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DIFFERENCES OF DIFFERENTIAL OPERATORS BETWEEN WEIGHTED-TYPE SPACES

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Abstract. Let $\mathcal{H}(\mathbb{D})$ be the space of analytic functions on the unit disc $\mathbb{D}$. Let $\psi = (\psi_j)_{j=0}^n$ and $\Phi = (\Phi_j)_{j=0}^n$ be such that $\psi_j, \Phi_j \in \mathcal{H}(\mathbb{D})$. The linear differential operator is defined by $T_\psi(f) = \sum_{j=0}^n \psi_j f^{(j)}$, $f \in \mathcal{H}(\mathbb{D})$. We characterize the boundedness and compactness of the difference operator $(T_\psi - T_\Phi)(f) = \sum_{j=0}^n (\psi_j - \Phi_j) f^{(j)}$ between weighted-type spaces of analytic functions. As applications, we obtained boundedness and compactness of the difference of multiplication operators between weighted-type and Bloch-type spaces. Also, we give examples of unbounded (non compact) differential operators such that their difference is bounded (compact).

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let $\mu, \varphi \in \mathcal{H}(\mathbb{D})$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The weighted composition operator $W_{\mu,\varphi} : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ is a linear operator defined by

$$W_{\mu,\varphi} f = \mu \cdot f \circ \varphi, \quad f \in \mathcal{H}(\mathbb{D}).$$

These operators have been appearing in a natural way on different function spaces and play a significant role in the isometry theory of Banach spaces. For more information on isometries we refer to the monographs of Fleming and Jamison [9, 10]. In case $\mu(z) = 1$ for all $z \in \mathbb{D}$, then the weighted composition operator $W_{\mu,\varphi}$ reduces to the composition operator which we denote by C_φ. If $\varphi(z) = z$ for all $z \in \mathbb{D}$, then $W_{\mu,\varphi}$ reduces to the multiplication operator which we denote by M_μ. For more information on composition operators, multiplication operators and weighted composition operators, we refer to the monographs [7, 12, 29, 31].

Let $n \in \mathbb{N}_0$. The nth-differentiation operator on $\mathcal{H}(\mathbb{D})$ is defined by $\mathcal{D}^n f = f^{(n)}$, where $\mathcal{D}^0 f = f$ and $\mathcal{D} f = f'$. The differentiation operator is typically

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unbounded on many analytic function spaces. The boundedness and compactness of the products $\mathcal{D}C_\varphi$ and $C_\varphi\mathcal{D}$ of composition operators and differentiation operators between Bergman spaces and Hardy spaces were first studied by Hibschweiler and Portnoy in [15] and then on Hardy spaces by Ohno [25]. Further, in this direction Zhu [43] investigated the boundedness and compactness of the product operators $\mathcal{D}C_\varphi$, $C_\varphi\mathcal{D}$, $\mathcal{D}M_\mu$ and $M_\mu\mathcal{D}$ from Bergman-type spaces to Bers-type spaces. Also, in [40, 41] Yanyan, Xiaoman and Yongmin characterized the boundedness and compactness of the product operators $\mathcal{D}M_\mu$ and $M_\mu\mathcal{D}$ from H^∞ and mixed norm spaces to Zygmund spaces and the Bloch-type spaces. The study of the product operators $\mathcal{D}M_\mu$ and $M_\mu\mathcal{D}$ is further extended to the product operators $\mathcal{D}^nM_\mu$ and $M_\mu\mathcal{D}^n$ which are explored by many authors while studying the products of multiplication operators, composition operators and differentiation operators on different spaces of analytic functions. For more information on these operators, we refer to [17, 23, 33] and the references therein. Also, the product operators $\mathcal{D}^nM_\mu$ and $M_\mu\mathcal{D}^n$ are included in the general class of linear differential operators which are defined as

$$T_\psi(f) = \sum_{j=0}^{n} \psi_j f^{(j)}, \qquad f \in \mathcal{H}(\mathbb{D}), \tag{1}$$

where $\psi = (\psi_j)_{j=0}^n$ such that $\psi_j \in \mathcal{H}(\mathbb{D})$.

These linear differential operators are extensively studied by many authors during the last several decades and have many applications in different areas such as mathematical physics, PDEs, harmonic maps theory, and Markov processes. For more information on these operators and their applications, we refer to [8, 11, 14, 27].

Let $\mu \in \mathcal{H}(\mathbb{D})$ and $\psi_j = \binom{n}{j}\mu^{(n-j)}$ for every $j = 0, \ldots, n$. Then by using Leibniz formula, from (1) we have

$$T_\psi(f) = \sum_{j=0}^{n} \binom{n}{j}\mu^{(n-j)} f^{(j)} = \mathcal{D}^n M_\mu(f), \qquad f \in \mathcal{H}(\mathbb{D}). \tag{2}$$

Also, in (1) if $\psi_j = 0$ for all $j = 0, \ldots, n-1$ and $\psi_n = \mu$, then we have

$$T_\psi(f) = \mu f^{(n)} = M_\mu\mathcal{D}^n(f), \quad f \in \mathcal{H}(\mathbb{D}). \tag{3}$$

For $n = 1$ in (2) and (3), the operators $\mathcal{D}M_\mu$ and $M_\mu\mathcal{D}$ are studied by Xiaoman and Yanyan [40], Yu and Liu [41] and Zhu [43].

In recent years, many authors are attracted towards the study of boundedness and compactness of the differences of composition operators, weighted composition operators and weighted differentiation composition operators acting on different Banach spaces of analytic functions as it leads to understand the topological structure of the sets of these operators in the operator norm topology. In 1981, Berkson [1] initiated the study of the topological structure by obtaining his isolation result in the space of composition operators on Hardy spaces. Further, Shapiro and Sundberg [30] and MacCluer, Ohno and Zhao [21] studied the topological structure of the set of composition operators on Hilbert-Hardy spaces and H^{∞}, respectively. In [16], Hosokawa, Izuchi and Ohno characterized the compactness of the differences of weighted composition operators on H^{∞}. Subsequently, many results of boundedness and compactness of the differences of these operators have been obtained by Moorhouse [24] on Bergman spaces, Song and Zhou [32] from Bloch space to H^{∞}, Wang, Yao and Chen [37] on weighted Bergman spaces, Tien and Khoi [35] between Fock spaces. Recently, Wang et al. [36] characterized the boundedness, compactness and order boundedness of the difference of Stević-Sharma operators between Banach spaces of holomorphic functions. Fore more information on the differences of these operators, we refer to [6, 13, 18, 22, 26, 34, 38] and references therein.

Motivated by these results, in this paper we characterize the boundedness and compactness of the differences of differential operators defined below in (4) between weighted Banach spaces of analytic functions.

$$(T_{\psi} - T_{\Phi})(f) = \sum_{j=0}^{n} (\psi_j - \Phi_j) f^{(j)}, \ f \in \mathcal{H}(\mathbb{D}), \tag{4}$$

for $\psi = (\psi_j)_{j=0}^n$ and $\Phi = (\Phi_j)_{j=0}^n$ such that $\psi_j,\ \Phi_j \in \mathcal{H}(\mathbb{D})$. As applications, we obtained boundedness and compactness of the differences of multiplication operators between weighted Banach spaces and weighted Bloch-type spaces of analytic functions. Also, we give examples of unbounded (non compact) differential operators such that their difference is bounded (compact).

2. Preliminaries

Let v be a strictly positive, continuous and bounded function on $\mathbb{D}$. We will call such a function v as a weight function or simply a weight. We define the weighted-type spaces of analytic functions as follows:

$$H_v^{\infty} = \{f \in \mathcal{H}(\mathbb{D}) : ||f||_v := \sup_{z \in \mathbb{D}} v(z)\ |f(z)| < \infty\},$$

and

$$H_{v,0}^{\infty} = \{f \in \mathcal{H}(\mathbb{D}) : \lim_{|z| \to 1} v(z)\ |f(z)| = 0\}.$$

Clearly H_v^{∞} is a Banach space under the norm $||f||_v := \sup_{z \in \mathbb{D}} v(z)\ |f(z)|$ and it is a natural space in the sense that the norm convergence in H_v^{∞} implies uniform convergence on compact subsets of $\mathbb{D}$. Also, $H_{v,0}^{\infty}$ is a closed subspace of H_v^{∞}. In case $v(z) = 1$, then $H_v^{\infty} = H^{\infty}$. The weighted Bloch space $\mathcal{B}_v$ is defined as

$$\mathcal{B}_v = \{f \in \mathcal{H}(\mathbb{D}) : ||f||_0 = \sup_{z \in \mathbb{D}} v(z)|f'(z)| < \infty\}.$$

If we define the norm $||\cdot||_v$ as

$$||f||_v = |f(0)| + ||f||_0, \quad f \in \mathcal{B}_v,$$

then $\mathcal{B}_v$ with $||\cdot||_v$ is a Banach space. We also define the weighted little Bloch space $\mathcal{B}_{v,0}$ as

$$\mathcal{B}_{v,0} = \{f \in \mathcal{H}(\mathbb{D}) : \lim_{|z|\longrightarrow 1} v(z)f'(z) = 0\}.$$

It is clear that $\mathcal{B}_{v,0}$ is a closed subspace of $\mathcal{B}_v$. For more details on the Bloch spaces, we refer to [42].

The associated weight $\widetilde{v}$ for a given weight v is defined as follows:

$$\widetilde{v}(z) = \left(\sup_{z\in\mathbb{D}} \{|f(z)| : f \in H_v^\infty,\ \|f\|_v \leq 1\} \right)^{-1} = \frac{1}{\|\delta_z\|_v},$$

where $\delta_z : H_v^\infty \to \mathbb{C}$ is the point evaluation linear functional. In the setting of general weighted spaces of analytic functions, the associated weight plays an important role. It has been seen in [3] that the following relations between v and $\widetilde{v}$ hold:

$$0 < v \leq \widetilde{v}, \text{ and } \widetilde{v} \text{ is bounded and continuous;} \tag{5}$$

$$\|f\|_v \leq 1 \text{ if and only if } \|f\|_{\widetilde{v}} \leq 1; \tag{6}$$

for each $z \in \mathbb{D}$ there exists f_z in the closed unit ball B_v^∞ of H_v^∞ such that

$$|f_z(z)| = \frac{1}{\widetilde{v}(z)}. \tag{7}$$

A weight v is said to be radial if $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. This radial and non-increasing weight is called typical if $\lim_{|z|\to 1} v(z) = 0$. Also, a weight v is called essential if there is a constant $k > 0$ such that

$$v(z) \leq \widetilde{v}(z) \leq kv(z) \tag{8}$$

for every $z \in \mathbb{D}$.

In [20], Lusky introduced the following condition (L1) which plays an important role in this paper:

$$\inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0. \tag{L1}$$

Radial weights which satisfy condition (L1) are always essential [5].

The standard weights $v_\alpha(z) = (1-|z|^2)^\alpha$, where $\alpha > 0$, and the logarithmic weights $v_\beta(z) = (1-\log(1-|z|^2))^\beta, \beta < 0$ satisfy condition (L1). For more details on the weighted-type spaces of analytic functions which have important applications in functional analysis, complex analysis, partial differential equations, convolution equations and distribution theory, we refer to [2–4, 19, 20].

For $a \in \mathbb{D}$, we define the functions $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$ and $\rho(z,a) = |\varphi_a(z)|$ for every $z \in \mathbb{D}$. φ_a is called a Mobius transformation that interchanges a and 0 and ρ is known as the pseudohyperbolic metric on $\mathbb{D}$. Note that $\varphi_a(\varphi_a(z)) = z$ and

$$\varphi_a'(z) = -\frac{1-|a|^2}{(1-\overline{a}z)^2},\ z \in \mathbb{D}.$$

We use the notation $A \preceq B$ for non-negative quantities A and B to mean that there is a positive constant C independent of the quantities A and B such that $A \leq CB$. In this paper the notation $A \asymp B$ means that $A \preceq B$ and $B \preceq A$.

Lemma 2.1 ([39]). *Let v be a radial weight satisfying condition* (L1). *Then there is $C_v > 0$ such that for every $f \in H_v^\infty$*

$$|f^{(n)}(z)| \leq \frac{C_v \|f\|_v}{(1-|z|^2)^n v(z)}$$

for every $z \in \mathbb{D}$ and every $n \in \mathbb{N}_0$.

In order to prove the boundedness and compactness of the operator $T_\psi - T_\Phi$, we need the following lemma whose proof is similar to Proposition 3.11 in [7].

Lemma 2.2. *Let v and w be arbitrary weights on $\mathbb{D}$. Let $\psi = (\psi_j)_{j=0}^n$ and $\Phi = (\Phi_j)_{j=0}^n$ be such that $\psi_j, \Phi_j \in \mathcal{H}(\mathbb{D})$. Then $T_\psi - T_\Phi : H_v^\infty \longrightarrow H_w^\infty$ is compact if and only if $T_\psi - T_\Phi : H_v^\infty \longrightarrow H_w^\infty$ is bounded and for any bounded sequence $\{f_n\}$ in H_v^∞ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, $||(T_\psi - T_\Phi)f_n||_w \to 0$ as $n \to \infty$.*

3. Boundedness and compactness of the operators $T_\psi - T_\Phi : H_v^\infty \to H_w^\infty$

In this section we characterize the boundedness and compactness of the operators $T_\psi - T_\Phi : H_v^\infty \to H_w^\infty$.

Theorem 3.1. *Let v be a radial weight satisfying condition* (L1) *and w be an arbitrary weight on $\mathbb{D}$. Let $\psi = (\psi_j)_{j=0}^n$ and $\Phi = (\Phi_j)_{j=0}^n$ be such that $\psi_j, \Phi_j \in \mathcal{H}(\mathbb{D})$. Then $T_\psi - T_\Phi : H_v^\infty \longrightarrow H_w^\infty$ is bounded if and only if*

$$A_j = \sup_{z\in\mathbb{D}} \frac{w(z)\big|\left(\psi_j - \Phi_j\right)(z)\big|}{(1-|z|^2)^j v(z)} < \infty \quad \text{for all } j = 0, 1, \ldots, n. \tag{9}$$

Moreover, if the operator $T_\psi - T_\Phi : H_v^\infty \longrightarrow H_w^\infty$ is bounded, then

$$\|T_\psi - T_\Phi\|_{H_v^\infty \longrightarrow H_w^\infty} \asymp \sum_{j=0}^n A_j. \tag{10}$$

Proof. First, suppose that the operator $T_\psi - T_\Phi$ is bounded. Fix $a \in \mathbb{D}$. According to (7), there exists $f_a \in B_v^\infty$ such that

$$|f_a(a)| = \frac{1}{\widetilde{v}(a)}.$$

Since v satisfies (L1), it is essential and we can replace $\widetilde{v}$ by v. Define

$$g_a(z) = (\varphi_a(z))^n f_a(z) \quad \text{for all } z \in \mathbb{D}.$$

Then $g_a^{(j)}(a) = 0$ for all $j < n$ and

$$|g_a^{(n)}(a)| = \frac{n!}{(1-|a|^2)^n v(a)}.$$

Thus

$$\begin{aligned}
\|T_\psi - T_\Phi\|_{H_v^\infty \to H_w^\infty} &\geq \|(T_\psi - T_\Phi)(g_a)\|_w \\
&\geq w(a)\left|\sum_{j=0}^{n} (\psi_j - \Phi_j)(a) g_a^{(j)}(a)\right| \\
&= w(a)|(\psi_n - \Phi_n)(a)||g_a^{(n)}(a)| \\
&\geq \frac{w(a)|(\psi_n - \Phi_n)(a)|}{(1-|a|^2)^n v(a)}. \qquad (11)
\end{aligned}$$

Hence from (11), we have

$$A_n = \sup_{a\in\mathbb{D}} \frac{w(a)\big|(\psi_n - \Phi_n)(a)\big|}{(1-|a|^2)^n v(a)} \leq \|T_\psi - T_\Phi\|_{H_v^\infty \to H_w^\infty} < \infty. \qquad (12)$$

This prove the condition (9) when $j = n$. Now for $j = n-1$, define

$$h_a(z) = (\varphi_a(z))^{n-1} f_a(z) \quad \text{for all } z \in \mathbb{D}.$$

Then $h_a^{(j)}(a) = 0$ for all $j < n-1$ and

$$|h_a^{(n-1)}(a)| = \frac{(n-1)!}{(1-|a|^2)^{n-1} v(a)}. \qquad (13)$$

Thus by using (13), we have

$$\begin{aligned}
\|T_\psi - T_\Phi\|_{H_v^\infty \to H_w^\infty} &\geq \|(T_\psi - T_\Phi)(h_a)\|_w \\
&\geq w(a)\left|\sum_{j=0}^{n} (\psi_j - \Phi_j)(a) h_a^{(j)}(a)\right| \\
&\geq \frac{w(a)|(\psi_{n-1} - \Phi_{n-1})(a)|}{(1-|a|^2)^{n-1} v(a)} - w(a)|(\psi_n - \Phi_n)(a)||h_a^{(n)}(a)|. \qquad (14)
\end{aligned}$$

By using Lemma 2.1, (14) implies

$$\begin{aligned}
\frac{w(a)|(\psi_{n-1} - \Phi_{n-1})(a)|}{(1-|a|^2)^{n-1} v(a)} &\leq w(a)|(\psi_n - \Phi_n)(a)||h_a^{(n)}(a)| + \|T_\psi - T_\Phi\|_{H_v^\infty \to H_w^\infty} \\
&\leq \frac{w(a)|(\psi_n - \Phi_n)(a)|C_v||h_a||_v}{v(a)(1-|a|^2)^n} + \|T_\psi - T_\Phi\|_{H_v^\infty \to H_w^\infty}. \qquad (15)
\end{aligned}$$

Thus from (12) and (15), we have

$$\begin{aligned} A_{n-1} &= \sup_{a\in\mathbb{D}} \frac{w(a)|(\psi_{n-1}-\Phi_{n-1})(a)|}{(1-|a|^2)^{n-1}\,v(a)} \\ &\le \sup_{a\in\mathbb{D}} \frac{w(a)|(\psi_n-\Phi_n)(a)|C_v}{v(a)\,(1-|a|^2)^n} + \|T_\psi - T_\Phi\|_{H_v^\infty\to H_w^\infty} \\ &\le (C_v+1)\|T_\psi - T_\Phi\|_{H_v^\infty\to H_w^\infty} < \infty. \end{aligned} \tag{16}$$

This prove the condition (9) when $j = n-1$. Similarly, using (12) and (16), we can have the following estimate for $j = n-2$,

$$A_{n-2} = \sup_{a\in\mathbb{D}} \frac{w(a)\big|(\psi_{n-2}-\Phi_{n-2})(a)\big|}{(1-|a|^2)^{n-2}\,v(a)} \le (C_v+1)^2\|T_\psi - T_\Phi\|_{H_v^\infty\to H_w^\infty} < \infty. \tag{17}$$

Further, using the same argument as in (12), (16) and (17), we can assume that

$$A_k = \sup_{a\in\mathbb{D}} \frac{w(a)|(\psi_k-\Phi_k)(a)|}{(1-|a|^2)^k\,v(a)} \le (C_v+1)^{n-k}\|T_\psi - T_\Phi\|_{H_v^\infty\to H_w^\infty} < \infty,\ k = j+1,\dots,n, \tag{18}$$

and prove the following estimate for $k = j$

$$A_j = \sup_{a\in\mathbb{D}} \frac{w(a)|(\psi_j-\Phi_j)(a)|}{(1-|a|^2)^j\,v(a)} \le (C_v+1)^{n-j}\|T_\psi - T_\Phi\|_{H_v^\infty\to H_w^\infty} < \infty. \tag{19}$$

Define

$$h_a(z) = (\varphi_a(z))^j\, f_a(z) \quad \text{for all } z\in\mathbb{D}.$$

Then $h_a^{(m)}(a) = 0$ for all $m < j$ and

$$|h_a^{(j)}(a)| = \frac{j!}{(1-|a|^2)^j\,v(a)}.$$

Thus

$$\begin{aligned} \|T_\psi - T_\Phi\|_{H_v^\infty\to H_w^\infty} &\ge \|(T_\psi - T_\Phi)(h_a)\|_w \\ &\ge w(a)\left|\sum_{k=0}^n (\psi_k-\Phi_k)(a) h_a^{(k)}(a)\right| \\ &\ge \frac{w(a)|(\psi_j-\Phi_j)(a)|}{(1-|a|^2)^j\,v(a)} - w(a)\sum_{k=j+1}^n \left|(\psi_k-\Phi_k)(a) h_a^{(k)}(a)\right|. \end{aligned}$$

Thus Lemma 2.1 implies that

$$\begin{aligned} A_j &= \sup_{a\in\mathbb{D}} \frac{w(a)|(\psi_j-\Phi_j)(a)|}{(1-|a|^2)^j\,v(a)} \\ &\le w(a)\sum_{k=j+1}^n \left|(\psi_k-\Phi_k)(a) h_a^{(k)}(a)\right| + \|T_\psi - T_\Phi\|_{H_v^\infty\to H_w^\infty} \end{aligned}$$

$$\leq \sum_{k=j+1}^{n} \frac{C_v \|h_a\|_v w(a) \left|\left(\psi_k - \Phi_k\right)(a)\right|}{\left(1-|a|^2\right)^k v(a)} + \|T_\psi - T_\Phi\|_{H_v^\infty \to H_w^\infty}. \tag{20}$$

Using (18) in (20), we have

$$\begin{aligned} A_j &= \sup_{a\in\mathbb{D}} \frac{w(a)|(\psi_j - \Phi_j)(a)|}{(1-|a|^2)^j v(a)} \\ &\leq \|T_\psi - T_\Phi\|_{H_v^\infty \to H_w^\infty} + \sum_{k=j+1}^{n} C_v A_k \\ &\leq \left(1 + \sum_{k=j+1}^{n} C_v (C_v+1)^{n-k}\right) \|T_\psi - T_\Phi\|_{H_v^\infty \to H_w^\infty} \\ &= (C_v+1)^{n-j} \|T_\psi - T_\Phi\|_{H_v^\infty \to H_w^\infty} < \infty. \end{aligned} \tag{21}$$

This proves (9).

Conversely, assume that (9) holds. Let $f \in H_v^\infty$. Then using Lemma 2.1, we have

$$\begin{aligned} ||(T_\psi - T_\Phi)(f)||_w &= \sup_{z\in\mathbb{D}} w(z) \left| \sum_{j=0}^{n} \left(\psi_j - \Phi_j\right)(z) f^{(j)}(z) \right| \\ &\leq \sum_{j=0}^{n} \sup_{z\in\mathbb{D}} w(z) \left| \left(\psi_j - \Phi_j\right)(z) \right| \left| f^{(j)}(z) \right| \\ &\leq C_v \|f\|_v \sum_{j=0}^{n} \sup_{z\in\mathbb{D}} \frac{w(z) \left| \left(\psi_j - \Phi_j\right)(z) \right|}{(1-|z|^2)^j v(z)}. \end{aligned}$$

Thus

$$||T_\psi - T_\Phi||_{H_v^\infty \longrightarrow H_w^\infty} \leq C_v \sum_{j=0}^{n} A_j. \tag{22}$$

This proves that the operator $T_\psi - T_\Phi : H_v^\infty \longrightarrow H_w^\infty$ is bounded. Also, from (21), we have

$$\sum_{j=0}^{n} A_j \leq \sum_{i=0}^{n} (C_v+1)^{n-i} ||T_\psi - T_\Phi||_{H_v^\infty \longrightarrow H_w^\infty}. \tag{23}$$

The asymptotic relation (10) follows from (22) and (23). □

Theorem 3.2. *Let v be a radial weight satisfying condition* (L1) *and w be an arbitrary weight on $\mathbb{D}$. Let $\psi = (\psi_j)_{j=0}^n$ and $\Phi = (\Phi_j)_{j=0}^n$ be such that $\psi_j, \Phi_j \in \mathcal{H}(\mathbb{D})$. Then the operator $T_\psi - T_\Phi : H_v^\infty \longrightarrow H_w^\infty$ is compact if and only if*

$$M_j = \lim_{|z|\to 1} \frac{w(z) \left| \left(\psi_j - \Phi_j\right)(z) \right|}{(1-|z|^2)^j v(z)} = 0 \quad \textit{for all } j = 0, 1, \ldots, n. \tag{24}$$

Proof. First, we assume that the operator $T_\psi - T_\Phi$ is compact. We begin with proving condition (24) for $j = n$. Let $\{z_k\}$ be a sequence with $|z_k| \to 1$ such that

$$\lim_{|z|\to 1} \frac{w(z)\left|\left(\psi_n - \Phi_n\right)(z)\right|}{(1-|z|^2)^n v(z)} = \lim_{k\to\infty} \frac{w(z_k)\left|\left(\psi_n - \Phi_n\right)(z_k)\right|}{(1-|z_k|^2)^n v(z_k)}.$$

Now by choosing a subsequence we may assume that there exists an $n_0 \in \mathbb{N}$ such that, $|z_k|^k \geq \frac{1}{2}$ for all $k \geq n_0$. According to (7) there exist $f_k \in B_v^\infty$ such that

$$f_k(z_k) = \frac{1}{\widetilde{v}(z_k)}.$$

Since v satisfies (L1), it is essential and we can replace $\widetilde{v}$ by v. Define

$$g_k(z) = \left(\varphi_{z_k}(z)\right)^n z^k f_k(z), \quad z \in \mathbb{D}.$$

Then $\|g_k\|_v \leq 1$, $g_k^{(m)}(z_k) = 0$ for all $m < n$ and

$$|g_k^{(n)}(z_k)| = \frac{n!|z_k|^k}{v(z_k)(1-|z_k|^2)^n}.$$

Since $\{g_k\}$ is a bounded sequence in H_v^∞ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $||(T_\psi - T_\Phi)(g_k)||_w \to 0$ as $k \to \infty$. Now

$$\begin{aligned} ||(T_\psi - T_\Phi)(g_k)||_w &\geq w(z_k)\left|\sum_{j=0}^{n} \left(\psi_j - \Phi_j\right)(z_k) g_k^{(j)}(z_k)\right| \\ &\geq \frac{w(z_k)|\left(\psi_n - \Phi_n\right)(z_k)||z_k|^k}{v(z_k)(1-|z_k|^2)^n} \\ &\geq \frac{w(z_k)|\left(\psi_n - \Phi_n\right)(z_k)|}{2v(z_k)(1-|z_k|^2)^n}. \end{aligned} \tag{25}$$

Thus (25) implies that

$$M_n = \lim_{k\to\infty} \frac{w(z_k)|\left(\psi_n - \Phi_n\right)(z_k)|}{v(z_k)(1-|z_k|^2)^n} = 0. \tag{26}$$

This proves the condition (24) for $j = n$.

Next, we establish the condition (24) for $j = n - 1$. For this, we define

$$h_k(z) = \left(\varphi_{z_k}(z)\right)^{n-1} z^k f_k(z), \quad z \in \mathbb{D}.$$

Then $\|h_k\|_v \leq 1$, $h_k^{(m)}(z_k) = 0$ for all $m < n - 1$ and

$$|h_k^{(n-1)}(z_k)| = \frac{(n-1)!|z_k|^k}{v(z_k)\left(1-|z_k|^2\right)^{n-1}}. \tag{27}$$

Clearly, $\{h_k\}$ is a bounded sequence in H_v^∞ which converges to zero uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.2, $||(T_\psi - T_\Phi)(h_k)||_w \to 0$ as $k \to \infty$. Now by using (27), we have

$$||(T_\psi - T_\Phi)(h_k)||_w \geq w(z_k)\left|\sum_{j=0}^{n} \left(\psi_j - \Phi_j\right)(z_k) h_k^{(j)}(z_k)\right|$$

$$\geq \frac{w(z_k)|\left(\psi_{n-1}-\Phi_{n-1}\right)(z_k)||z_k|^k}{v(z_k)(1-|z_k|^2)^{n-1}}$$
$$- w(z_k)|(\psi_n-\Phi_n)(z_k)||h_k^{(n)}(z_k)|$$
$$\geq \frac{w(z_k)|\left(\psi_{n-1}-\Phi_{n-1}\right)(z_k)|}{2v(z_k)(1-|z_k|^2)^{n-1}}$$
$$- w(z_k)|(\psi_n-\Phi_n)(z_k)||h_k^{(n)}(z_k)|. \tag{28}$$

Further, using Lemma 2.1, (28) implies that

$$\begin{aligned} &\frac{w(z_k)|\left(\psi_{n-1}-\Phi_{n-1}\right)(z_k)|}{v(z_k)(1-|z_k|^2)^{n-1}} \\ &\leq \frac{w(z_k)|(\psi_n-\Phi_n)(z_k)|C_v}{v(z_k)(1-|z_k|^2)^n} + ||(T_\psi - T_\Phi)(h_k)||_w. \end{aligned} \tag{29}$$

From (26) and (29), we have

$$M_{n-1} = \lim_{k\to\infty} \frac{w(z_k)|\left(\psi_{n-1}-\Phi_{n-1}\right)(z_k)|}{v(z_k)(1-|z_k|^2)^{n-1}} = 0. \tag{30}$$

This proves the condition (24) when $j = n-1$. By repeating the arguments used for (26) and (30), we can obtained the following estimate for $j = n-2$.

$$M_{n-2} = \lim_{k\to\infty} \frac{w(z_k)|\left(\psi_{n-2}-\Phi_{n-2}\right)(z_k)|}{v(z_k)(1-|z_k|^2)^{n-2}} = 0. \tag{31}$$

Continuing in this manner, we can assume

$$M_i = \lim_{k\to\infty} \frac{w(z_k)|\left(\psi_i-\Phi_i\right)(z_k)|}{v(z_k)(1-|z_k|^2)^i} = 0, \qquad i = j+1,\ldots,n \tag{32}$$

for proving

$$M_j = \lim_{k\to\infty} \frac{w(z_k)|\left(\psi_j-\Phi_j\right)(z_k)|}{v(z_k)(1-|z_k|^2)^j} = 0. \tag{33}$$

Define

$$J_k(z) = (\varphi_{z_k}(z))^j\, z^k f_k(z), \quad z \in \mathbb{D}.$$

Clearly, $\|J_k\|_v \leq 1$, $J_k^{(m)}(z_k) = 0$ for all $m < j$ and

$$|J_k^{(j)}(z_k)| = \frac{j!|z_k|^k}{v(z_k)(1-|z_k|^2)^j}. \tag{34}$$

Thus $\{J_k\}$ is a bounded sequence in H_v^∞ which converges to zero uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.2, $||(T_\psi - T_\Phi)(J_k)||_w \to 0$ as $k \to \infty$. Using (34), we have

$$\begin{aligned} ||(T_\psi - T_\Phi)(J_k)||_w &\geq w(z_k)\left|\sum_{i=0}^{n}\left(\psi_i-\Phi_i\right)(z_k) J_k^{(i)}(z_k)\right| \\ &\geq \frac{w(z_k)|\left(\psi_j-\Phi_j\right)(z_k)|}{2v(z_k)(1-|z_k|^2)^j} \end{aligned}$$

$$- w(z_k) \sum_{i=j+1}^{n} \left| \left(\psi_i - \Phi_i\right)(z_k)\right| \left|J_k^{(i)}(z_k)\right|. \tag{35}$$

Thus by using Lemma 2.1 and (32), (35) implies that

$$\begin{aligned}
&\frac{w(z_k)\left|\left(\psi_j - \Phi_j\right)(z_k)\right|}{v(z_k)(1-|z_k|^2)^j} \\
&\leq C_v\|J_k\|_v \sum_{i=j+1}^{n} \frac{w(z_k)\left|\left(\psi_i - \Phi_i\right)(z_k)\right|}{\left(1-|z_k|^2\right)^i v(z_k)} + ||(T_\psi - T_\Phi)(J_k)||_w \\
&= C_v \sum_{i=j+1}^{n} M_i + ||(T_\psi - T_\Phi)(J_k)||_w \to 0 \text{ as } k \to \infty.
\end{aligned}$$

This proves the condition (24) for every $j = 0, 1, \ldots, n$.

Conversely, suppose that the condition (24) holds. Clearly the condition (9) of Theorem 3.1 is satisfied. Thus the operator $T_\psi - T_\Phi$ is bounded. Also, the condition (9) of Theorem 3.1 implies that

$$S_j = \sup_{z\in\mathbb{D}} w(z)|(\psi_j - \Phi_j)(z)| < \infty \quad \text{for all } j = 0, 1, \ldots, n. \tag{36}$$

Let $\{f_k\}$ be a bounded sequence in H_v^∞ that converges to zero uniformly on the compact subsets of $\mathbb{D}$ and let $K = \sup_k \|f_k\|_v$. To prove that the operator $T_\psi - T_\Phi$ is compact, we need to show that $||(T_\psi - T_\Phi)f_k||_w \to 0$ as $k \to \infty$. Let $\epsilon > 0$. Then for each $j = 0, \ldots, n$ there exist $r_j \in (0, 1)$ such that whenever $r_j < |z| < 1$, we have the following

$$H_j = \frac{w(z)\left|(\psi_j - \Phi_j)(z)\right|}{(1-|z|^2)^j v(z)} < \epsilon. \tag{37}$$

Let $r_{\max} = \max\{r_0, r_1, \ldots, r_n\}$. Then (37) also true for $r_{\max} < |z| < 1$. Now since $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimates implies that the derivative $\{f_k^{(j)}\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, for each $j = 1, 2, \ldots, n$. Hence for each $j = 0, 1, \ldots, n$, there is an $m_j \in \mathbb{N}$ such that whenever $|z| \leq r_{\max}$ and $k \geq m = \max\{m_0, m_1, \ldots, m_n\}$, we have

$$|f_k^{(j)}(z)| < \epsilon. \tag{38}$$

Using (36) and (38), we have

$$\begin{aligned}
\sup_{|z|\leq r_{\max}} w(z)\left|\sum_{j=0}^{n} \left(\psi_j - \Phi_j\right)(z) f_k^{(j)}(z)\right| &\leq \epsilon \sum_{j=0}^{n} \sup_{|z|\leq r_{\max}} w(z)\left|\left(\psi_j - \Phi_j\right)(z)\right| \\
&= \epsilon \sum_{j=0}^{n} S_j.
\end{aligned} \tag{39}$$

Further, using Lemma 2.1, (37) and (39), we have

$$\begin{aligned}||(T_\psi - T_\Phi)(f_k)||_w &= \sup_{z\in\mathbb{D}} w(z)\left|\sum_{j=0}^{n} (\psi_j - \Phi_j)(z) f_k^{(j)}(z)\right| \\ &= \sup_{r_{\max}<|z|<1} w(z)\left|\sum_{j=0}^{n} (\psi_j - \Phi_j)(z) f_k^{(j)}(z)\right| \\ &\quad + \sup_{|z|\le r_{\max}} w(z)\left|\sum_{j=0}^{n} (\psi_j - \Phi_j)(z) f_k^{(j)}(z)\right| \\ &\le C_v ||f_k||_v \sum_{j=0}^{n} \sup_{r_{\max}<|z|<1} \frac{w(z)\big|(\psi_j - \Phi_j)(z)\big|}{(1-|z|^2)^j v(z)} + \epsilon \sum_{j=0}^{n} S_j \\ &\le \left((n+1)C_v K + \sum_{j=0}^{n} S_j\right)\epsilon.\end{aligned}$$

This proves that the operator $T_\psi - T_\Phi : H_v^\infty \longrightarrow H_w^\infty$ is compact. □

In Theorem 3.1 and Theorem 3.2, if we take $\Phi_j = 0$ for all $j = 0, 1, \ldots, n$, then we get the boundedness and compactness of the differential operator T_ψ which we state in the following corollary.

Corollary 3.3. *Let v be a radial weight satisfying condition* (L1) *and w be an arbitrary weight on $\mathbb{D}$. Let $\psi = (\psi_j)_{j=0}^n$ be such that $\psi_j \in \mathcal{H}(\mathbb{D})$. Then*

(a) *the operator $T_\psi : H_v^\infty \longrightarrow H_w^\infty$ is bounded if and only if*

(40) $$M_j = \sup_{z\in\mathbb{D}} \frac{w(z)|\psi_j(z)|}{(1-|z|^2)^j\, v(z)} < \infty \quad \text{for all } j = 0, 1, \ldots, n.$$

(b) *the operator $T_\psi : H_v^\infty \longrightarrow H_w^\infty$ is compact if and only if*

(41) $$B_j = \lim_{|z|\to 1} \frac{w(z)|\psi_j(z)|}{(1-|z|^2)^j\, v(z)} = 0 \quad \text{for all } j = 0, 1, \ldots, n.$$

Corollary 3.4. *Let v be a radial weight satisfying condition* (L1) *and w be an arbitrary weight on $\mathbb{D}$ such that $\frac{v}{w}$ is bounded. Let $\psi = (\psi_j)_{j=0}^n$ be such that $\psi_j \in \mathcal{H}(\mathbb{D})$. Then the operator $T_\psi : H_v^\infty \longrightarrow H_w^\infty$ is bounded if and only if T_ψ is compact if and only if $\psi_j = 0$ for all $j = 0, \ldots, n$.*

Proof. Suppose that the operator T_ψ is bounded. Then Corollary 3.3(a) implies that

(42) $$\frac{w(z)|\psi_j(z)|}{(1-|z|^2)^j\, v(z)} \le M_j, \quad z \in \mathbb{D},\ j = 0, 1, \ldots, n.$$

Since $\frac{v}{w}$ is bounded, there exists $\lambda > 0$ such that $\frac{v(z)}{w(z)} \le \lambda$ for all $z \in \mathbb{D}$. Thus from (42), we get

$$|\psi_j(z)| \le \lambda M_j \left(1-|z|^2\right)^j, \quad j = 0, 1, \ldots, n.$$

By the maximum modulus theorem, $\psi_j \equiv 0$ for all $j = 0, 1, \ldots, n$. □

Let $\eta \in \mathcal{H}(\mathbb{D})$ and $\Phi_j = \binom{n}{j}\eta^{(n-j)}$ for every $j = 0, 1, \ldots, n$. Then by using Leibniz formula, from (1) we have

$$T_\Phi(f) = \sum_{j=0}^{n} \binom{n}{j} \eta^{(n-j)} f^{(j)} = \mathcal{D}^n M_\eta(f) \quad f \in \mathcal{H}(\mathbb{D}). \tag{43}$$

Further, using (2) and (43), we have the following difference operator

$$\begin{aligned}(\mathcal{D}^n M_\mu - \mathcal{D}^n M_\eta)(f) &= ((\mu - \eta)f)^{(n)} \\ &= \sum_{j=0}^{n} \binom{n}{j} (\mu - \eta)^{(n-j)} f^{(j)} = (T_\psi - T_\Phi)(f),\end{aligned} \tag{44}$$

which is the difference of product of multiplication operators and nth-differentiation operators.

In view of (44), Theorem 3.1 and Theorem 3.2 give the boundedness and compactness of the difference operator $\mathcal{D}^n M_\mu - \mathcal{D}^n M_\eta$, which we state in the following corollary.

Corollary 3.5. *Let v be a radial weight satisfying condition* (L1) *and w be an arbitrary weight on $\mathbb{D}$. Let $\mu, \eta \in \mathcal{H}(\mathbb{D})$. Then*

(a) *the operator $\mathcal{D}^n M_\mu - \mathcal{D}^n M_\eta : H_v^\infty \longrightarrow H_w^\infty$ is bounded if and only if*

$$A_j = \sup_{z \in \mathbb{D}} \frac{w(z) \left| (\mu - \eta)^{(n-j)}(z) \right|}{(1 - |z|^2)^j v(z)} < \infty \quad \text{for all } j = 0, 1, \ldots, n. \tag{45}$$

Moreover, if the operator $\mathcal{D}^n M_\mu - \mathcal{D}^n M_\eta : H_v^\infty \longrightarrow H_w^\infty$ is bounded, then

$$\|\mathcal{D}^n M_\mu - \mathcal{D}^n M_\eta\|_{H_v^\infty \longrightarrow H_w^\infty} \asymp \sum_{j=0}^{n} A_j. \tag{46}$$

(b) *the operator $\mathcal{D}^n M_\mu - \mathcal{D}^n M_\eta : H_v^\infty \longrightarrow H_w^\infty$ is compact if and only if*

$$M_j = \lim_{|z| \to 1} \frac{w(z) \left| (\mu - \eta)^{(n-j)}(z) \right|}{(1 - |z|^2)^j v(z)} = 0 \quad \text{for all } j = 0, 1, \ldots, n. \tag{47}$$

In (1), if we put $\Phi_j = 0$ for all $j = 0, \ldots, n-1$ and $\Phi_n = \eta$, then we have

$$T_\Phi(f) = \eta f^{(n)} = M_\eta \mathcal{D}^n(f), \quad f \in \mathcal{H}(\mathbb{D}). \tag{48}$$

From (3) and (48), we have the following difference operator

$$(M_\mu \mathcal{D}^n - M_\eta \mathcal{D}^n)(f) = (\mu - \eta) f^{(n)} = (T_\psi - T_\Phi)(f). \tag{49}$$

In view of (49), Theorem 3.1 and Theorem 3.2 reduces to the following corollary.

Corollary 3.6. *Let v be a radial weight satisfying condition* (L1) *and w be an arbitrary weight on $\mathbb{D}$. Let $\mu, \eta \in \mathcal{H}(\mathbb{D})$. Then*

(a) *the operator* $M_\mu \mathcal{D}^n - M_\eta \mathcal{D}^n : H_v^\infty \longrightarrow H_w^\infty$ *is bounded if and only if*

$$M = \sup_{z\in\mathbb{D}} \frac{w(z)|(\mu-\eta)(z)|}{(1-|z|^2)^n v(z)} < \infty.$$

(b) *the operator* $M_\mu \mathcal{D}^n - M_\eta \mathcal{D}^n : H_v^\infty \longrightarrow H_w^\infty$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{w(z)|(\mu-\eta)(z)|}{(1-|z|^2)^n v(z)} = 0.$$

Since the differentiation operator $\mathcal{D}$ is an isometry from $\mathcal{B}_w$ to H_v^∞, the following corollary follows immediately from Corollary 3.5 for $n = 1$.

Corollary 3.7. *Let* v *be a typical weight satisfying condition* (L1) *and* w *be an arbitrary weight on* $\mathbb{D}$. *Let* $\mu, \eta \in \mathcal{H}(\mathbb{D})$. *Then*

(a) *the operator* $M_\mu - M_\eta : H_v^\infty \longrightarrow \mathcal{B}_w$ *is bounded if and only if*

(i) $A_0 = \sup_{z\in\mathbb{D}} \frac{w(z)\left|(\mu-\eta)'(z)\right|}{v(z)} < \infty,$

(ii) $A_1 = \sup_{z\in\mathbb{D}} \frac{w(z)\left|(\mu-\eta)(z)\right|}{(1-|z|^2)v(z)} < \infty.$

(b) *the operator* $M_\mu - M_\eta : H_v^\infty \longrightarrow \mathcal{B}_w$ *is compact if and only if*

(i) $M_0 = \lim_{|z|\to 1} \frac{w(z)\left|(\mu-\eta)'(z)\right|}{v(z)} = 0,$

(ii) $M_1 = \lim_{|z|\to 1} \frac{w(z)\left|(\mu-\eta)(z)\right|}{(1-|z|^2)v(z)} = 0.$

In the following example, we give two unbounded operators T_ψ and T_Φ such that their difference $T_\psi - T_\Phi$ is bounded.

Example 3.8. Let $\psi = (\psi_0, \psi_1)$, $\Phi = (\Phi_0, \Phi_1)$ where $\psi_0(z) = \frac{1}{(1-z)^2}$, $\psi_1(z) = \frac{1}{1-z}$, $\Phi_0(z) = \frac{1}{(1-z)^2}$ and $\Phi_1(z) = \frac{z}{1-z}$. Let $v(z) = 1 - |z|^2$ and $w(z) = (1-|z|^2)^2$. Clearly v and w are radial weights satisfying condition (L1). Now for $z = r$, we have

$$\frac{w(r)|\psi_1(r)|}{v(r)(1-r^2)} = \frac{1}{1-r} \to \infty \quad \text{as } r \to 1.$$

Hence the condition (40) of Corollary 3.3(a) is not satisfied for $j = 1$. This shows that T_ψ is not bounded. Also, for $z = r$, we have

$$\frac{w(r)|\Phi_1(r)|}{v(r)(1-r^2)} = \frac{r}{1-r} \quad \text{as } r \to 1.$$

Again, the condition (40) of Corollary 3.3(a) is not satisfied for the case $j = 1$. Thus T_Φ is not bounded. Clearly the condition (9) of Theorem 3.1 holds for $j = 0$. Now we consider

$$\frac{w(z)\left|(\psi_1 - \Phi_1)(z)\right|}{(1-|z|^2)v(z)} = \frac{(1-|z|^2)^2}{(1-|z|^2)(1-|z|^2)}\left|\frac{1}{1-z} - \frac{z}{1-z}\right| = 1 \tag{50}$$

for all $z \in \mathbb{D}$. Hence the condition (9) of Theorem 3.1 is satisfied for $j = 1$. Thus by Theorem 3.1, $T_\psi - T_\Phi$ is bounded. But $T_\psi - T_\Phi$ is not compact because (50) implies that condition (24) of Theorem 3.2 is not satisfied for $j = 1$.

In the next example, we give two non compact operators T_ψ and T_Φ such that their difference $T_\psi - T_\Phi$ is compact.

Example 3.9. Let v, ψ, and Φ be same as defined in Example 3.8. Let $w(z) = (1 - |z|^2)^3$. Now for $z = r$, it follows that

$$\lim_{r\to1} \frac{w(r)\big|\psi_1(r)\big|}{v(r)(1-r^2)} = \lim_{r\to1} \frac{1-r^2}{(1-r)} = \lim_{r\to1}(1+r) = 2.$$

Hence the condition (41) of Corollary 3.3(b) is not satisfied for $j = 1$. Thus T_ψ is not compact. Also, for $z = r$, we have

$$\lim_{r\to1} \frac{w(r)|\Phi_1(r)|}{v(r)(1-r^2)} = \lim_{r\to1} \frac{(1-r^2)r}{1-r} = \lim_{r\to1}(1+r)r = 2.$$

Again, the condition (41) of Corollary 3.3(b) is not satisfied for $j = 1$ and hence T_Φ is not compact. Now we show that $T_\psi - T_\Phi$ is compact. Since $\psi_0(z) = \Phi_0(z) = \frac{1}{(1-z)^2}$, $z \in \mathbb{D}$, the condition (24) of Theorem 3.2 is obviously satisfied for $j = 0$. Now we consider

$$\text{(51)} \qquad \frac{w(z)\big|(\psi_1 - \Phi_1)(z)\big|}{(1-|z|^2)v(z)} = \frac{(1-|z|^2)^3}{(1-|z|^2)(1-|z|^2)}\left|\frac{1}{1-z} - \frac{z}{1-z}\right| = 1 - |z|^2.$$

From (51), it follows that the condition (24) of Theorem 3.2 is satisfied for $j = 1$. Hence the operator $T_\psi - T_\Phi$ is compact.

4. Boundedness and compactness of the operators $T_\psi - T_\Phi : H^\infty_{v,0} \to H^\infty_{w,0}$

In this section we characterize the boundedness and compactness of the operators $T_\psi - T_\Phi$ between little weighted-type spaces.

Theorem 4.1. *Let v and w be typical weights such that v satisfying the condition* (L1). *Let $\psi = (\psi_j)_{j=0}^n$ and $\Phi = (\Phi_j)_{j=0}^n$ be such that $\psi_j, \Phi_j \in \mathcal{H}(\mathbb{D})$. Then the following statements are equivalent.*

(a) *The operator $T_\psi - T_\Phi : H^\infty_{v,0} \longrightarrow H^\infty_{w,0}$ is bounded.*
(b) *The operator $T_\psi - T_\Phi : H^\infty_v \longrightarrow H^\infty_w$ is bounded.*

Proof. First, assume that the operator $T_\psi - T_\Phi : H^\infty_{v,0} \longrightarrow H^\infty_{w,0}$ is bounded. Since $\left(H^\infty_{v,0}\right)^{**} = H^\infty_v$ and $\left(H^\infty_{w,0}\right)^{**} = H^\infty_w$ (for the proof see [28] and [4]), we have $(T_\psi - T_\Phi)^{**} = T_\psi - T_\Phi$. Thus the operator $T_\psi - T_\Phi : H^\infty_v \longrightarrow H^\infty_w$ is bounded.

Conversely, suppose that the operator $T_\psi - T_\Phi : H^\infty_v \longrightarrow H^\infty_w$ is bounded. Then clearly $T_\psi - T_\Phi : H^\infty_{v,0} \longrightarrow H^\infty_w$ is bounded. Further, the condition (9) of

Theorem 3.1 implies that

$$
\begin{aligned}
L_j &= \lim_{|z|\to 1} w(z)|(\psi_j - \Phi_j)(z)| \\
&= \lim_{|z|\to 1} \frac{w(z)|(\psi_j - \Phi_j)(z)|(1-|z|^2)^j v(z)}{(1-|z|^2)^j v(z)} \\
&\leq \lim_{|z|\to 1} A_j (1-|z|^2)^j v(z) = 0 \quad \text{for all } j = 0, 1, \ldots, n. \qquad (52)
\end{aligned}
$$

Let $f \in H^\infty_{v,0}$. Then to complete the proof we need to show that $(T_\psi - T_\Phi)f \in H^\infty_{w,0}$. For each polynomial $p(z)$, (52) implies that

$$
\begin{aligned}
\lim_{|z|\to 1} w(z)|(T_\psi - T_\Phi)p(z)| &= \lim_{|z|\to 1} w(z)\left|\sum_{j=0}^{n} (\psi_j - \Phi_j)(z) p^{(j)}(z)\right| \\
&\leq \lim_{|z|\to 1} \sum_{j=0}^{n} w(z)|(\psi_j - \Phi_j)(z)| \|p^{(j)}\|_\infty = 0.
\end{aligned}
$$

Thus $(T_\psi - T_\Phi)p \in H^\infty_{w,0}$. Since the polynomials are dense in $H^\infty_{v,0}$ (see [2]), there exists a sequence of polynomials $\{p_k\}$ such that $\|f - p_k\|_v \to 0$ as $k \to \infty$. Hence

$$
\|(T_\psi - T_\Phi)f - (T_\psi - T_\Phi)p_k\| \leq \|(T_\psi - T_\Phi)\|_{H^\infty_v \to H^\infty_w} \|f - p_k\|_v \to 0 \ \text{ as } k \to \infty.
$$

Since $H^\infty_{w,0}$ is a closed subspace of H^∞_w, we have $(T_\psi - T_\Phi)(f) \in H^\infty_{w,0}$. Hence the operator $T_\psi - T_\Phi : H^\infty_{v,0} \longrightarrow H^\infty_{w,0}$ is bounded. ☐

Similarly to the proof of Theorem 3.2, we get the following result, so we omit the proof.

Theorem 4.2. *Let v and w be typical weights such that v satisfying the condition* (L1). *Let $\psi = (\psi_j)_{j=0}^n$ and $\Phi = (\Phi_j)_{j=0}^n$ be such that $\psi_j, \Phi_j \in \mathcal{H}(\mathbb{D})$. Then the differential operator $T_\psi - T_\Phi : H^\infty_{v,0} \longrightarrow H^\infty_{w,0}$ is compact if and only if*

$$
E_j = \lim_{|z|\to 1} \frac{w(z)\left|(\psi_j - \Phi_j)(z)\right|}{(1-|z|^2)^j v(z)} = 0 \quad \textit{for all } j = 0, 1, \ldots, n. \qquad (53)
$$

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