}$$

$$\nabla_X \tilde{\omega} = \widetilde{\nabla_X \omega}, \tag{11}$$

$$X g^{-1}(\omega, \theta) = g^{-1}(\nabla_X \omega, \theta) + g^{-1}(\omega, \nabla_X \theta), \tag{12}$$

for all $X \in \Im^1_0(M)$, $\omega, \theta \in \Im^0_1(M)$ and $\varphi \in \Im^1_1(M)$, where ∇ is the Levi-Civita connection of (M, g).

Proof. (i) The formula (9) is easy.

(ii) Let $\varphi = \varphi_h^j \frac{\partial}{\partial x^j} \otimes dx^h$, $\omega = \omega_k dx^k$ and $\theta = \theta_i dx^i$. Then we have

$$\begin{aligned} g^{-1}(\omega, \theta\varphi) &= g^{-1}(\omega_k dx^k, \theta_i \varphi_h^i dx^h) = g^{kh}\omega_k \theta_i \varphi_h^i = g^{kh}\tilde{\omega}^t g_{tk} \tilde{\theta}^s g_{is}\varphi_h^i \\ &= \delta_t^h \tilde{\omega}^t \tilde{\theta}^s g_{is} \varphi_h^i = g_{is}\varphi_h^i \tilde{\omega}^h \tilde{\theta}^s = g(\varphi_h^i \tilde{\omega}^h \frac{\partial}{\partial x^i}, \tilde{\theta}^s \frac{\partial}{\partial x^s}) = g(\varphi\tilde{\omega}, \tilde{\theta}). \end{aligned}$$

(iii) For all $Y \in \Im_0^1(M)$, we have

$$\begin{aligned} g(\nabla_X \tilde{\omega}, Y) &= X(g(\tilde{\omega}, Y)) - g(\tilde{\omega}, \nabla_X Y) \\ &= X(\omega(Y)) - \omega(\nabla_X Y) \\ &= (\nabla_X \omega)(Y) \\ &= g(\widetilde{\nabla_X \omega}, Y). \end{aligned}$$

(iv)
$$\begin{aligned} X g^{-1}(\omega, \theta) &= X g(\tilde{\omega}, \tilde{\theta}) \\ &= g(\nabla_X \tilde{\omega}, \tilde{\theta}) + g(\tilde{\omega}, \nabla_X \tilde{\theta}) \\ &= g(\widetilde{\nabla_X \omega}, \tilde{\theta}) + g(\tilde{\omega}, \widetilde{\nabla_X \theta}) \\ &= g^{-1}(\nabla_X \omega, \theta) + g^{-1}(\omega, \nabla_X \theta). \end{aligned}$$ □

3. Berger type deformed Sasaki metric

3.1. Berger type deformed Sasaki metric

Let M be a $2m$-dimensional Riemannian manifold with a Riemannian metric g. An almost paracomplex manifold is an almost product manifold (M^{2m}, φ), $\varphi^2 = id$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank.

Let (M^{2m}, φ) be an almost paracomplex manifold. A Riemannian metric g is said to be an anti-paraHermitian metric (B-metric) [9] if

$$g(\varphi X, \varphi Y) = g(X, Y) \Leftrightarrow g(\varphi X, Y) = g(X, \varphi Y), \tag{13}$$

or from (10) equivalently

$$g^{-1}(\omega\varphi, \theta\varphi) = g^{-1}(\omega, \theta) \Leftrightarrow g^{-1}(\omega\varphi, \theta) = g^{-1}(\omega, \theta\varphi) \tag{14}$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$.

If (M^{2m}, φ) is an almost paracomplex manifold with an anti-paraHermitian metric g, then the triple (M^{2m}, φ, g) is said to be an almost anti-paraHermitian manifold (an almost B-manifold) [9]. Moreover, (M^{2m}, φ, g) is said to be anti-paraKähler manifold (B-manifold) [9] if φ is parallel with respect to the Levi-Civita connection ∇ of g, i.e., $(\nabla\varphi = 0)$.

It is well known that if (M^{2m}, φ, g) is an anti-paraKähler manifold, the Riemannian curvature tensor is pure [9], and for all $Y, Z \in \Im_0^1(M)$,

$$\begin{cases} R(\varphi Y, Z) = R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) = R(Y, Z). \end{cases} \tag{15}$$

Definition. Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold and T^*M be its cotangent bundle. A fiber-wise Berger type deformation of the Sasaki metric denoted by $\tilde{g}$ is defined on T^*M by

$$\tilde{g}(X^H, Y^H) = g(X, Y) = g(X, Y) \circ \pi, \tag{16}$$

$$\tilde{g}(X^H, \theta^V) = 0, \tag{17}$$

$$\tilde{g}(\omega^V, \theta^V) = g^{-1}(\omega, \theta) + \delta^2 g^{-1}(\omega, p\varphi) g^{-1}(\theta, p\varphi), \tag{18}$$

for all $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$, where δ is some constant [3].

Since any tensor field of type $(0, s)$ on T^*M where $s \geq 1$ is completely determined with the vector fields of type X^H and ω^V where $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$ (see [12]). In the particular case the metric $\tilde{g}$ is a tensor field of type $(0, 2)$ on T^*M. It follows that $\tilde{g}$ is completely determined by its formulas (16), (17) and (18).

By means of (2) and (3), the complete lift X^C of $X \in \Im_0^1(M)$ is given by

$$X^C = X^H - (p(\nabla X))^V, \tag{19}$$

where $p(\nabla X) = (p(\nabla X))_i dx^i = p_h(\nabla X)_i^h dx^i = p_h(\frac{\partial X^h}{\partial x^i} + \Gamma_{ij}^h X^j) dx^i$.

Taking account of (16), (17), (18) and (19), we obtain

$$\begin{aligned} \tilde{g}(X^C, Y^C) &= g(X, Y)^V + g^{-1}(p(\nabla X), p(\nabla Y)) \\ &\quad + \delta^2 g^{-1}(p(\nabla X), p\varphi) g^{-1}(p(\nabla Y), p\varphi), \end{aligned} \tag{20}$$

where

$$g^{-1}(p(\nabla X), p(\nabla Y)) = g^{ij}(p(\nabla X))_i (p(\nabla Y))_j = g^{ij} p_h p_k (\nabla X)_i^h (\nabla Y)_j^k,$$

$$g^{-1}(p(\nabla X), p\varphi) = g^{ij}(p(\nabla X))_i (p\varphi)_j = g^{ij} p_h p_k (\nabla X)_i^h \varphi_j^k.$$

Since the tensor field $\tilde{g} \in \Im_2^0(T^*M)$ is completely determined also by its action on vector fields of type X^C and Y^C (see [12]), we say that formula (20) is an alternative characterization of $\tilde{g}$.

Remark 3.1. From formulas (16), (17), (18), the Berger type deformed Sasaki metric $\tilde{g}$ has components

$$\tilde{g} = \begin{pmatrix} g_{ij} & 0 \\ 0 & g^{ij} + \delta^2 g^{it} g^{js} (p\varphi)_t (p\varphi)_s \end{pmatrix} \tag{21}$$

with respect to the adapted frame $\{\tilde{e}_\alpha\}_{\alpha=\overline{1,2m}}$.

Lemma 3.2. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then, we have the followings*:

(1) $X^H(f(\alpha)) = 0$,
(2) $\omega^V(f(\alpha)) = 2f'(\alpha) g^{-1}(\omega, p)$,
(3) $X^H g(Y, Z) = X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,
(4) $X^H g^{-1}(\theta, \eta) = X g^{-1}(\theta, \eta) = g^{-1}(\nabla_X \theta, \eta) + g^{-1}(\theta, \nabla_X \eta)$,

(5) $X^H g^{-1}(\theta, p\varphi) = g^{-1}(\nabla_X \theta, p\varphi)$,
(6) $\omega^V g^{-1}(\theta, p\varphi) = g^{-1}(\theta, \omega\varphi)$,
(7) $\omega^V g^{-1}(\theta, \eta) = 0$,
(8) $\omega^V g(Y, Z) = 0$,

for all $X, Y, Z \in \Im^1_0(M)$ *and* $\omega, \theta, \eta \in \Im^0_1(M)$, *where* $\alpha = g^{-1}(p, p)$.

Lemma 3.3. *Let* (M^{2m}, φ, g) *be an anti-paraKähler manifold and* $(T^*M, \tilde{g})$ *be its cotangent bundle equipped with the Berger type deformed Sasaki metric. Then, we have the followings*:

(1) $X^H \tilde{g}(\theta^V, \eta^V) = \tilde{g}((\nabla_X \theta)^V, \eta^V) + \tilde{g}(\theta^V, (\nabla_X \eta)^V)$,
(2) $\omega^V \tilde{g}(\theta^V, \eta^V) = \delta^2 g^{-1}(\theta, \omega\varphi) g^{-1}(\eta, p\varphi) + \delta^2 g^{-1}(\theta, p\varphi) g^{-1}(\eta, \omega\varphi)$,

for all $X \in \Im^1_0(M)$ *and* $\omega, \theta, \eta \in \Im^0_1(M)$.

Proof. From Lemma 3.2, we have

$$\begin{aligned}
X^H \tilde{g}(\theta^V, \eta^V) &= X^H \left[g^{-1}(\theta, \eta) + \delta^2 g^{-1}(\theta, p\varphi) g^{-1}(\eta, p\varphi) \right] \\
&= X^H (g^{-1}(\theta, \eta)) + \delta^2 X^H (g^{-1}(\theta, p\varphi)) g^{-1}(\eta, p\varphi) \\
&\quad + \delta^2 g^{-1}(\theta, p\varphi) X^H (g^{-1}(\eta, p\varphi)) \\
&= g^{-1}(\nabla_X \theta, \eta) + g^{-1}(\theta, \nabla_X \eta) + \delta^2 g^{-1}(\nabla_X \theta, p\varphi) g^{-1}(\eta, p\varphi) \\
&\quad + \delta^2 g^{-1}(\theta, p\varphi) g^{-1}(\nabla_X \eta, p\varphi) \\
&= \tilde{g}((\nabla_X \theta)^V, \eta^V) + \tilde{g}(\theta^V, (\nabla_X \eta)^V), \\
\omega^V \tilde{g}(\theta^V, \eta^V) &= \omega^V \left[g^{-1}(\theta, \eta) + \delta^2 g^{-1}(\theta, p\varphi) g^{-1}(\eta, p\varphi) \right] \\
&= \omega^V (g^{-1}(\theta, \eta)) + \delta^2 \omega^V (g^{-1}(\theta, p\varphi)) g^{-1}(\eta, p\varphi) \\
&\quad + \delta^2 g^{-1}(\theta, p\varphi) \omega^V (g^{-1}(\eta, p\varphi)) \\
&= \delta^2 g^{-1}(\theta, \omega\varphi) g^{-1}(\eta, p\varphi) + \delta^2 g^{-1}(\theta, p\varphi) g^{-1}(\eta, \omega\varphi). \qquad \square
\end{aligned}$$

3.2. Levi-Civita connection of Berger type deformed Sasaki metric

We shall calculate the Levi-Civita connection $\widetilde{\nabla}$ of T^*M with Berger type deformed Sasaki metric $\tilde{g}$. This connection is characterized by the Koszul formula:

$$\begin{aligned}
2\tilde{g}(\widetilde{\nabla}_U V, W) &= U\tilde{g}(V, W) + V\tilde{g}(W, U) - W\tilde{g}(U, V) \\
&\quad + \tilde{g}(W, [U, V]) + \tilde{g}(V, [W, U]) - \tilde{g}(U, [V, W])
\end{aligned} \tag{22}$$

for all $U, V, W \in \Im^1_0(T^*M)$.

Theorem 3.4. *Let* (M^{2m}, φ, g) *be an anti-paraKähler manifold and* $(T^*M, \tilde{g})$ *its cotangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the following formulas.*

(1) $\widetilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H + \dfrac{1}{2}(pR(X, Y))^V$,
(2) $\widetilde{\nabla}_{X^H} \theta^V = (\nabla_X \theta)^V + \dfrac{1}{2}(R(\tilde{p}, \tilde{\theta})X)^H$,

(3) $\widetilde{\nabla}_{\omega^V} Y^H = \frac{1}{2}(R(\tilde{p},\tilde{\omega})Y)^H$,

(4) $\widetilde{\nabla}_{\omega^V} \theta^V = \frac{\delta^2}{1+\delta^2\alpha} g^{-1}(\omega,\theta\varphi)(p\varphi)^V$,

for all $X, Y \in \Im^1_0(M)$, $\omega,\theta \in \Im^0_1(M)$ *and* $\alpha = g^{-1}(p,p)$, *where* ∇ *and* R *denote the Levi-Civita connection and the curvature tensor of* (M^{2m},φ,g), *respectively.*

Proof. The proof of Theorem 3.4 follows directly from Kozul formula (22), Lemma 2.1, Definition 3.1 and Lemma 3.3.

(1) Direct calculations give,

$$\begin{aligned}
\text{(i) } 2\tilde{g}(\widetilde{\nabla}_{X^H} Y^H, Z^H) &= X^H\tilde{g}(Y^H,Z^H) + Y^H\tilde{g}(Z^H,X^H) - Z^H\tilde{g}(X^H,Y^H)\\
&\quad + \tilde{g}(Z^H,[X^H,Y^H]) + \tilde{g}(Y^H,[Z^H,X^H])\\
&\quad - \tilde{g}(X^H,[Y^H,Z^H])\\
&= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y])\\
&\quad + g(Y,[Z,X]) - g(X,[Y,Z])\\
&= 2g(\nabla_X Y, Z) = 2\tilde{g}((\nabla_X Y)^H, Z^H).
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } 2\tilde{g}(\widetilde{\nabla}_{X^H} Y^H, \eta^V) &= X^H\tilde{g}(Y^H,\eta^V) + Y^H\tilde{g}(\eta^V,X^H) - \eta^V\tilde{g}(X^H,Y^H)\\
&\quad + \tilde{g}(\eta^V,[X^H,Y^H]) + \tilde{g}(Y^H,[\eta^V,X^H])\\
&\quad - \tilde{g}(X^H,[Y^H,\eta^V])\\
&= \tilde{g}(\eta^V,[X^H,Y^H]) = \tilde{g}((pR(X,Y))^V,\eta^V),
\end{aligned}$$

then,

$$\widetilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H + \frac{1}{2}(pR(X,Y))^V.$$

(2) By straightforward calculations,

$$\begin{aligned}
\text{(iii) } 2\tilde{g}(\widetilde{\nabla}_{X^H}\theta^V, Z^H) &= X^H\tilde{g}(\theta^V,Z^H) + \theta^V\tilde{g}(Z^H,X^H) - Z^H\tilde{g}(X^H,\theta^V)\\
&\quad + \tilde{g}(Z^H,[X^H,\theta^V]) + \tilde{g}(\theta^V,[Z^H,X^H])\\
&\quad - \tilde{g}(X^H,[\theta^V,Z^H])\\
&= \tilde{g}(\theta^V,[Z^H,X^H])\\
&= \tilde{g}((pR(Z,X))^V,\theta^V)\\
&= g^{-1}(pR(Z,X),\theta) + \delta^2 g^{-1}(pR(Z,X),p\varphi)g^{-1}(\theta,p\varphi).
\end{aligned}$$

Using the formula,

$$\widetilde{\omega R(Z,X)} = R(X,Z)\tilde{\omega}. \tag{23}$$

Where, for any $Y \in \Im^1_0(M)$, we get:

$$\begin{aligned}
g(\widetilde{\omega R(Z,X)}, Y) &= g_{st}(\widetilde{\omega R(Z,X)})^s Y^t = g_{st}g^{ks}(\omega R(Z,X))_k Y^t\\
&= \delta^k_t \omega_a R^a_{ijk} Z^i X^j Y^t = g_{ab}\tilde{\omega}^b R^a_{ijk} Z^i X^j Y^k\\
&= g(R(Z,X)Y,\tilde{\omega}) = g(R(X,Z)\tilde{\omega}, Y).
\end{aligned}$$

From (23), we have

$$g^{-1}(pR(Z,X),\theta) = g(\widetilde{\mathrm{pR}(Z,X)},\tilde{\theta}) = g(R(X,Z)\tilde{p},\tilde{\theta}) = g(R(\tilde{p},\tilde{\theta})X,Z)$$
$$= \tilde{g}((R(\tilde{p},\tilde{\theta})X)^H, Z^H).$$

From (23) and (15), we have

$$g^{-1}(pR(Z,X),p\varphi) = g(\varphi(\widetilde{\mathrm{pR}(Z,X)}),\tilde{p}) = g(\varphi R(X,Z)\tilde{p},\tilde{p})$$
$$= g(R(\varphi X,Z)\tilde{p},\tilde{p}) = 0, \tag{24}$$

then,

$$2\tilde{g}(\widetilde{\nabla}_{X^H}\theta^V, Z^H) = \tilde{g}((R(\tilde{p},\tilde{\theta})X)^H, Z^H),$$

and also with direct calculations we get,

(iv) $2\tilde{g}(\widetilde{\nabla}_{X^H}\theta^V,\eta^V) = X^H\tilde{g}(\theta^V,\eta^V) + \theta^V\tilde{g}(\eta^V,X^H) - \eta^V\tilde{g}(X^H,\theta^V)$
$+ \tilde{g}(\eta^V,[X^H,\theta^V]) + \tilde{g}(\theta^V,[\eta^V,X^H])$
$- \tilde{g}(X^H,[\theta^V,\eta^V])$
$= X^H\tilde{g}(\theta^V,\eta^V) + \tilde{g}(\eta^V,[X^H,\theta^V]) + \tilde{g}(\theta^V,[\eta^V,X^H]).$

Using the first formula of Lemma 3.3 we have:

$$2\tilde{g}(\widetilde{\nabla}_{X^H}\theta^V,\eta^V) = \tilde{g}((\nabla_X\theta)^V,\eta^V) + \tilde{g}(\theta^V,(\nabla_X\eta)^V)$$
$$+ \tilde{g}(\eta^V,(\nabla_X\theta)^V) - \tilde{g}(\theta^V,(\nabla_X\eta)^V)$$
$$= 2\tilde{g}((\nabla_X\theta)^V,\eta^V).$$

So, we have:

$$\widetilde{\nabla}_{X^H}\theta^V = (\nabla_X\theta)^V + \frac{1}{2}(R(\tilde{p},\tilde{\theta})X)^H.$$

The other formulas are obtained by a similar calculations. □

Lemma 3.5. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T^*M,\tilde{g})$ be its cotangent bundle equipped with the Berger type deformed Sasaki metric. Then*

(1) $\widetilde{\nabla}_{X^H}(p\varphi)^V = 0,$

(2) $\widetilde{\nabla}_{\omega^V}(p\varphi)^V = (\omega\varphi)^V + \dfrac{\delta^2}{1+\delta^2\alpha}g^{-1}(\omega,p)(p\varphi)^V,$

for all $X \in \Im^1_0(M)$, $\omega \in \Im^0_1(M)$ and $\alpha = g^{-1}(p,p)$.

Definition. Let (M^{2m},φ,g) be an anti-paraKähler manifold and F is a tensor field of type $(1,1)$ on M. Then the vertical and horizontal vector fields F^V and F^H respectively are defined on T^*M by

$$\begin{array}{rcl} F^V : T^*M & \to & TT^*M \\ (x,p) & \mapsto & F^V(x,p) = (pF)^V, \end{array}$$

$$\begin{array}{rcl} F^H : T^*M & \to & TT^*M \\ (x,p) & \mapsto & F^H(x,p) = (F(\tilde{p}))^H, \end{array}$$

locally we have

$$F^V = p_i(dx^i F)^V, \tag{25}$$

$$F^H = \tilde{p}^i(F(\frac{\partial}{\partial x^i}))^H, \tag{26}$$

where $p = p_j dx^j$ and $\tilde{p} = \tilde{p}^i \frac{\partial}{\partial x^i} = p_j g^{ij} \frac{\partial}{\partial x^i}$.

Proposition 3.6. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(T^*M, \tilde{g})$ be its tangent bundle equipped with the Berger type deformed Sasaki metric and F be a tensor field of type $(1,1)$ on M. Then:*

(1) $\widetilde{\nabla}_{X^H} F^H = (\nabla_X F)^H + \frac{1}{2}(pR(X, F(\tilde{p})))^V$,

(2) $\widetilde{\nabla}_{X^H} F^V = (\nabla_X F)^V + \frac{1}{2}(R(\tilde{p}, \widetilde{pF})X)^H$,

(3) $\widetilde{\nabla}_{\omega^V} F^H = (F(\widetilde{\omega}))^H + \frac{1}{2}(R(\tilde{p}, \tilde{\omega})F(\tilde{p}))^H$,

(4) $\widetilde{\nabla}_{\omega^V} F^V = (\omega F)^V + \dfrac{\delta^2}{1+\delta^2\alpha} g^{-1}(\omega\varphi, pF)(p\varphi)^V$,

for all $X \in \Im^1_0(M)$, $\omega \in \Im^0_1(M)$ and $\widetilde{pF} = g^{-1} \circ (pF)$, where ∇ and R denote the Levi-Civita connection and the curvature tensor of (M^{2m}, φ, g), respectively.

Proof. By Definition 3.2 and Theorem 3.4 we have:

$$\begin{aligned}
(1)\ \widetilde{\nabla}_{X^H} F^H &= \widetilde{\nabla}_{X^H}(\tilde{p}^k(F(\frac{\partial}{\partial x^k}))^H) \\
&= X^H(\tilde{p}^k)(F(\frac{\partial}{\partial x^k}))^H + \tilde{p}^k \widetilde{\nabla}_{X^H} F(\frac{\partial}{\partial_k})^H \\
&= X^i\big[\frac{\partial}{\partial x^i}(\tilde{p}^k) + p_a\Gamma^a_{hi}\frac{\partial}{\partial p_h}(\tilde{p}^k)\big](F(\frac{\partial}{\partial x^k}))^H + \tilde{p}^k(\nabla_X F(\frac{\partial}{\partial_k}))^H \\
&\quad + \frac{\tilde{p}^k}{2}\big(pR(X, F(\frac{\partial}{\partial x^k}))\big)^V \\
&= (X(\tilde{p}^k)F(\frac{\partial}{\partial x^k}) + \tilde{p}^k\nabla_X F(\frac{\partial}{\partial x^k}))^H \\
&\quad + X^i p_a \Gamma^a_{hi}\frac{\partial}{\partial p_h}(p_j g^{jk})(F(\frac{\partial}{\partial x^k}))^H + \frac{1}{2}\big(pR(X, F(\tilde{p}))\big)^V \\
&= (\nabla_X(\tilde{p}^k F(\frac{\partial}{\partial x_k})))^H + X^i p_a \Gamma^a_{ji} g^{jk}(F(\frac{\partial}{\partial x^k}))^H \\
&\quad + \frac{1}{2}\big(pR(X, F(\tilde{p}))\big)^V.
\end{aligned}$$

Since the $(\nabla_X p)_j = -X^i p_a \Gamma^a_{ji}$, $\widetilde{dx_j} = g^{jk}\frac{\partial}{\partial x^k}$, $\widetilde{\nabla_X p} = (\nabla_X p)_j \widetilde{dx_j}$ and (15) then

$$\begin{aligned}
\widetilde{\nabla}_{X^H} F^H &= (\nabla_X F(\tilde{p}))^H - (F(\widetilde{\nabla_X p}))^H + \frac{1}{2}\big(pR(X, F(\tilde{p}))\big)^V \\
&= (\nabla_X F(\tilde{p}))^H - (F(\nabla_X \tilde{p}))^H + \frac{1}{2}\big(pR(X, F(\tilde{p}))\big)^V
\end{aligned}$$

$$= ((\nabla_X F)(\tilde{p}))^H + \frac{1}{2}\big(pR(X, F(\tilde{p}))\big)^V$$
$$= (\nabla_X F)^H + \frac{1}{2}\big(pR(X, F(\tilde{p}))\big)^V.$$

The other formulas are obtained by a similar calculation. □

4. Curvatures of Berger type deformed Sasaki metric

We shall calculate the Riemannian curvature tensor $\widetilde{R}$ of T^*M with the Berger type deformed Sasaki metric $\tilde{g}$. This curvature tensor is characterized by the formula,

$$\widetilde{R}(U, V)W = \widetilde{\nabla}_U \widetilde{\nabla}_V W - \widetilde{\nabla}_V \widetilde{\nabla}_U W - \widetilde{\nabla}_{[U,V]} W \tag{27}$$

for all $U, V, W \in \Im_0^1(T^*M)$.

Theorem 4.1. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(T^*M, \tilde{g})$ be its cotangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the following formulas*:

$$\begin{aligned} \widetilde{R}(X^H, Y^H)Z^H = {} & (R(X, Y)Z)^H + \frac{1}{4}(R(\tilde{p}, R(Z, Y)\tilde{p})X)^H \\ & - \frac{1}{4}(R(\tilde{p}, R(Z, X)\tilde{p})Y)^H + \frac{1}{2}(R(\tilde{p}, R(X, Y)\tilde{p})Z)^H \\ & + \frac{1}{2}\big(\widetilde{(\nabla_Z R)(X, Y)\tilde{p}}\,\big)^V, \end{aligned} \tag{28}$$

$$\begin{aligned} \widetilde{R}(X^H, \theta^V)Z^H = {} & \frac{1}{2}((\nabla_X R)(\tilde{p}, \tilde{\theta})Z)^H + \frac{1}{4}(pR(X, R(\tilde{p}, \tilde{\theta})Z))^V \\ & - \frac{\delta^2}{2(1+\delta^2\alpha)} g^{-1}(\theta\varphi, pR(X, Z))(p\varphi)^V \\ & - \frac{1}{2}(\theta R(X, Z))^V, \end{aligned} \tag{29}$$

$$\begin{aligned} \widetilde{R}(X^H, Y^H)\eta^V = {} & \frac{1}{2}((\nabla_X R)(\tilde{p}, \tilde{\eta})Y)^H - \frac{1}{2}((\nabla_Y R)(\tilde{p}, \tilde{\eta})X)^H \\ & + \frac{1}{4}(pR(X, R(\tilde{p}, \tilde{\eta})Y))^V - \frac{1}{4}(pR(Y, R(\tilde{p}, \tilde{\eta})X))^V \\ & - (\eta R(X, Y))^V - \frac{\delta^2}{1+\delta^2\alpha} g^{-1}(pR(X, Y), \eta\varphi)(p\varphi)^V, \end{aligned} \tag{30}$$

$$\widetilde{R}(X^H, \theta^V)\eta^V = \frac{-1}{2}(R(\tilde{\theta}, \tilde{\eta})X)^H - \frac{1}{4}(R(\tilde{p}, \tilde{\theta})R(\tilde{p}, \tilde{\eta})X)^H, \tag{31}$$

$$\begin{aligned} \widetilde{R}(\omega^V, \theta^V)Z^H = {} & (R(\tilde{\omega}, \tilde{\theta})Z)^H + \frac{1}{4}(R(\tilde{p}, \tilde{\omega})R(\tilde{p}, \tilde{\theta})Z)^H \\ & - \frac{1}{4}(R(\tilde{p}, \tilde{\theta})R(\tilde{p}, \tilde{\omega})Z)^H, \end{aligned} \tag{32}$$

$$\begin{aligned}\widetilde{R}(\omega^V,\theta^V)\eta^V &= \frac{-\delta^4}{(1+\delta^2\alpha)^2}g^{-1}(\omega,p)g^{-1}(\theta,\eta\varphi)(p\varphi)^V\\ &+\frac{\delta^4}{(1+\delta^2\alpha)^2}g^{-1}(\theta,p)g^{-1}(\omega,\eta\varphi)(p\varphi)^V\\ &+\frac{\delta^2}{1+\delta^2\alpha}\big[g^{-1}(\theta,\eta\varphi)(\omega\varphi)^V-g^{-1}(\omega,\eta\varphi)(\theta\varphi)^V\big],\end{aligned}\tag{33}$$

for all $X,Y,Z\in\Im^1_0(M)$ *and* $\omega,\theta,\eta\in\Im^0_1(M)$*, where* ∇ *and* R *denote respectively the Levi-Civita connection and the curvature tensor of* (M^{2m},φ,g) *and* $\widetilde{(\nabla_Z R)(X,Y)\tilde{p}}=g\circ(\nabla_Z R)(X,Y)\tilde{p}\in\Im^0_1(M)$.

Proof. Let $X,Y,Z\in\Im^1_0(M)$ and $\omega,\theta,\eta\in\Im^0_1(M)$. By applying Theorem 3.4 and Proposition 3.6 we have:

(1) $\widetilde{R}(X^H,Y^H)Z^H=\widetilde{\nabla}_{X^H}\widetilde{\nabla}_{Y^H}Z^H-\widetilde{\nabla}_{Y^H}\widetilde{\nabla}_{X^H}Z^H-\widetilde{\nabla}_{[X^H,Y^H]}Z^H$.

(i) Let $F:T^*M\to T^*M$ be the bundle map given by $pF=pR(Y,Z)$.
Direct calculations and using (23), we have

$$\begin{aligned}\widetilde{\nabla}_{X^H}\widetilde{\nabla}_{Y^H}Z^H &= \widetilde{\nabla}_{X^H}\big[(\nabla_Y Z)^H+\frac{1}{2}F^V\big]\\ &=(\nabla_X\nabla_Y Z)^H+\frac{1}{2}(pR(X,\nabla_Y Z))^V+\frac{1}{2}(\nabla_X(pR(Y,Z)))^V\\ &-\frac{1}{2}((\nabla_X p)R(Y,Z))^V+\frac{1}{4}(R(\tilde{p},\widetilde{\mathrm{pR}(Y,Z)})X)^H,\\ &=(\nabla_X\nabla_Y Z)^H+\frac{1}{2}(pR(X,\nabla_Y Z))^V+\frac{1}{2}(\nabla_X(pR(Y,Z)))^V\\ &-\frac{1}{2}((\nabla_X p)R(Y,Z))^V+\frac{1}{4}(R(\tilde{p},R(Z,Y)\tilde{p})X)^H.\end{aligned}$$

(ii) With permutation of X by Y, we have

$$\begin{aligned}\widetilde{\nabla}_{Y^H}\widetilde{\nabla}_{X^H}Z^H &=(\nabla_Y\nabla_X Z)^H+\frac{1}{2}(pR(Y,\nabla_X Z))^V+\frac{1}{2}(\nabla_Y(pR(X,Z)))^V\\ &-\frac{1}{2}((\nabla_Y p)R(X,Z))^V+\frac{1}{4}(R(\tilde{p},R(Z,X)\tilde{p})Y)^H.\end{aligned}$$

(iii) Direct calculations give

$$\begin{aligned}\widetilde{\nabla}_{[X^H,Y^H]}Z^H &=\widetilde{\nabla}_{[X,Y]^H}Z^H+\widetilde{\nabla}_{(pR(X,Y))^V}Z^H\\ &=(\nabla_{[X,Y]}Z)^H+\frac{1}{2}(pR([X,Y],Z))^V+\frac{1}{2}(R(\tilde{p},R(Y,X)\tilde{p})Z)^H.\end{aligned}$$

Hence, we have:

$$\begin{aligned}\widetilde{R}(X^H,Y^H)Z^H &=(R(X,Y)Z)^H+\frac{1}{4}(R(\tilde{p},R(Z,Y)\tilde{p})X)^H\\ &-\frac{1}{4}(R(\tilde{p},R(Z,X)\tilde{p})Y)^H+\frac{1}{2}(R(\tilde{p},R(X,Y)\tilde{p})Z)^H\\ &+\frac{1}{2}\big((\nabla_X(pR(Y,Z))^V-(\nabla_X p)R(Y,Z))^V\end{aligned}$$

$$
\begin{aligned}
&- pR((\nabla_X Y), Z))^V - pR(Y, (\nabla_X Z)))^V\big)\\
&- \frac{1}{2}\big((\nabla_Y(pR(X,Z))^V - (\nabla_Y p)R(X,Z))^V\\
&- pR((\nabla_Y X), Z))^V - pR(X, (\nabla_Y Z)))^V\big)\\
=\ & (R(X,Y)Z)^H + \frac{1}{4}(R(\tilde{p}, R(Z,Y)\tilde{p})X)^H\\
&- \frac{1}{4}(R(\tilde{p}, R(Z,X)\tilde{p})Y)^H + \frac{1}{2}(R(\tilde{p}, R(X,Y)\tilde{p})Z)^H\\
&+ \frac{1}{2}\big(\widetilde{(\nabla_X R)(Z,Y)\tilde{p}}\,\big)^V - \frac{1}{2}\big(\widetilde{(\nabla_Y R)(Z,X)\tilde{p}}\,\big)^V,
\end{aligned}
$$

where

$$
\begin{aligned}
&\big(\nabla_X(pR(Y,Z)) - (\nabla_X p)R(Y,Z) - pR(\nabla_X Y, Z) - pR(Y, \nabla_X Z)\big)^V\\
=\ & \big(\nabla_X(\widetilde{R(Z,Y)\tilde{p}}) - \widetilde{R(Z,Y)\widetilde{\nabla_X p}} - \widetilde{R(Z,\nabla_X Y)\tilde{p}} - \widetilde{R(\nabla_X Z, Y)\tilde{p}}\,\big)^V\\
=\ & \big(\widetilde{\nabla_X(R(Z,Y)\tilde{p})} - \widetilde{R(Z,Y)(\nabla_X\tilde{p})} - \widetilde{R(Z,\nabla_X Y)\tilde{p}} - \widetilde{R(\nabla_X Z, Y)\tilde{p}}\,\big)^V\\
=\ & \big(\widetilde{(\nabla_X R)(Z,Y)\tilde{p}}\,\big)^V.
\end{aligned}
$$

Using the second Bianchi identity, we obtain the formula (28).

(2) $\widetilde{R}(X^H, \theta^V)Z^H = \widetilde{\nabla}_{X^H}\widetilde{\nabla}_{\theta^V}Z^H - \widetilde{\nabla}_{\theta^V}\widetilde{\nabla}_{X^H}Z^H - \widetilde{\nabla}_{[X^H,\theta^V]}Z^H$.

(i) Let $F : TM \to TM$ be the bundle map given by $F(\tilde{p}) = R(\tilde{p}, \tilde{\theta})Z$.

$$
\begin{aligned}
\widetilde{\nabla}_{X^H}\widetilde{\nabla}_{\theta^V}Z^H &= \frac{1}{2}\widetilde{\nabla}_{X^H}F^H\\
&= \frac{1}{2}\big(\nabla_X(R(\tilde{p},\tilde{\theta})Z) - R(\nabla_X\tilde{p},\tilde{\theta})Z\big)^H + \frac{1}{4}(pR(X, R(\tilde{p},\tilde{\theta})Z))^V.
\end{aligned}
$$

(ii) Let $F : T^*M \to T^*M$ be the bundle map given by $pF = pR(X,Z)$.

$$
\begin{aligned}
\widetilde{\nabla}_{\theta^V}\widetilde{\nabla}_{X^H}Z^H &= \widetilde{\nabla}_{\theta^V}\big[(\nabla_X Z)^H + \frac{1}{2}F^V\big]\\
&= \frac{1}{2}(R(\tilde{p},\tilde{\theta})(\nabla_X Z))^H + \frac{1}{2}(\theta R(X,Z))^V\\
&\quad + \frac{\delta^2}{2(1+\delta^2\alpha)}g^{-1}(\theta\varphi, pR(X,Z))(p\varphi)^V.
\end{aligned}
$$

(iii) Direct calculations give

$$
\widetilde{\nabla}_{[X^H,\theta^V]}Z^H = \widetilde{\nabla}_{(\nabla_X\theta)^V}Z^H = \frac{1}{2}(R(\tilde{p}, \widetilde{\nabla_X\theta})Z)^H,
$$

we obtain the formula (29).

(3) Applying formula (29) and 1^{st} Bianchi identity, we get

$$
\begin{aligned}
\widetilde{R}(X^H, Y^H)\eta^V &= \widetilde{R}(X^H, \eta^V)Y^H - \widetilde{R}(Y^H, \eta^V)X^H\\
&= \frac{1}{2}((\nabla_X R)(\tilde{p},\tilde{\eta})Y)^H + \frac{1}{4}(pR(X, R(\tilde{p},\tilde{\eta})Y))^V
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}(\eta R(X,Y))^V - \frac{\delta^2}{2(1+\delta^2\alpha)} g^{-1}(\eta\varphi, pR(X,Y))(p\varphi)^V \\
&- \frac{1}{2}((\nabla_Y R)(\tilde{p},\tilde{\eta})X)^H - \frac{1}{4}(pR(Y, R(\tilde{p},\tilde{\eta})X))^V \\
&+ \frac{1}{2}(\eta R(Y,X))^V + \frac{\delta^2}{2(1+\delta^2\alpha)} g^{-1}(\eta\varphi, pR(Y,X))(p\varphi)^V \\
= &\frac{1}{2}((\nabla_X R)(\tilde{p},\tilde{\eta})Y)^H - \frac{1}{2}((\nabla_Y R)(\tilde{p},\tilde{\eta})X)^H \\
&+ \frac{1}{4}(pR(X, R(\tilde{p},\tilde{\eta})Y))^V - \frac{1}{4}(pR(Y, R(\tilde{p},\tilde{\eta})X))^V \\
&- (\eta R(X,Y))^V - \frac{\delta^2}{1+\delta^2\alpha} g^{-1}(pR(X,Y), \eta\varphi)(p\varphi)^V .
\end{aligned}
$$

The other formulas are obtained by a similar calculation. □

4.1. Sectional curvature of Berger type deformed Sasaki metric

It is known that the sectional curvature $\widetilde{K}$ on $(T^*M, \tilde{g})$ for P is given by

$$
\widetilde{K}(V,W) = \frac{\tilde{g}(\widetilde{R}(V,W)W,V)}{\tilde{g}(V,V)\tilde{g}(W,W) - \tilde{g}(V,W)^2}, \tag{34}
$$

where $P = P(V,W)$ denotes the plane spanned by $\{V, W\}$, for all, linearly independent vector fields $V, W \in \Im^1_0(T^*M)$. Let $\widetilde{K}(X^H, Y^H)$, $\widetilde{K}(X^H, \theta^V)$ and $\widetilde{K}(\omega^V, \theta^V)$ denote the sectional curvatures of the planes spanned by $\{X^H, Y^H\}$, $\{X^H, \theta^V\}$ and $\{\omega^V, \theta^V\}$ on $(T^*M, \tilde{g})$ respectively, where X, Y are orthonormal vector fields and ω, θ are orthonormal covector fields on M.

Proposition 4.2. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(T^*M, \tilde{g})$ be its cotangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the following*:

(i) $\tilde{g}(\widetilde{R}(X^H, Y^H)Y^H, X^H) = g(R(X,Y)Y,X) - \frac{3}{4}\|R(X,Y)\tilde{p}\|^2$,

(ii) $\tilde{g}(\widetilde{R}(X^H, \theta^V)\theta^V, X^H) = \frac{1}{4}\|R(\tilde{p},\tilde{\theta})X\|^2$,

(iii) $\tilde{g}(\widetilde{R}(\omega^V, \theta^V)\theta^V, \omega^V) = \frac{\delta^2}{1+\delta^2\alpha}\left[g^{-1}(\omega,\omega\varphi)g^{-1}(\theta,\theta\varphi) - g^{-1}(\omega,\theta\varphi)^2\right]$.

Proof. (i) From the formula (28), we have

$$
\begin{aligned}
\tilde{g}(\widetilde{R}(X^H, Y^H)Y^H, X^H) &= g(R(X,Y)Y,X) + \frac{1}{4}g(R(\tilde{p}, R(Y,Y)\tilde{p})X,X) \\
&\quad - \frac{1}{4}g(R(\tilde{p}, R(Y,X)\tilde{p})Y,X) \\
&\quad + \frac{1}{2}g(R(\tilde{p}, R(X,Y)\tilde{p})Y,X) \\
&= g(R(X,Y)Y,X) - \frac{3}{4}\|R(X,Y)\tilde{p}\|^2.
\end{aligned}
$$

(ii) From the formula (31), we have

$$\tilde{g}(\widetilde{R}(X^H,\theta^V)\theta^V,X^H) = -\frac{1}{2}g(R(\tilde{\theta},\tilde{\theta})X,X) - \frac{1}{4}g(R(\tilde{p},\tilde{\theta})R(\tilde{p},\tilde{\theta})X,X)$$
$$= \frac{1}{4}\|R(\tilde{p},\tilde{\theta})X\|^2.$$

(iii) The result follows immediately from the formula (33)

$$\begin{aligned}\tilde{g}(\widetilde{R}(\omega^V,\theta^V)\theta^V,\omega^V) &= \frac{-\delta^4}{(1+\delta^2\alpha)^2}g^{-1}(\omega,p)g^{-1}(\theta,\theta\varphi)\tilde{g}((p\varphi)^V,\omega^V)\\ &\quad+\frac{\delta^4}{(1+\delta^2\alpha)^2}g^{-1}(\theta,p)g^{-1}(\omega,\theta\varphi)\tilde{g}((p\varphi)^V,\omega^V)\\ &\quad+\frac{\delta^2}{1+\delta^2\alpha}g^{-1}(\theta,\theta\varphi)\tilde{g}((\omega\varphi)^V,\omega^V)\\ &\quad-\frac{\delta^2}{1+\delta^2\alpha}g^{-1}(\omega,\theta\varphi)\tilde{g}((\theta\varphi)^V,\omega^V)\\ &= \frac{\delta^2}{1+\delta^2\alpha}\big[g^{-1}(\omega,\omega\varphi)g^{-1}(\theta,\theta\varphi)-g^{-1}(\omega,\theta\varphi)^2\big].\end{aligned}$$ □

Theorem 4.3. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T^*M,\tilde{g})$ be its cotangent bundle equipped with the Berger type deformed Sasaki metric. Then the sectional curvature $\widetilde{K}$ satisfy the following equations*:

(1) $\widetilde{K}(X^H,Y^H) = K(X,Y) - \frac{3}{4}\|R(X,Y)\tilde{p}\|^2$,

(2) $\widetilde{K}(X^H,\theta^V) = \dfrac{\|R(\tilde{p},\tilde{\theta})X\|^2}{4(1+\delta^2 g^{-1}(\theta,p\varphi)^2)}$,

(3) $\widetilde{K}(\omega^V,\theta^V) = \dfrac{\delta^2\big[g^{-1}(\omega,\omega\varphi)g^{-1}(\theta,\theta\varphi)-g^{-1}(\omega,\theta\varphi)^2\big]}{(1+\delta^2\alpha)\big[1+\delta^2(g^{-1}(\omega,p\varphi)^2+g^{-1}(\theta,p\varphi)^2)\big]}$,

where K denote the sectional curvature tensor of (M^{2m},φ,g).

Proof. Using Proposition 4.2 we have

(1)
$$\begin{aligned}\widetilde{K}(X^H,Y^H) &= \frac{\tilde{g}(\widetilde{R}(X^H,Y^H)Y^H,X^H)}{\tilde{g}(X^H,X^H)\tilde{g}(Y^H,Y^H)-\tilde{g}(X^H,Y^H)^2}\\ &= \tilde{g}(\widetilde{R}(X^H,Y^H)Y^H,X^H)\\ &= K(X,Y)-\frac{3}{4}\|R(X,Y)\tilde{p}\|^2.\end{aligned}$$

(2)
$$\begin{aligned}\widetilde{K}(X^H,\theta^V) &= \frac{\tilde{g}(\widetilde{R}(X^H,\theta^V)\theta^V,X^H)}{\tilde{g}(X^H,X^H)\tilde{g}(\theta^V,\theta^V)-\tilde{g}(X^H,\theta^V)^2}\\ &= \frac{1}{1+\delta^2 g^{-1}(\theta,p\varphi)^2}\tilde{g}(\widetilde{R}(X^H,\theta^V)\theta^V,X^H)\\ &= \frac{\|R(\tilde{p},\tilde{\theta})X\|^2}{4(1+\delta^2 g^{-1}(\theta,p\varphi)^2)}.\end{aligned}$$

$$(3)\ \widetilde{K}(\omega^V,\theta^V) = \frac{\tilde{g}(\widetilde{R}(\omega^V,\theta^V)\theta^V,\omega^V)}{\tilde{g}(\omega^V,\omega^V)\tilde{g}(\theta^V,\theta^V)-\tilde{g}(\omega^V,\theta^V)^2}$$
$$= \frac{1}{1+\delta^2(g^{-1}(\omega,p\varphi)^2+g^{-1}(\theta,p\varphi)^2)}\tilde{g}(\widetilde{R}(\omega^V,\theta^V)\theta^V,\omega^V)$$
$$= \frac{\delta^2\big[g^{-1}(\omega,\omega\varphi)g^{-1}(\theta,\theta\varphi)-g^{-1}(\omega,\theta\varphi)^2\big]}{(1+\delta^2\alpha)\big[1+\delta^2(g^{-1}(\omega,p\varphi)^2+g^{-1}(\theta,p\varphi)^2)\big]}. \qquad \square$$

Proposition 4.4. *Let (M^{2m},φ,g) be an anti-paraKähler manifold of constant sectional curvature λ and $(T^*M,\tilde{g})$ its cotangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the following*:

(1) $\widetilde{K}(X^H,Y^H) = \lambda - \dfrac{3\lambda^2}{4}\big[g(X,\tilde{p})^2+g(Y,\tilde{p})^2\big]$,

(2) $\widetilde{K}(X^H,\theta^V) = \dfrac{\lambda^2\big[\alpha g(X,\tilde{\theta})^2-2g(X,\tilde{\theta})g(X,\tilde{p})g^{-1}(\theta,p)+g(X,\tilde{p})^2\big]}{4(1+\delta^2 g^{-1}(\theta,p\varphi)^2)}$,

(3) $\widetilde{K}(\omega^V,\theta^V) = \dfrac{\delta^2\big[g^{-1}(\omega,\omega\varphi)g^{-1}(\theta,\theta\varphi)-g^{-1}(\omega,\theta\varphi)^2\big]}{(1+\delta^2\alpha)\big[1+\delta^2(g^{-1}(\omega,p\varphi)^2+g^{-1}(\theta,p\varphi)^2)\big]}$.

Proof. Since M has a constant curvature λ, it follows that

$$R(X,Y)Z = \lambda\big[g(Y,Z)X-g(X,Z)Y\big]$$

and by a direct calculation we get

$$\|R(X,Y)\tilde{p}\|^2 = \lambda^2\big[g(X,\tilde{p})^2+g(Y,\tilde{p})^2\big],$$
$$\|R(\tilde{p},\tilde{\theta})X\|^2 = \lambda^2\big[\alpha g(X,\tilde{\theta})^2-2g(X,\tilde{\theta})g(X,\tilde{p})g^{-1}(\theta,p)+g(X,\tilde{p})^2\big].$$

This completes the proof. $\square$

Remark 4.5. Let $(x,p)\in T^*M$ with $p\neq 0$, $\{E_i\}_{i=\overline{1,2m}}$ and $\{\omega^i\}_{i=\overline{1,2m}}$ be a local orthonormal frame and coframe on M, respectively, such that $\omega^1=\frac{p}{\|p\|}$. Then

$$\big\{F_i=E_i^H, F_{n+1}=\frac{1}{\sqrt{1+\delta^2\alpha}}(\omega^1\varphi)^V, F_{n+j}=(\omega^j\varphi)^V\big\}_{i=\overline{1,n},j=\overline{2,n}} \tag{35}$$

is a local orthonormal frame on T^*M, where $n=2m$.

Lemma 4.6. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(T^*M,\tilde{g})$ its cotangent bundle equipped with the Berger type deformed Sasaki metric and $(F_a)_{a=\overline{1,2n}}$ be a local orthonormal frame on $(T^*M,\tilde{g})$ defined by* (35). *Then the sectional curvatures $\widetilde{K}$ satisfy the following equations*:

$$\widetilde{K}(F_i,F_j) = K(E_i,E_j)-\frac{3}{4}\|R(E_i,E_j)\tilde{p}\|^2,$$
$$\widetilde{K}(F_i,F_{n+1}) = 0,$$
$$\widetilde{K}(F_i,F_{n+l}) = \frac{1}{4}\|R(\tilde{p},\widetilde{\omega^l\varphi})E_i\|^2,$$
$$\widetilde{K}(F_{n+k},F_{n+1}) = \frac{\delta^2}{(1+\delta^2\alpha)^2}\big[g^{-1}(\omega^1,\omega^1\varphi)g^{-1}(\omega^k,\omega^k\varphi)-g^{-1}(\omega^1,\omega^k\varphi)^2\big],$$

$$\widetilde{K}(F_{n+k},F_{n+l})=\frac{\delta^2}{1+\delta^2\alpha}\big[g^{-1}(\omega^l,\omega^l\varphi)g^{-1}(\omega^k,\omega^k\varphi)-g^{-1}(\omega^l,\omega^k\varphi)^2\big],$$

for $i,j=\overline{1,n}$ *and* $k,l=\overline{2,n}$, *where* K *is a sectional curvature of* M.

Proof. The results comes directly from Theorem 4.3 and formula (35). □

4.2. Scalar curvature of Berger type deformed Sasaki metric

Theorem 4.7. *Let* (M^{2m},φ,g) *be an anti-paraKähler manifold and* $(T^*M,\tilde{g})$ *be its cotangent bundle equipped with the Berger type deformed Sasaki metric. If* σ *(resp.,* $\widetilde{\sigma}$*) denotes the scalar curvature of* (M^{2m},φ,g) *(resp.,* $(T^*M,\tilde{g})$*), then we have*

$$\begin{aligned}\widetilde{\sigma}=\sigma&-\frac{3}{4}\sum_{i,j=1}^{2m}\|R(E_i,(E_j)\tilde{p}\|^2+\frac{1}{2}\sum_{i,j=1}^{2m}\|R(\tilde{p},\widetilde{\omega^j\varphi})E_i\|^2\\&+\frac{\delta^2}{1+\delta^2\alpha}\Big(\frac{-2}{1+\delta^2\alpha}+W^2-2m+2-\frac{2\delta^2}{1+\delta^2\alpha}g^{-1}(p,p\varphi)W\Big),\end{aligned}\tag{36}$$

where $W=\sum_{i=1}^{2m}g^{-1}(\omega^i,\omega^i\varphi)$, $(E_i)_{i=\overline{1,2m}}$ *and* $(\omega^i)_{i=\overline{1,2m}}$ *is a local orthonormal frame and coframe on* M *respectively.*

Proof. Let $(F_a)_{a=\overline{1,2n}}$ be a local orthonormal frame on $(T^*M,\tilde{g})$ defined by (35). Using Theorem 4.3 and definition of scalar curvature, we have

$$\begin{aligned}\widetilde{\sigma}&=\sum_{i,j=1}^{n}\widetilde{K}(F_i,F_j)+2\sum_{i,j=1}^{n}\widetilde{K}(F_i,F_{n+j})+\sum_{i,j=1}^{n}\widetilde{K}(F_{n+i},F_{n+j})\\&=\sum_{i,j=1}^{n}\widetilde{K}(F_i,F_j)+2\sum_{i=1}^{n}\widetilde{K}(F_i,F_{n+1})+2\sum_{i=1,j=2}^{n}\widetilde{K}(F_i,F_{n+j})\\&\quad+2\sum_{i=2}^{n}\widetilde{K}(F_{n+i},F_{n+1})+\sum_{i,j=2}^{n}\widetilde{K}(F_{n+i},F_{n+j})\\&=\sum_{i,j=1}^{n}\big[K(E_i,E_j)-\frac{3}{4}\|R(E_i,E_j)\tilde{p}\|^2\big]+2\sum_{i=1,j=2}^{n}\frac{1}{4}\|R(\tilde{p},\widetilde{\omega^j\varphi})E_i\|^2\\&\quad+2\sum_{i=2}^{n}\frac{\delta^2}{(1+\delta^2\alpha)^2}\big[g^{-1}(\omega^1,\omega^1\varphi)g^{-1}(\omega^i,\omega^i\varphi)-g^{-1}(\omega^1,\omega^i\varphi)^2\big]\\&\quad+\sum_{i,j=2}^{n}\frac{\delta^2}{1+\delta^2\alpha}\big[g^{-1}(\omega^i,\omega^i\varphi)g^{-1}(\omega^j,\omega^j\varphi)-g^{-1}(\omega^i,\omega^j\varphi)^2\big]\\&=\sigma-\frac{3}{4}\sum_{i,j=1}^{n}\|R(E_i,E_j)\tilde{p}\|^2+\frac{1}{2}\sum_{i,j=1}^{n}\|R(\tilde{p},\widetilde{\omega^j\varphi})E_i\|^2\\&\quad+\frac{2\delta^2}{\alpha(1+\delta^2\alpha)^2}g^{-1}(p,p\varphi)\sum_{i=2}^{n}g^{-1}(\omega^i,\omega^i\varphi)\end{aligned}$$

$$
\begin{aligned}
&- \frac{2\delta^2}{(1+\delta^2\alpha)^2} \sum_{i=2}^{n} g^{-1}(\omega^1, \omega^i\varphi)^2 \\
&+ \frac{\delta^2}{1+\delta^2\alpha} \sum_{j=2}^{n} g^{-1}(\omega^j, \omega^j\varphi) \sum_{i=1}^{n} g^{-1}(\omega^i, \omega^i\varphi) \\
&- \frac{\delta^2}{1+\delta^2\alpha} \sum_{j=2}^{n} \sum_{i=2}^{n} g^{-1}(\omega^i, \omega^j\varphi)^2.
\end{aligned}
$$

We put $W = \sum_{i=1}^{n} g^{-1}(\omega^i, \omega^i\varphi)$, then

$$
\begin{aligned}
\widetilde{\sigma} = \sigma &- \frac{3}{4} \sum_{i,j=1}^{2m} \|R(E_i,(E_j)\tilde{p}\|^2 + \frac{1}{2} \sum_{i,j=1}^{2m} \|R(\tilde{p}, \widetilde{\omega^j\varphi})E_i\|^2 \\
&+ \frac{\delta^2}{1+\delta^2\alpha} \Big(\frac{-2}{1+\delta^2\alpha} + W^2 - 2m + 2 - \frac{2\delta^2}{1+\delta^2\alpha} g^{-1}(p, p\varphi) W \Big).
\end{aligned}
$$

□

Corollary 4.8. *Let (M^{2m}, φ, g) be a flat anti-paraKähler manifold and $(T^*M, \tilde{g})$ be its cotangent bundle equipped with the Berger type deformed Sasaki metric. Then the scalar curvature $\widetilde{\sigma}$ satisfy the following equations*:

$$
\widetilde{\sigma} = \frac{\delta^2}{1+\delta^2\alpha} \Big(\frac{-2}{1+\delta^2\alpha} + W^2 - 2m + 2 - \frac{2\delta^2}{1+\delta^2\alpha} g^{-1}(p, p\varphi) W \Big). \tag{37}
$$

Proposition 4.9. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold of constant sectional curvature λ and $(T^*M, \tilde{g})$ be its cotangent bundle equipped with the Berger type deformed Sasaki metric. Then the scalar curvature $\widetilde{\sigma}$ satisfy the following equations*:

$$
\begin{aligned}
\widetilde{\sigma} &= (2m-1)\lambda \left(2m - \frac{\alpha\lambda}{2} \right) \\
&+ \frac{\delta^2}{1+\delta^2\alpha} \Big(\frac{-2}{1+\delta^2\alpha} + W^2 - 2m + 2 - \frac{2\delta^2}{1+\delta^2\alpha} g^{-1}(p, p\varphi) W \Big).
\end{aligned} \tag{38}
$$

Proof. Since M has a constant curvature λ, it follows that

$$
R(X,Y)Z = \lambda\big[g(Y,Z)X - g(X,Z)Y\big]
$$

and

$$
\sigma = 2m(2m-1)\lambda,
$$

and

$$
\sum_{i,j=1}^{2m} \|R(E_i,(E_j)\tilde{p}\|^2 = \sum_{i,j=1}^{2m} \|R(\tilde{p}, \widetilde{\omega^j\varphi})E_i\|^2 = 2\alpha\lambda^2(2m-1).
$$

This completes the proof. □

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