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GLOBAL THEORY OF VERTICAL RECURRENT FINSLER CONNECTION

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Abstract. The aim of the present paper is to establish an *intrinsic* generalization of Cartan connection in Finsler geometry. This connection is called the vertical recurrent Finsler connection. An intrinsic proof of the existence and uniqueness theorem for such connection is investigated. Moreover, it is shown that for such connection, the associated semi-spray coincides with the canonical spray and the associated nonlinear connection coincides with the Barthel connection. Explicit intrinsic expression relating this connection and Cartan connection is deduced. We also investigate some applications concerning the fundamental geometric objects associated with this connection. Finally, three important results concerning the curvature tensors associated to a special vertical recurrent Finsler connection are studied.

1. Introduction

Studying Finsler geometry, one encounters substantial difficulties trying to seek analogues of classical global, or sometimes even local, results of Riemannian geometry. These difficulties arise mainly from the fact that in Finsler geometry all geometric objects depend not only on positional coordinate as in Riemannian geometry, but also on directional argument. Moreover, in Riemannian geometry, there is a canonical linear connection on the manifold M, whereas in Finsler geometry there are at least four canonical linear connections: The Cartan connection, the Berwald connection (as previously mentioned), in addition to the Chern (Rund) and the Hashiguchi connections. However, these are not connections on M but on TTM, the tangent bundle of TM, or on $\pi^{-1}(TM)$, the pullback of the tangent bundle TM by $\pi : TM \longrightarrow M$.

The theory of connections is an important field of research of differential geometry. It was initially developed to solve pure geometrical problems. The most important linear connections in Finsler geometry were studied in [2, 8, 9, 11], etc. In [14, 16, 17], we have established new intrinsic proofs of intrinsic

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versions of the existence and uniqueness theorems for the fundamental linear connections on the pullback bundle of a Finsler manifold. Recently, we studied the horizontal recurrent Finsler connections in [18].

The present paper is a continuation of [16–18], where we investigate an *intrinsic* generalization of Cartan connection in Finsler geometry. This connection is called the vertical recurrent Finsler connection. An intrinsic proof of the existence and uniqueness theorem for such connection is investigated. Moreover, It is shown that for such connection, the associated semi-spray coincides with the canonical spray and the associated nonlinear connection coincides with the Barthel connection. Explicit intrinsic expression relating this connection and Cartan connection is deduced. We also investigate some important relations and properties concerning the fundamental geometric objects associated with this connection. Finally, we study three important results concerning the curvature tensors associated to a special vertical recurrent Finsler connection.

It is worth mentioning that the present work is formulated in a coordinate-free form.

2. Notation and preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1, 3, 8, 10, 12, 16, 17]. We shall use the notations of [16].

Let M be a differential manifold of dimension n, T_xM is the tangent space at $x \in M$ and $TM := \cup_x T_xM$ is the tangent bundle of M. In what follows, we denote by $\pi : \mathcal{T}M \longrightarrow M$ the subbundle of nonzero vectors tangent to M, $\mathfrak{F}(TM)$ the algebra of C^∞ functions on TM, $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(TM)$-module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. The tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundles

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

with the well known definitions of the bundle morphisms ρ and γ. The vector space $V_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : d\pi(X) = 0\}$ is the vertical space to M at u.

Let D be a linear connection on the pullback bundle $\pi^{-1}(TM)$. We associate with D the map $K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta}$, called the connection map of D. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ is called the horizontal space to M at u . The connection D is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \;\; \forall\, u \in \mathcal{T}M.$$

If M is endowed with a regular connection, then the vector bundle maps γ, $\rho|_{H(\mathcal{T}M)}$ and $K|_{V(\mathcal{T}M)}$ are vector bundle isomorphisms. The map $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$ will be called the horizontal map of the connection D.

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors of D, denoted by Q and T respectively, are defined by

$$Q(\overline{X},\overline{Y}) = \mathbf{T}(\beta\overline{X},\beta\overline{Y}),\quad T(\overline{X},\overline{Y}) = \mathbf{T}(\gamma\overline{X},\beta\overline{Y})\quad \forall\, \overline{X},\overline{Y}\in\mathfrak{X}(\pi(M)),$$

where $\mathbf{T}(X,Y) = D_X\rho Y - D_Y\rho X - \rho[X,Y], \forall\, X,Y\in\mathfrak{X}(TM)$ is the (classical) torsion tensor field associated with D.

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors of D, denoted by R, P and S respectively, are defined by

$$R(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X},\beta\overline{Y})\overline{Z},$$
$$P(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X},\gamma\overline{Y})\overline{Z},$$
$$S(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\gamma\overline{X},\gamma\overline{Y})\overline{Z},$$

where $\mathbf{K}(X,Y)\rho Z = -D_XD_Y\rho Z + D_YD_X\rho Z + D_{[X,Y]}\rho Z, \forall\, X,Y,Z\in\mathfrak{X}(TM)$ is the (classical) curvature tensor field associated with D.

The contracted curvature tensors of D, denoted by $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$ (known also as the (v)h-, (v)hv- and (v)v-torsion tensors respectively), are defined by

$$\widehat{R}(\overline{X},\overline{Y}) = R(\overline{X},\overline{Y})\overline{\eta},\quad \widehat{P}(\overline{X},\overline{Y}) = P(\overline{X},\overline{Y})\overline{\eta},\quad \widehat{S}(\overline{X},\overline{Y}) = S(\overline{X},\overline{Y})\overline{\eta}.$$

Definition 1 ([11, 16])**.** A Finsler manifold of dimension n is a pair (M,L), where M is a differentiable manifold of dimension n and L is a map

$$L: TM \longrightarrow \mathbb{R},$$

satisfying the axioms:

(a) $L(u) > 0$ for all $u\in\mathcal{T}M$ and $L(0) = 0$,
(b) L is C^∞ on $\mathcal{T}M$, C^1 on TM,
(c) L is homogenous of degree 1 in the directional argument y,
(d) The exterior 2-form $\Omega := dd_JE$ has maximal rank (nondegenerate), where $J := \gamma\circ\rho$ is the natural almost tangent structure of TM and $E := L^2/2$.

In this case, L is called the Lagrangian, E is the energy function associated with L and $g(\rho X,\rho Y) := \Omega(JX,Y),\ \forall\ X,Y\in\mathfrak{X}(TM)$ is called the Finsler metric defined on $\pi^{-1}(TM)$ by L.

On a Finsler manifold (M,L), there are *canonically* associated four linear connections on $\pi^{-1}(TM)$ [16]: the Cartan connection, the Chern (Rund) connection, the Hashiguchi connection and the Berwald connection. Each of these connections is regular with (h)hv-torsion T satisfying $T(\overline{X},\overline{\eta}) = 0$. The following theorem guarantees the existence and uniqueness of the Cartan connection on the pullback bundle.

Theorem 2.1 ([11, 14])**.** *Let (M,L) be a Finsler manifold and g the Finsler metric defined by L. There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that*

(a) ∇ *is metric*: $\nabla g = 0$,

(b) *The* $(h)h$*-torsion of* ∇ *vanishes*: $Q = 0$,

(c) *The* $(h)hv$*-torsion* T *of* ∇ *satisfies*: $g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y})$.

Such a connection is called the Cartan connection associated with the Finsler manifold (M, L).

We terminate this section by some concepts and results concerning the Klein-Grifone approach to intrinsic Finsler geometry. For more details, we refer to [4, 5, 7, 13].

A semi-spray is a vector field X on TM, C^{∞} on $\mathcal{T}M$, C^1 on TM, such that $\rho \circ X = \overline{\eta}$. A semispray X which is homogeneous of degree 2 in the directional argument ($[\mathcal{C}, X] = X\,; \mathcal{C} := \gamma\overline{\eta}$) is called a spray.

Proposition 2.2 ([7]). *Let* (M, L) *be a Finsler manifold. The vector field* G *on* TM *defined by* $i_G\Omega = -dE$ *is a spray, where* $E := \frac{1}{2}L^2$ *is the energy function and* $\Omega := dd_JE$. *Such a spray is called the canonical spray.*

A nonlinear connection on M is a vector 1-form Γ on TM, C^{∞} on $\mathcal{T}M$, C^0 on TM, such that

$$J\Gamma = J, \qquad \Gamma J = -J.$$

The horizontal and vertical projectors h_{Γ} and v_{Γ} associated with Γ are defined by $h_{\Gamma} := \frac{1}{2}(I + \Gamma)$ and $v_{\Gamma} := \frac{1}{2}(I - \Gamma)$.

Theorem 2.3 ([5]). *On a Finsler manifold* (M, L), *there exists a unique conservative homogenous nonlinear connection with zero torsion. It is given by*:

$$\Gamma = [J, G],$$

where G *is the canonical spray.*

Such a nonlinear connection is called the canonical connection, the Barthel connection or the Cartan nonlinear connection associated with (M, L).

3. Vertical recurrent Finsler connection

In this section, we establish an *intrinsic* generalization of Cartan connection. This connection is called the vertical recurrent Finsler connection. An intrinsic proof of the existence and uniqueness theorem for such connection is investigated. Moreover, It is shown that for such connection, the associated semi-spray coincides with the canonical spray and the associated nonlinear connection coincides with the Barthel connection.

The following definition and three lemmas are useful for subsequence use.

Definition 2 ([11, 14]). Let D be a regular connection on $\pi^{-1}(TM)$ with horizontal map β. Then, the semispray $S = \beta\overline{\eta}$ will be called the semispray associated with D. Moreover, the nonlinear connection $\Gamma = 2\beta \circ \rho - I$ will be called the nonlinear connection associated with D and will be denoted by Γ_D.

Lemma 3.1 ([11, 14]). *Let* D *be a regular connection on* $\pi^{-1}(TM)$ *whose connection map is* K *and whose horizontal map is* β. *Then, the* $(h)hv$*-torsion* T *of* D *has the property that* $T(\overline{X}, \overline{\eta}) = 0$ *is equivalent to that* $\Gamma := \beta \circ \rho - \gamma \circ K$

is a nonlinear connection on M. Consequently, Γ coincides with the nonlinear connection associated with D: $\Gamma = \Gamma_D = 2\beta \circ \rho - I$, and in this case $h_\Gamma = h_D = \beta \circ \rho$ and $v_\Gamma = v_D = \gamma \circ K$.

Lemma 3.2 ([14])**.** *The Cartan connection ∇, associated with the Finsler manifold (M, L), is uniquely determined by the relations*:

(a) $2g(\nabla_{\gamma\overline{X}}\overline{Y}, \overline{Z}) = \gamma\overline{X} \cdot g(\overline{Y}, \overline{Z}) + g(\overline{Y}, \rho[\beta\overline{Z}, \gamma\overline{X}]) + g(\overline{Z}, \rho[\gamma\overline{X}, \beta\overline{Y}])$.

(b) $$2g(\nabla_{\beta\overline{X}}\rho Y, \rho Z) = \beta\overline{X} \cdot g(\overline{Y}, \overline{Z}) + \beta\overline{Y} \cdot g(\overline{Z}, \overline{X}) - \beta\overline{Z} \cdot g(\overline{X}, \overline{Y}) - g(\overline{X}, \rho[\beta\overline{Y}, \beta\overline{Z}]) + g(\overline{Y}, \rho[\beta\overline{Z}, \beta\overline{X}]) + g(\overline{Z}, \rho[\beta\overline{X}, \beta\overline{Y}]),$$

where β is the horizontal map associated with Cartan connection ∇.

Lemma 3.3 ([11, 14])**.** *The Berwald connection D° is explicitly expressed in terms of Cartan connection ∇ in the form*:

(a) $D^\circ_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X}, \overline{Y}) = \rho[\gamma\overline{X}, \beta\overline{Y}]$.

(b) $D^\circ_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y} + \widehat{P}(\overline{X}, \overline{Y}) = K[\beta\overline{X}, \gamma\overline{Y}]$.

Now, we announce the main result of this paper, namely, the existence and uniqueness theorem of the vertical recurrent Finsler connection.

Theorem 3.4. *Let (M, L) be a Finsler manifold, g the Finsler metric defined by L and B a non-zero scalar 1-form on $\pi^{-1}(TM)$. There exists a unique regular connection $\overline{D}(B)$ (or simply $\overline{D}$) on $\pi^{-1}(TM)$ such that*

(C1) *The metric g is $\overline{D}$-horizontally parallel*: $\overline{D}_{\bar{\beta}\overline{X}}g = 0$,

(C2) *The metric g is $\overline{D}$-vertically recurrent*: $\overline{D}_{\gamma\overline{X}}g = B(\overline{X})g$,

(C3) *The $(h)h$-torsion $\overline{Q}$ of $\overline{D}$ vanishes*: $\overline{Q} = 0$,

(C4) *The $(h)hv$-torsion $\overline{T}$ of $\overline{D}$ is symmetric*: $\overline{T}(\overline{X}, \overline{Y}) = \overline{T}(\overline{Y}, \overline{X})$,

where $\bar{\beta}$ is the horizontal map associated with $\overline{D}$.

Such a connection is called the vertical recurrent Finsler connection associated with the Finsler manifold (M, L), with respect to the scalar 1-form B.

Proof. First we prove the *uniqueness*. Since $\overline{D}$ is a regular connection, then, using Definition 2, its horizontal (vertical) projector is defined by $\bar{h} := \bar{\beta} \circ \rho$ ($\bar{v} := I - \bar{\beta} \circ \rho$); $\bar{\beta}$ being the horizontal map associated with $\overline{D}$.

By using axiom (C2), we obtain

$$\gamma\overline{X} \cdot g(\overline{Y}, \overline{Z}) = B(\overline{X})g(\overline{Y}, \overline{Z}) + g(\overline{D}_{\gamma\overline{X}}\overline{Y}, \overline{Z}) + g(\overline{Y}, \overline{D}_{\gamma\overline{X}}\overline{Z}).$$

From which together with the same expressions for $\gamma\overline{Y} \cdot g(\overline{Z}, \overline{X})$ and $\gamma\overline{Z} \cdot g(\overline{X}, \overline{Y})$, we get

$$\begin{aligned} &\gamma\overline{X} \cdot g(\overline{Y}, \overline{Z}) + \gamma\overline{Y} \cdot g(\overline{Z}, \overline{X}) - \gamma\overline{Z} \cdot g(\overline{X}, \overline{Y}) \\ &= B(\overline{X})g(\overline{Y}, \overline{Z}) + g(\overline{D}_{\gamma\overline{X}}\overline{Y}, \overline{Z}) + g(\overline{Y}, \overline{D}_{\gamma\overline{X}}\overline{Z}) + B(\overline{Y})g(\overline{Z}, \overline{X}) + g(\overline{D}_{\gamma\overline{Y}}\overline{Z}, \overline{X}) \\ &\quad + g(\overline{Z}, \overline{D}_{\gamma\overline{Y}}\overline{X}) - B(\overline{Z})g(\overline{X}, \overline{Y}) - g(\overline{D}_{\gamma\overline{Z}}\overline{X}, \overline{Y}) - g(\overline{X}, \overline{D}_{\gamma\overline{Z}}\overline{Y}). \end{aligned}$$

Applying axiom (C4), the above relation becomes

$$
\begin{aligned}
2g(\overline{D}_{\gamma\overline{X}}\overline{Y},\overline{Z}) &= \gamma\overline{X}\cdot g(\overline{Y},\overline{Z})+\gamma\overline{Y}\cdot g(\overline{Z},\overline{X})-\gamma\overline{Z}\cdot g(\overline{X},\overline{Y})\\
&\quad - g(\rho[\gamma\overline{Y},\bar{\beta}\overline{X}]-\rho[\gamma\overline{X},\bar{\beta}\overline{Y}],\overline{Z})-g(\rho[\gamma\overline{Y},\bar{\beta}\overline{Z}]-\rho[\gamma\overline{Z},\bar{\beta}\overline{Y}],\overline{X})\\
&\quad - g(\rho[\gamma\overline{X},\bar{\beta}\overline{Z}]-\rho[\gamma\overline{Z},\bar{\beta}\overline{X}],\overline{Y})-B(\overline{X})g(\overline{Y},\overline{Z})\\
&\quad - B(\overline{Y})g(\overline{Z},\overline{X})+B(\overline{Z})g(\overline{X},\overline{Y}). \qquad (1)
\end{aligned}
$$

As $J^2=0=[J,J]$ and $\rho\circ J=0$, we get

$$
\rho[\gamma\overline{X},\bar{\beta}\overline{Y}]=\rho[\gamma\overline{X},\beta\overline{Y}], \qquad (2)
$$

where β is the horizontal map associated with Cartan connection ∇. From (2) taking into account Lemma 3.3 and the fact that the Cartan torsion tensor $\mathbf{T}(\overline{X},\overline{Y},\overline{Z}):=g(T(\overline{X},\overline{Y}),\overline{Z})$ is totally symmetric [17], Equation (1) implies that

$$
\begin{aligned}
&2g(\overline{D}_{\gamma\overline{X}}\overline{Y},\overline{Z})\\
&= 2g({D^\circ}_{\gamma\overline{X}}\overline{Y},\overline{Z})+({D^\circ}_{\gamma\overline{X}}g)(\overline{Y},\overline{Z})+({D^\circ}_{\gamma\overline{Y}}g)(\overline{Z},\overline{X})-({D^\circ}_{\gamma\overline{Z}}g)(\overline{X},\overline{Y})\\
&\quad - B(\overline{X})g(\overline{Y},\overline{Z})-B(\overline{Y})g(\overline{Z},\overline{X})+B(\overline{Z})g(\overline{X},\overline{Y})\\
&= 2g({D^\circ}_{\gamma\overline{X}}\overline{Y},\overline{Z})+2g(T(\overline{X},\overline{Y}),\overline{Z})+2g(T(\overline{Y},\overline{Z}),\overline{X})-2g(T(\overline{Z},\overline{X}),\overline{Y})\\
&\quad - B(\overline{X})g(\overline{Y},\overline{Z})-B(\overline{Y})g(\overline{Z},\overline{X})+B(\overline{Z})g(\overline{X},\overline{Y})\\
&= 2g({D^\circ}_{\gamma\overline{X}}\overline{Y},\overline{Z})+2g(T(\overline{X},\overline{Y}),\overline{Z})-B(\overline{X})g(\overline{Y},\overline{Z})-B(\overline{Y})g(\overline{Z},\overline{X})\\
&\quad + g(\bar{b},\overline{Z})g(\overline{X},\overline{Y}),
\end{aligned}
$$

where $g(\bar{b},\overline{X}):=B(\overline{X})$. Consequently, using again Lemma 3.3 and the fact that g is non-degenerate, the above equation reduces to

$$
\overline{D}_{\gamma\overline{X}}\overline{Y}=\nabla_{\gamma\overline{X}}\overline{Y}-\frac{1}{2}\{B(\overline{X})\overline{Y}+B(\overline{Y})\overline{X}-g(\overline{X},\overline{Y})\bar{b}\}. \qquad (3)
$$

Similarly, using axioms (C1) and (C3), we obtain, for all $\overline{X},\overline{Y},\overline{Z}\in\mathfrak{X}(\pi(M))$

$$
\begin{aligned}
2g(\overline{D}_{\bar{\beta}\overline{X}}\overline{Y},\overline{Z}) &= \bar{\beta}\overline{X}\cdot g(\overline{Y},\overline{Z})+\bar{\beta}\overline{Y}\cdot g(\overline{Z},\overline{X})-\bar{\beta}\overline{Z}\cdot g(\overline{X},\overline{Y})\\
&\quad - g(\overline{X},\rho[\bar{\beta}\overline{Y},\bar{\beta}\overline{Z}])-g(\overline{Y},\rho[\bar{\beta}\overline{Z},\bar{\beta}\overline{X}])+g(\overline{Z},\rho[\bar{\beta}\overline{X},\bar{\beta}\overline{Y}]). \qquad (4)
\end{aligned}
$$

As the difference between two nonlinear connections is a semibasic form, setting $\bar{\beta}\overline{X}=\beta\overline{X}+\gamma\overline{X}_t$ and using the fact that

$$
\rho[\bar{\beta}\overline{X},\bar{\beta}\overline{Y}]=\rho[\beta\overline{X},\beta\overline{Y}]+\rho[\beta\overline{X},\gamma\overline{Y}_t]+\rho[\gamma\overline{X}_t,\beta\overline{Y}],
$$

taking into account Lemma 3.2 and Lemma 3.3, the above equation becomes

$$
\begin{aligned}
2g(\overline{D}_{\bar{\beta}\overline{X}}\overline{Y},\overline{Z}) &= 2g(\nabla_{\beta\overline{X}}\overline{Y},\overline{Z})+2g(D^\circ_{\bar{\beta}\overline{X}}\overline{Y},\overline{Z})+({D^\circ}_{\gamma\overline{X}_t}g)(\overline{Y},\overline{Z})\\
&\quad + ({D^\circ}_{\gamma\overline{Y}_t}g)(\overline{Z},\overline{X})-({D^\circ}_{\gamma\overline{Z}_t}g)(\overline{X},\overline{Y})\\
&= 2g(\nabla_{\beta\overline{X}}\overline{Y},\overline{Z})+2g(D^\circ_{\gamma\overline{X}_t}\overline{Y},\overline{Z})+2\mathbf{T}(\overline{X}_t,\overline{Y},\overline{Z})\\
&\quad + 2\mathbf{T}(\overline{Y}_t,\overline{Z},\overline{X})-2\mathbf{T}(\overline{Z}_t,\overline{X},\overline{Y}). \qquad (5)
\end{aligned}
$$

From which, by setting $\overline{X} = \overline{Y} = \overline{\eta}$, noting that $\bar{K} \circ \bar{\beta} = K \circ \beta = 0$ and $i_{\overline{\eta}}\mathbf{T} = 0$, we obtain $\overline{\eta}_t = 0$. Hence, by setting $\overline{Y} = \overline{\eta}$ again into Equation (5), we have $\overline{X}_t = 0$ and hence $\bar{\beta} = \beta, \bar{K} = K$ and the given 1-form B satisfies

$$B(\overline{X})\overline{\eta} + B(\overline{\eta})\overline{X} - L\ell(\overline{X})\bar{b} = 0, \tag{6}$$

where $\ell := L^{-1} i_{\overline{\eta}}\, g$. Consequently, Equation (5) reduces to

$$\overline{D}_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y}. \tag{7}$$

Now, from (3) and (7), the full expression of $\overline{D}_X\overline{Y}$ is given by

$$\overline{D}_X\overline{Y} = \nabla_X\overline{Y} - \frac{1}{2}\{B(KX)\overline{Y} + B(\overline{Y})KX - g(KX, \overline{Y})\bar{b}\}. \tag{8}$$

Hence $\overline{D}_X\overline{Y}$ is uniquely determined by the right-hand side of (8).

To prove the *existence* of $\overline{D}$, we define $\overline{D}$ by the requirement that (3) and (7) hold for all $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$. Now, we have to prove that the connection $\overline{D}$ satisfies the conditions of Theorem 3.4:

$\overline{D}$ *satisfies condition* (C1): Follows from (7), taking into account the fact that g is ∇-horizontally parallel (Theorem 2.1(a)).

$\overline{D}$ *satisfies condition* (C2): From (3), taking into account the fact that g is ∇-vertically parallel (Theorem 2.1(a)), we have

$$\begin{aligned}
&(\overline{D}_{\gamma\overline{X}}g)(\overline{Y}, \overline{Z})\\
&= (\nabla_{\gamma\overline{X}}g)(\overline{Y}, \overline{Z}) + \frac{1}{2}\{B(\overline{X})g(\overline{Y}, \overline{Z}) + B(\overline{Y})g(\overline{X}, \overline{Z}) - B(\overline{Z})g(\overline{X}, \overline{Y})\}\\
&\quad + \frac{1}{2}\{B(\overline{X})g(\overline{Y}, \overline{Z}) + B(\overline{Z})g(\overline{Y}, \overline{X}) - B(\overline{Y})g(\overline{X}, \overline{Z})\}\\
&= B(\overline{X})g(\overline{Y}, \overline{Z}).
\end{aligned}$$

Hence, the result follows.

$\overline{D}$ *satisfies condition* (C3): Follows immediately from (7) and (Theorem 2.1(b)).

$\overline{D}$ *satisfies condition* (C4): From (3), one can show that

$$\overline{T}(\overline{X}, \overline{Y}) = T(\overline{X}, \overline{Y}) - \frac{1}{2}\{B(\overline{X})\overline{Y} + B(\overline{Y})\overline{X} - g(\overline{X}, \overline{Y})\bar{b}\}. \tag{9}$$

Hence, $\overline{T}$ is symmetric. This completes the proof. □

In view of the above Theorem, we have:

Theorem 3.5. *The nonlinear connection associated with the vertical recurrent Finsler connection $\overline{D}$ coincides with the Barthel connection*: $\Gamma_{\overline{D}} = [J, G]$.

Moreover, $\bar{\beta} = \beta$ and $\bar{K} = K$ where β and K are the horizontal and the connection maps of Cartan connection ∇, respectively.

Remark 3.6. It is clear that, if the given scalar form B in Theorem 3.4 vanishes, then the vertical recurrent Finsler connection $\overline{D}$ coincides with the Cartan connection ∇. Consequently, the vertical recurrent Finsler connection is a generalization for the Cartan connection.

Corollary 3.7. *The vertical recurrent Finsler connection $\overline{D}$, with respect to a given non-zero scalar 1-form B, is explicitly expressed in terms of the Cartan connection ∇ in the form:*

$$\overline{D}_X \overline{Y} = \nabla_X \overline{Y} - \frac{1}{2}\{B(KX)\overline{Y} + B(\overline{Y})KX - g(KX, \overline{Y})\bar{b}\}. \tag{10}$$

In particular, we have

(a) $\overline{D}_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} - \frac{1}{2}\{B(\overline{X})\overline{Y} + B(\overline{Y})\overline{X} - g(\overline{X}, \overline{Y})\bar{b}\}$,

(b) $\overline{D}_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y}$,

where $g(\bar{b}, \overline{X}) := B(\overline{X})$.

Proposition 3.8. *Let $\overline{D}$ be the vertical recurrent Finsler connection with respect to a given non-zero scalar 1-form B. Then, we have*

(a) *the (h)hv-torsion $\overline{T}$ of $\overline{D}$ is symmetric indicatory tensor satisfying*

$$\overline{T}(\overline{X}, \overline{Y}) = T(\overline{X}, \overline{Y}) - \frac{1}{2}\{B(\overline{X})\overline{Y} + B(\overline{Y})\overline{X} - g(\overline{X}, \overline{Y})\bar{b}\}.$$

(b) *the (v)v-torsion $\widehat{\overline{S}}$ of $\overline{D}$ vanishes.*

(c) *the (v)hv-torsion $\widehat{\overline{P}}$ of $\overline{D}$ is symmetric indicatory tensor satisfying*

$$\widehat{\overline{P}}(\overline{X}, \overline{Y}) = \widehat{P}(\overline{X}, \overline{Y}).$$

(d) *the (v)h-torsion $\widehat{\overline{R}}$ of $\overline{D}$ satisfies $\widehat{\overline{R}}(\overline{X}, \overline{Y}) = \widehat{R}(\overline{X}, \overline{Y})$,*

where $\widehat{P}$ and $\widehat{R}$ are respectively the (v)hv-torsion and the (v)h-torsion of Cartan connection.

Proof. (a) Follows from Eqs. (9), (6) and the fact that Cartan torsion T is a symmetric indicatory tensor.

(b) Using (a) above and Proposition 2.5 of [17], we obtain

$$\begin{aligned}\overline{S}(\overline{X}, \overline{Y})\overline{Z} = (\overline{D}_{\gamma\overline{Y}}\overline{T})(\overline{X}, \overline{Z}) - (\overline{D}_{\gamma\overline{X}}\overline{T})(\overline{Y}, \overline{Z}) + \overline{T}(\overline{X}, \overline{T}(\overline{Y}, \overline{Z})) \\ - \overline{T}(\overline{Y}, \overline{T}(\overline{X}, \overline{Z})) + \overline{T}(\widehat{\overline{S}}(\overline{Y}, \overline{X}), \overline{Z}).\end{aligned}$$

From which, by setting $\overline{Z} = \overline{\eta}$, taking into account again (a) above and the fact that $K \circ \gamma = id_{\mathfrak{X}(\pi(M))}$, the result follows.

(c) From Theorem 3.5 and Corollary 3.7 together with $K \circ \beta = 0$, we get

$$\begin{aligned}\widehat{\overline{P}}(\overline{X}, \overline{Y}) &= -\overline{D}_{\beta\overline{X}}\overline{D}_{\gamma\overline{Y}}\overline{\eta} + \overline{D}_{\gamma\overline{Y}}\overline{D}_{\beta\overline{X}}\overline{\eta} + \overline{D}_{[\beta\overline{X}, \gamma\overline{Y}]}\overline{\eta} \\ &= -\overline{D}_{\beta\overline{X}}\overline{Y} + K[\beta\overline{X}, \gamma\overline{Y}] \\ &= -\nabla_{\beta\overline{X}}\overline{Y} + K[\beta\overline{X}, \gamma\overline{Y}] \\ &= \widehat{P}(\overline{X}, \overline{Y}).\end{aligned}$$

Hence, $\widehat{\overline{P}}$ is a symmetric indicatory tensor.

(d) The proof is similar to that of (c) above. □

Remark 3.9. In view of the above Proposition and the fact that $\widehat{R}(\overline{X},\overline{Y}) = -K[\beta\overline{X},\beta\overline{Y}]$, the (v)h-torsion tensor $\widehat{\overline{R}}$ of vertical recurrent Finsler connection vanishes if and only if the horizontal distribution is completely integrable.

Theorem 3.10. *Let $\overline{D}$ be the vertical recurrent Finsler connection with respect to a given non-zero scalar* 1*-form B. The classical curvature tensor $\overline{\boldsymbol{K}}$ of $\overline{D}$ and the corresponding classical curvature tensor $\boldsymbol{K}$ of the Cartan connection ∇ are related by* [1]

$$\begin{aligned}
&\overline{\boldsymbol{K}}(X,Y)\overline{Z}\\
&= \boldsymbol{K}(X,Y)\overline{Z} - \frac{1}{2}\{B(\widehat{\boldsymbol{K}}(X,Y))\overline{Z} + B(\overline{Z})\widehat{\boldsymbol{K}}(X,Y) - g(\widehat{\boldsymbol{K}}(X,Y),\overline{Z})\bar{b}\}\\
&\quad - \frac{1}{2}\mathfrak{U}_{X,Y}\{(\nabla_Y B)(\overline{Z})KX + (\nabla_Y B)(KX)\overline{Z} + g(KY,\overline{Z})\nabla_X \bar{b}\\
&\quad - \frac{1}{2}\{B(\overline{Z})B(KX)KY + g(KY,\overline{Z})B(\bar{b})KX + g(KX,\overline{Z})B(KY)\bar{b}\}\}.
\end{aligned}$$

In particular, we have

(a) $\overline{S}(\overline{X},\overline{Y})\overline{Z}$

$$\begin{aligned}
&= S(\overline{X},\overline{Y})\overline{Z} - \frac{1}{2}\mathfrak{U}_{\overline{X},\overline{Y}}\{(\nabla_{\gamma\overline{Y}}B)(\overline{Z})\overline{X} + (\nabla_{\gamma\overline{Y}}B)(\overline{X})\overline{Z} + g(\overline{Y},\overline{Z})\nabla_{\gamma\overline{X}}\bar{b}\\
&\quad - \frac{1}{2}\{B(\overline{Z})B(\overline{X})\overline{Y} + g(\overline{Y},\overline{Z})B(\bar{b})\overline{X} + g(\overline{X},\overline{Z})B(\overline{Y})\bar{b}\}\}.
\end{aligned}$$

(b) $\overline{P}(\overline{X},\overline{Y})\overline{Z}$

$$\begin{aligned}
&= P(\overline{X},\overline{Y})\overline{Z} - \frac{1}{2}\{B(\widehat{P}(\overline{X},\overline{Y}))\overline{Z} + B(\overline{Z})\widehat{P}(\overline{X},\overline{Y}) - g(\widehat{P}(\overline{X},\overline{Y}),\overline{Z})\bar{b}\}\\
&\quad + \frac{1}{2}\{(\nabla_{\beta\overline{X}}B)(\overline{Z})\overline{Y} + (\nabla_{\beta\overline{X}}B)(\overline{Y})\overline{Z} - g(\overline{Y},\overline{Z})\nabla_{\beta\overline{X}}\bar{b}\}.
\end{aligned}$$

(c) $\overline{R}(\overline{X},\overline{Y})\overline{Z}$

$$= R(\overline{X},\overline{Y})\overline{Z} - \frac{1}{2}\{B(\widehat{R}(\overline{X},\overline{Y}))\overline{Z} + B(\overline{Z})\widehat{R}(\overline{X},\overline{Y}) - g(\widehat{R}(\overline{X},\overline{Y}),\overline{Z})\bar{b}\}.$$

Proof. From Theorem 3.7, we have

$$\begin{aligned}
\overline{D}_X\overline{D}_Y\overline{Z} &= \overline{D}_X\{\nabla_Y\overline{Z} - \frac{1}{2}\{B(KY)\overline{Z} + B(\overline{Z})KY - g(KY,\overline{Z})\bar{b}\}\}\\
&= \nabla_X\nabla_Y\overline{Z} - \frac{1}{2}\{B(KX)\nabla_Y\overline{Z} + B(\nabla_Y\overline{Z})KX - g(KX,\nabla_Y\overline{Z})\bar{b}\}\\
&\quad - \frac{1}{2}\{X\cdot B(KY) + B(KY)\{\nabla_X\overline{Z}
\end{aligned}$$

[1] $\mathfrak{U}_{X,Y}\{H(X,Y)\} := H(X,Y) - H(Y,X)$

$$
\begin{aligned}
&\qquad - \frac{1}{2}\{B(KX)\overline{Z} + B(\overline{Z})KX - g(KX,\overline{Z})\bar{b}\}\}\} \\
&- \frac{1}{2}\{X \cdot B(\overline{Z})KY + B(\overline{Z})\{\nabla_X KY \\
&\qquad - \frac{1}{2}\{B(KX)KY + B(KY)KX - g(KX,KY)\bar{b}\}\}\} \\
&+ \frac{1}{2}\{X \cdot g(KY,\overline{Z})\bar{b} + g(KY,\overline{Z})\{\nabla_X \bar{b} \\
&\qquad - \frac{1}{2}\{B(KX)\bar{b} + B(\bar{b})KX - B(KX)\bar{b}\}\}\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\overline{D}_Y \overline{D}_X \overline{Z} &= \overline{D}_Y\{\nabla_X \overline{Z} - \frac{1}{2}\{B(KX)\overline{Z} + B(\overline{Z})KX - g(KX,\overline{Z})\bar{b}\}\} \\
&= \nabla_Y \nabla_X \overline{Z} - \frac{1}{2}\{B(KY)\nabla_X \overline{Z} + B(\nabla_X \overline{Z})KY - g(KY,\nabla_X \overline{Z})\bar{b}\} \\
&\quad - \frac{1}{2}\{Y \cdot B(KX) + B(KX)\{\nabla_Y \overline{Z} \\
&\qquad - \frac{1}{2}\{B(KY)\overline{Z} + B(\overline{Z})KY - g(KY,\overline{Z})\bar{b}\}\}\} \\
&\quad - \frac{1}{2}\{Y \cdot B(\overline{Z})KX + B(\overline{Z})\{\nabla_Y KX \\
&\qquad - \frac{1}{2}\{B(KY)KX + B(KX)KY - g(KY,KX)\bar{b}\}\}\} \\
&\quad + \frac{1}{2}\{Y \cdot g(KX,\overline{Z})\bar{b} + g(KX,\overline{Z})\{\nabla_Y \bar{b} \\
&\qquad - \frac{1}{2}\{B(KY)\bar{b} + B(\bar{b})KY - B(KY)\bar{b}\}\}\}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\overline{D}_{[X,Y]}\overline{Z} &= \nabla_{[X,Y]}\overline{Z} - \frac{1}{2}\{B(K[X,Y])\overline{Z} + B(\overline{Z})K[X,Y] - g(K[X,Y],\overline{Z})\bar{b}\} \\
&= \nabla_{[X,Y]}\overline{Z} - \frac{1}{2}\{B(\widehat{\mathbf{K}}(X,Y) + \nabla_X KY - \nabla_Y KX)\overline{Z} \\
&\quad + B(\overline{Z})\{\widehat{\mathbf{K}}(X,Y) + \nabla_X KY - \nabla_Y KX\} \\
&\quad - g(\widehat{\mathbf{K}}(X,Y) + \nabla_X KY - \nabla_Y KX, \overline{Z})\bar{b}\}.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\overline{\mathbf{K}}(X,Y)\overline{Z} = \mathbf{K}(X,Y)\overline{Z} &- \frac{1}{2}\{B(\widehat{\mathbf{K}}(X,Y))\overline{Z} + B(\overline{Z})\widehat{\mathbf{K}}(X,Y) - g(\widehat{\mathbf{K}}(X,Y),\overline{Z})\bar{b}\} \\
&- \frac{1}{2}\mathfrak{U}_{X,Y}\{(\nabla_Y B)(\overline{Z})KX + (\nabla_Y B)(KX)\overline{Z} + g(KY,\overline{Z})\nabla_X \bar{b} \\
&- \frac{1}{2}\{B(\overline{Z})B(KX)KY + g(KY,\overline{Z})B(\bar{b})KX + g(KX,\overline{Z})B(KY)\bar{b}\}\}. \qquad (11)
\end{aligned}
$$

Hence, part (a) follows from the above equation by setting $X = \gamma\overline{X}$ and $Y = \gamma\overline{Y}$, taking into account $K \circ \gamma = id_{\mathfrak{X}(\pi(M))}$ and that $\widehat{S}(\overline{X}, \overline{Y}) = 0$. Part (b) follows from (11) by substituting $X = \beta\overline{X}$ and $Y = \gamma\overline{Y}$, noting that $K \circ \beta = 0$. Finally, part (c) follows from the same equation by setting $X = \beta\overline{X}$ and $Y = \beta\overline{Y}$. □

The following results give a version of Bianchi identities with respect to the vertical recurrent Finsler connection $\overline{D}$.

Proposition 3.11. *For the vertical recurrent Finsler connection $\overline{D}$, we have*:

(a) $\overline{S}(\overline{X}, \overline{Y})\overline{Z} = (\overline{D}_{\gamma\overline{Y}}\overline{T})(\overline{X}, \overline{Z}) - (\overline{D}_{\gamma\overline{X}}\overline{T})(\overline{Y}, \overline{Z}) + \overline{T}(\overline{X}, \overline{T}(\overline{Y}, \overline{Z})) - \overline{T}(\overline{Y}, \overline{T}(\overline{X}, \overline{Z}))$,

(b) $\overline{P}(\overline{X}, \overline{Y})\overline{Z} - \overline{P}(\overline{Z}, \overline{Y})\overline{X} = (\overline{D}_{\beta\overline{Z}}\overline{T})(\overline{Y}, \overline{X}) - (\overline{D}_{\beta\overline{X}}\overline{T})(\overline{Y}, \overline{Z}) - \overline{T}(\widehat{P}(\overline{Z}, \overline{Y}), \overline{X}) + \overline{T}(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$,

(c) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{\overline{R}(\overline{X}, \overline{Y})\overline{Z} - \overline{T}(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z})\} = 0$.

Proof. The proof follows from Proposition 2.5 of [17], taking into account Theorem 3.5, Proposition 3.8 above and the fact that $\overline{Q}$ vanishes (Theorem 3.4(C3)). □

Proposition 3.12. *For the vertical recurrent Finsler connection $\overline{D}$, we have*:

(a) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{(\overline{D}_{\gamma\overline{X}}\overline{S})(\overline{Y}, \overline{Z}, \overline{W})\} = 0$.

(b) $(\overline{D}_{\beta\overline{Z}}\overline{S})(\overline{X}, \overline{Y}, \overline{W}) - (\overline{D}_{\gamma\overline{X}}\overline{P})(\overline{Z}, \overline{Y}, \overline{W}) + (\overline{D}_{\gamma\overline{Y}}\overline{P})(\overline{Z}, \overline{X}, \overline{W})$
$= \overline{P}(\overline{T}(\overline{X}, \overline{Z}), \overline{Y})\overline{W} - \overline{P}(\overline{T}(\overline{Y}, \overline{Z}), \overline{X})\overline{W} + \overline{S}(\widehat{P}(\overline{Z}, \overline{X}), \overline{Y})\overline{W} - \overline{S}(\widehat{P}(\overline{Z}, \overline{Y}), \overline{X})\overline{W}$.

(c) $(\overline{D}_{\gamma\overline{X}}\overline{R})(\overline{Y}, \overline{Z}, \overline{W}) + (\overline{D}_{\beta\overline{Y}}\overline{P})(\overline{Z}, \overline{X}, \overline{W}) - (\overline{D}_{\beta\overline{Z}}\overline{P})(\overline{Y}, \overline{X}, \overline{W})$
$= \overline{P}(\overline{Z}, \widehat{P}(\overline{Y}, \overline{X}))\overline{W} - \overline{P}(\overline{Y}, \widehat{P}(\overline{Z}, \overline{X}))\overline{W} + \overline{R}(\overline{T}(\overline{X}, \overline{Z}), \overline{Y})\overline{W} - \overline{R}(\overline{T}(\overline{X}, \overline{Y}), \overline{Z})\overline{W} + \overline{S}(\widehat{R}(\overline{Y}, \overline{Z}), \overline{X})\overline{W}$.

(d) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{(\overline{D}_{\beta\overline{X}}\overline{R})(\overline{Y}, \overline{Z}, \overline{W}) + \overline{P}(\overline{X}, \widehat{R}(\overline{Y}, \overline{Z}))\overline{W}\} = 0$.

Proof. The proof follows from Proposition 2.6 of [17], together with Theorem 3.5 and Proposition 3.8 above. □

4. Applications

In this section, we have three important results concerning the curvature tensors associated to special vertical recurrent Finsler connection $\overline{D}$

A Finsler manifold (M, L) is said to be ([6, 15]) S_3-like if $\dim M \geq 4$ and the v-curvature tensor S of Cartan connection ∇ has the form

$$S(\overline{X}, \overline{Y})\overline{Z} = s(x)\, L^{-2}\{\hbar(\overline{X}, \overline{Z})\phi(\overline{Y}) - \hbar(\overline{Y}, \overline{Z})\phi(\overline{X})\}, \tag{12}$$

where $\hbar := g - \ell \otimes \ell$, $\ell := L^{-1} i_{\overline{\eta}}\, g$ and $s(x)$ is a scalar function in the position alone.

Theorem 4.1. *Let $\overline{D}(B)$ be the vertical recurrent Finsler connection on a Finsler manifold (M,L) with respect to a given non-zero scalar 1-form $B(\overline{X}) := L^{-1}\ell(\overline{X})$. The v-curvature tensor $\overline{S}$ of $\overline{D}(B)$ vanishes if and only if (M,L) is S_3-like with scalar function $s = -\frac{3}{4}$.*

Proof. If $B(\overline{X}) := L^{-1}\ell(\overline{X})$ (or equivalently $\bar{b} := L^{-2}\overline{\eta}$), noting that $\nabla_{\beta\overline{X}}L = 0$, $(\nabla_{\beta\overline{X}}\ell)(\overline{Y}) = 0$, $\nabla_{\gamma\overline{X}}L = \ell(\overline{X})$ and

$$(\nabla_{\gamma\overline{X}}\ell)(\overline{Y}) = L^{-1}\hbar(\overline{X},\overline{Y}) =: L^{-1}g(\phi(\overline{X}),\overline{Y}),$$

then, one can show that

$$\begin{aligned}
(\nabla_{\beta\overline{X}}B)(\overline{Y}) = 0 \ &\Rightarrow\ \nabla_{\beta\overline{X}}\bar{b} = 0,\\
(\nabla_{\gamma\overline{X}}B)(\overline{Y}) &= L^{-2}\hbar(\overline{X},\overline{Y}) - L^{-2}\ell(\overline{X})\ell(\overline{Y}),\\
\nabla_{\gamma\overline{X}}\bar{b} &= L^{-2}\phi(\overline{X}) - L^{-3}\ell(\overline{X})\overline{\eta}.
\end{aligned}$$

Consequently, after some calculations taking into account Theorem 3.10(a), we have

$$\overline{S}(\overline{X},\overline{Y})\overline{Z} = S(\overline{X},\overline{Y})\overline{Z} + \frac{3}{4}L^{-2}\{\hbar(\overline{X},\overline{Z})\phi(\overline{Y}) - \hbar(\overline{Y},\overline{Z})\phi(\overline{X})\}.$$

From which, the result follows. □

Theorem 4.2. *Let $\overline{D}$ be the vertical recurrent Finsler connection with respect to a given non-zero scalar 1-form B. If B is ∇-horizontally parallel and $B(\overline{\eta}) \neq 1, 2$, then the hv-curvature tensor $\overline{P}$ of $\overline{D}$ vanishes if and only the hv-curvature tensor P of Cartan connection ∇ vanishes.*

Proof. Let $\overline{D}$ be the vertical recurrent Finsler connection with respect to a given non-zero scalar 1-form B. Since B is ∇-horizontally parallel, then from Theorem 3.10(b), we have

$$\begin{aligned}
\overline{P}(\overline{X},\overline{Y})\overline{Z} = P(\overline{X},\overline{Y})\overline{Z} - \frac{1}{2}\{&B(\widehat{P}(\overline{X},\overline{Y}))\overline{Z}\\
&+ B(\overline{Z})\widehat{P}(\overline{X},\overline{Y}) - g(\widehat{P}(\overline{X},\overline{Y}),\overline{Z})\bar{b}\}.
\end{aligned} \tag{13}$$

Firstly, if the hv-curvature tensor P of Cartan connection vanishes, then $\widehat{P} = 0$. Hence, from (13), the hv-curvature $\overline{P}$ of the vertical recurrent Finsler connection vanishes.

Conversely, if $\overline{P} = 0$, then from (13), we get

$$P(\overline{X},\overline{Y})\overline{Z} = \frac{1}{2}\{B(\widehat{P}(\overline{X},\overline{Y}))\overline{Z} + B(\overline{Z})\widehat{P}(\overline{X},\overline{Y}) - g(\widehat{P}(\overline{X},\overline{Y}),\overline{Z})\bar{b}\}. \tag{14}$$

One can show that, using $\widehat{P}(\overline{X},\overline{Y}) = (\nabla_{\beta\overline{\eta}}T)(\overline{X},\overline{Y})$, $\widehat{P}(\overline{X},\overline{\eta}) = 0$ and the fact that $g((\nabla_W T)(\overline{X},\overline{Y}),\overline{Z}) = g((\nabla_W T)(\overline{X},\overline{Z}),\overline{Y})$ ([17]),

$$g(\widehat{P}(\overline{X},\overline{Y}),\overline{\eta}) = g(\widehat{P}(\overline{X},\overline{\eta}),\overline{Y}) = 0.$$

Hence, by setting $\overline{Z} = \overline{\eta}$ into (14), we obtain

$$(2 - B(\overline{\eta}))\widehat{P}(\overline{X}, \overline{Y}) = B(\widehat{P}(\overline{X}, \overline{Y}))\overline{\eta}. \tag{15}$$

From which, tacking into account the fact that $i_{\overline{b}}\, g := B$, we have

$$(1 - B(\overline{\eta}))B(\widehat{P}(\overline{X}, \overline{Y})) = 0. \tag{16}$$

According to the given assumption that $B(\overline{\eta}) \neq 1, 2$, using Equations (15) and (16), we conclude that the (v)hv-torsion tensor $\widehat{P}$ vanishes. Consequently, from (14), the hv-curvature P also vanishes. □

Theorem 4.3. *Let $\overline{D}$ be the vertical recurrent Finsler connection with respect to a given non-zero scalar* 1*-form B. If B satisfies $4(1 - B(\overline{\eta})) + B(\overline{b})L^2 \neq 0$ and $B(\overline{\eta}) \neq 2$, then the h-curvature tensor $\overline{R}$ of $\overline{D}$ vanishes if and only the h-curvature tensor R of Cartan connection ∇ vanishes.*

Proof. Let $\overline{D}$ be the vertical recurrent Finsler connection with respect to a given non-zero scalar 1-form B satisfying $4(1 - B(\overline{\eta})) + B(\overline{b})L^2 \neq 0$ and $B(\overline{\eta}) \neq 2$. From Theorem 3.10(c), we have

$$\begin{aligned}\overline{R}(\overline{X}, \overline{Y})\overline{Z} = R(\overline{X}, \overline{Y})\overline{Z} - \frac{1}{2}\{&B(\widehat{R}(\overline{X}, \overline{Y}))\overline{Z} + B(\overline{Z})\widehat{R}(\overline{X}, \overline{Y}) \\ &- g(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z})\overline{b}\}.\end{aligned} \tag{17}$$

Firstly, if the h-curvature tensor R of Cartan connection vanishes, then $\widehat{R} = 0$. Hence, from (17), the h-curvature tensor $\overline{R}$ also vanishes.

Conversely, if $\overline{R} = 0$, then Equation (17) reduces to

$$R(\overline{X}, \overline{Y})\overline{Z} = \frac{1}{2}\{B(\widehat{R}(\overline{X}, \overline{Y}))\overline{Z} + B(\overline{Z})\widehat{R}(\overline{X}, \overline{Y}) - g(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z})\overline{b}\}. \tag{18}$$

Setting $\overline{Z} = \overline{\eta}$, we obtain

$$\widehat{R}(\overline{X}, \overline{Y}) = \frac{1}{2}\{B(\widehat{R}(\overline{X}, \overline{Y}))\overline{\eta} + B(\overline{\eta})\widehat{R}(\overline{X}, \overline{Y}) - g(\widehat{R}(\overline{X}, \overline{Y}), \overline{\eta})\, \overline{b}\}. \tag{19}$$

From which, noting that $g(\overline{\eta}, \overline{\eta}) = L^2$, we get

$$g(\widehat{R}(\overline{X}, \overline{Y}), \overline{\eta}) = \frac{1}{2}L^2\, B(\widehat{R}(\overline{X}, \overline{Y})). \tag{20}$$

Again from Equation (19), one can show that

$$(1 - B(\overline{\eta}))B(\widehat{R}(\overline{X}, \overline{Y})) = -\frac{1}{2}B(\overline{b})g(\widehat{R}(\overline{X}, \overline{Y}), \overline{\eta}). \tag{21}$$

Now, from (20) and (21), it follows that

$$\{4(1 - B(\overline{\eta})) + B(\overline{b})L^2\}B(\widehat{R}(\overline{X}, \overline{Y})) = 0.$$

From which, taking into account the given assumption $4(1 - B(\overline{\eta})) + B(\overline{b})L^2 \neq 0$ and Equation (20), we conclude that

$$B(\widehat{R}(\overline{X}, \overline{Y})) = 0 = g(\widehat{R}(\overline{X}, \overline{Y}), \overline{\eta}).$$

Hence, in view of Equation (19), we obtain

$$(2 - B(\overline{\eta}))\widehat{R}(\overline{X}, \overline{Y}) = 0.$$

Therefore, the (v)h-torsion tensor $\widehat{R}$ of Cartan connection vanishes provided that $B(\overline{\eta}) \neq 2$. Consequently, from Equation (18), the h-curvature tensor R of Cartan connection also vanishes. This completes the proof. □

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