Commun. Korean Math. Soc. **36** (2021), No. 3, pp. 557–573
https://doi.org/10.4134/CKMS.c200254
pISSN: 1225-1763 / eISSN: 2234-3024

FRACTIONAL CALCULUS OPERATORS OF THE PRODUCT OF GENERALIZED MODIFIED BESSEL FUNCTION OF THE SECOND TYPE

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Abstract. In this present paper, we consider four integrals and differentials containing the Gauss' hypergeometric ${}_2F_1(x)$ function in the kernels, which extend the classical Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential operators. Formulas (images) for compositions of such generalized fractional integrals and differential constructions with the n-times product of the generalized modified Bessel function of the second type are established. The results are obtained in terms of the generalized Lauricella function or Srivastava-Daoust hypergeometric function. Equivalent assertions for the Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential are also deduced.

1. Introduction

We recognize the Saigo fractional integral along with differential operators in conjunction with the hypergeometric function ${}_2F_1$ [4, 8, 9, 12, 15, 19]:

$$\left(I_{0+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}f\right)(x)=\frac{x^{-\mathfrak{a}-\mathfrak{b}}}{\Gamma(\mathfrak{a})}\int_0^x(x-t)^{\mathfrak{a}-1}\,{}_2F_1\left(\mathfrak{a}+\mathfrak{b},-\mathfrak{c};\mathfrak{a};1-\frac{t}{x}\right)f(t)\,\mathrm{d}t,\tag{1}$$

$$\left(I_{-}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}f\right)(x)=\frac{1}{\Gamma(\mathfrak{a})}\int_x^\infty(t-x)^{\mathfrak{a}-1}t^{-\mathfrak{a}-\mathfrak{b}}\,{}_2F_1\left(\mathfrak{a}+\mathfrak{b},-\mathfrak{c};\mathfrak{a};1-\frac{x}{t}\right)f(t)\,\mathrm{d}t,\tag{2}$$

and

$$\begin{aligned}\left(D_{0+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}f\right)(x)&=\left(I_{0+}^{-\mathfrak{a},-\mathfrak{b},\mathfrak{a}+\mathfrak{c}}f\right)(x)\\&=\left(\frac{d}{dx}\right)^n\left(I_{0+}^{-\mathfrak{a}+n,-\mathfrak{b}-n,\mathfrak{a}+\mathfrak{c}-n}f\right)(x)\quad(n=[\Re(\mathfrak{a})]+1),\end{aligned}\tag{3}$$

Received July 20, 2020; Revised September 13, 2020; Accepted December 16, 2020.
2010 *Mathematics Subject Classification.* 26A33, 33C10, 33C20, 33C50, 33C60, 26A09.
Key words and phrases. Saigo fractional calculus operators, generalized Lauricella function, Gauss' hypergeometric ${}_2F_1(x)$ function, generalized modified Bessel function of the second type.

$$\left(D_{-}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}f\right)(x) = \left(I_{-}^{-\mathfrak{a},-\mathfrak{b},\mathfrak{a}+\mathfrak{c}}f\right)(x)$$

$$= (-1)^n\left(\frac{d}{dx}\right)^n\left(I_{-}^{-\mathfrak{a}+n,-\mathfrak{b}-n,\mathfrak{a}+\mathfrak{c}}f\right)(x) \quad (n = [\Re(\mathfrak{a})]+1). \tag{4}$$

Here and in the following, $[x]$ specifies the greatest integer and $[x] \leq x$; $x \in \mathbb{R}$.

Setting $\mathfrak{b} = -\mathfrak{a}$ in (1), (2), (3), and (4) yields the familiar Riemann-Liouville (R-L) fractional integral along with differential operators of order $\mathfrak{a} \in \mathbb{C}$ with $\Re(\mathfrak{a}) > 0$ and $x \in \mathbb{R}^+$ (see, e.g., [8, 16]):

$$\left(I_{0+}^{\mathfrak{a},-\mathfrak{a},\mathfrak{c}}f\right)(x) = \left(I_{0+}^{\mathfrak{a}}f\right)(x) = \frac{1}{\Gamma(\mathfrak{a})}\int_0^x (x-t)^{\mathfrak{a}-1}f(t)\,\mathrm{d}t, \tag{5}$$

$$\left(I_{-}^{\mathfrak{a},-\mathfrak{a},\mathfrak{c}}f\right)(x) \equiv \left(I_{-}^{\mathfrak{a}}f\right)(x) = \frac{1}{\Gamma(\mathfrak{a})}\int_x^\infty (t-x)^{\mathfrak{a}-1}f(t)\,\mathrm{d}t, \tag{6}$$

and

$$\left(D_{0+}^{\mathfrak{a},-\mathfrak{a},\mathfrak{c}}f\right)(x) = \left(D_{0+}^{\mathfrak{a}}f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\mathfrak{a})}\int_0^x (x-t)^{n-\mathfrak{a}-1}f(t)\,\mathrm{d}t$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{n-\mathfrak{a}}f\right)(x) \qquad (n = [\Re(\mathfrak{a})]+1), \tag{7}$$

$$\left(D_{-}^{\mathfrak{a},-\mathfrak{a},\mathfrak{c}}f\right)(x) = \left(D_{-}^{\mathfrak{a}}f\right)(x) = (-1)^n\left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\mathfrak{a})}\int_x^\infty (t-y)^{n-\mathfrak{a}-1}f(t)\,\mathrm{d}t$$

$$= (-1)^n\left(\frac{d}{dx}\right)^n \left(I_{-}^{n-\mathfrak{a}}f\right)(x) \qquad (n = [\Re(\mathfrak{a})]+1). \tag{8}$$

Further, let $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{C}$ be the sets of positive integers, positive real numbers and complex numbers, respectively, and also let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Setting $\mathfrak{b} = 0$ in (1), (2), (3), and (4) yields the known Erdélyi-Kober (E-K) fractional integral and differential of order $\mathfrak{a} \in \mathbb{C}$, $(n = [\Re(\mathfrak{a})]+1)$ with $\Re(\mathfrak{a}) > 0$ and $x \in \mathbb{R}^+$ (see, e.g., [8, 16]):

$$\left(I_{0+}^{\mathfrak{a},0,\mathfrak{c}}f\right)(x) = \left(I_{\mathfrak{c},\mathfrak{a}}^{+}f\right)(x) = \frac{x^{-\mathfrak{a}-\mathfrak{c}}}{\Gamma(\mathfrak{a})}\int_0^x (x-t)^{\mathfrak{a}-1}t^{\mathfrak{c}}f(t)\,\mathrm{d}t, \tag{9}$$

$$\left(I_{-}^{\mathfrak{a},0,\mathfrak{c}}f\right)(x) = \left(K_{\mathfrak{c},\mathfrak{a}}^{-}f\right)(x) \equiv \frac{x^{\mathfrak{c}}}{\Gamma(\mathfrak{a})}\int_x^\infty (t-x)^{\mathfrak{a}-1}t^{-\mathfrak{a}-\mathfrak{c}}f(t)\,\mathrm{d}t, \tag{10}$$

and

$$\left(D_{0+}^{\mathfrak{a},0,\mathfrak{c}}f\right)(x) = \left(D_{\mathfrak{c},\mathfrak{a}}^{+}f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\mathfrak{a}+n,-\mathfrak{a},\mathfrak{a}+\mathfrak{c}-n}f\right)(x), \tag{11}$$

$$\left(D_{-}^{\mathfrak{a},0,\mathfrak{c}}f\right)(x) = \left(D_{\mathfrak{c},\mathfrak{a}}^{-}f\right)(x) = (-1)^n\left(\frac{d}{dx}\right)^n \left(I_{-}^{-\mathfrak{a}+n,-\mathfrak{a},\mathfrak{a}+\mathfrak{c}}f\right)(x), \tag{12}$$

$$\left(D_{\mathfrak{c},\mathfrak{a}}^{+}f\right)(x) = x^{-\mathfrak{c}}\left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\mathfrak{a})}\int_0^x t^{\mathfrak{a}+\mathfrak{c}}(x-t)^{n-\mathfrak{a}-1}f(t)\,\mathrm{d}t, \tag{13}$$

$$(14) \qquad \left(D^{-}_{\mathfrak{c},\mathfrak{a}}f\right)(x) = x^{\mathfrak{c}+\mathfrak{a}}\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\mathfrak{a})}\int_{x}^{\infty} t^{-\mathfrak{c}}(t-x)^{n-\mathfrak{a}-1}f(t)\,dt.$$

The generalized modified Bessel function $\tilde{I}^{k,s}_{\tau}(z)$ of the second type is defined by Griffiths *et al.* [2]

$$(15) \qquad \tilde{I}^{k,s}_{\tau}(z) = \left(\frac{z}{2}\right)^{\tau+k-s}\sum_{r=0}^{\infty}\frac{1}{(k(r+1)-s)!\quad\Gamma(\tau+r+1)}\left(\frac{z}{2}\right)^{r(k+1)},$$

where $s \in \{1,2,\dots,k\}, \tau = \{0,\pm1,\pm2,\dots\}$, $z \in \mathbb{C}$ and $k = \{1,2,3,\dots\}$.

We aim to investigate compositions of the generalized fractional integration and differentiation operators (1), (2), (3) and (4) with the n-times product of the generalized modified Bessel function of the second type $\tilde{I}^{k,s}_{\tau}(z)$. Also, those results for the well known Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential operators are developed.

We derive that such compositions are written in term of the generalized Lauricella function or Srivastava and Daoust hypergeometric function [18, 19] which is defined by

$$(16) \quad F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}[z_1,\dots,z_n]$$
$$= F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\left[\begin{array}{l}[(a):\theta',\dots,\theta^{(n)}],\ [(b'):\phi'];\cdots;[(b)^{(n)}:\phi^{(n)}]:\\ [(c):\psi',\dots,\psi^{(n)}],\ [(d'):\delta'];\cdots;[(d)^{(n)}:\delta^{(n)}]:\end{array}\ z_1,\dots,z_n\right]$$
$$= \sum_{k_1,\dots,k_n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{k_1\theta'_j+\cdots+k_n\theta(n)_j}}{\prod_{j=1}^{C}(c_j)_{k_1\psi'_j+\cdots+k_n\psi^{(n)}_j}}\frac{\prod_{j=1}^{B'}(b'_j)_{k_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b^{(n)}_j)_{k_n\phi^{(n)}_j}}{\prod_{j=1}^{D'}(d'_j)_{k_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d^{(n)}_j)_{k_n\delta^{(n)}_j}}\frac{z_1^{k_1}}{k_n!}\cdots\frac{z_1^{k_1}}{k_n!},$$

the coefficients

(i) $\theta^{m}_{j}(j=1,\dots,A),\ \phi^{m}_{J}(j=1,\dots,B^{(m)})$,
(ii) $\psi^{m}_{j}(j=1,\dots,C)$,
(iii) $\delta^{m}_{j}(j=1,\dots,D^{(m)})$,

$\forall\ m \in \{1,\dots,n\}$ are real and positive, and (a) abbreviates the array of A parameters $a_1,\dots,a_A$, $(b^{(m)})$ abbreviates the array of $B^{(m)}$ parameters $b^{(m)}_j$ $(j=1,\dots,B^{(m)}), \forall\ m \in \{1,\dots,n\}$, with similar interpretations for (c) and (d^m) $(m=1,\dots,n)$. $(y)_a$ is defined as Pochhammer symbol:

$$(17) \qquad (\delta)_a = \frac{\Gamma(\delta+a)}{\Gamma(a)},\quad \delta, a \in \mathbb{C}.$$

The multiple series (16) converges (absolutely) either $\Delta_i > 0$ $(i=1,\dots,n)$, $\forall\ z_1,\dots,z_n \in \mathbb{C}$ or $\Delta_i = 0$ $|z| < \varrho_i$ $(i=1,\dots,n)$, and divergences when $\Delta_i < 0$ except for the trivial case $z_1 = \cdots = z_n = 0$, where

$$(18) \qquad \Delta_i = 1 + \sum_{j=1}^{C}\psi^{(i)}_j + \sum_{j=1}^{D^{(i)}}\delta^{(i)}_j - \sum_{j=1}^{A}\theta^{(i)}_j - \sum_{j=1}^{B^{(i)}}\phi^{(i)}_j\ (i=1,\dots,n),$$

$$(19) \qquad \varrho_i = \min_{\mu_1,\dots,\mu_n>0}(E_i)\ (i=1,\dots,n),$$

with

$$E_i = (\mu_i)^{1+\sum_{j=1}^{D^{(i)}} \delta_j^i - \sum_{j=1}^{B^{(i)}} \phi_j^i} \frac{\left\{\prod_{j=1}^{C} \left(\sum_{i=1}^{n} \mu\psi_j^{(i)}\right)^{\psi_j^{(i)}}\right\} \left(\prod_{j=1}^{D^{(i)}} (\delta_j^{(i)})^{\delta_j^{(i)}}\right)}{\left\{\prod_{j=1}^{A} \left\{\sum_{i=1}^{n} \mu\theta_j^{(i)}\right\}^{\theta_j^{(i)}}\right\} \left\{\prod_{j=1}^{B^{(i)}} (\phi_j^{(i)})^{\phi_j^{(i)}}\right\}}. \tag{20}$$

The special cases of (16) reduce to the hypergeometric function of one variable ${}_pF_q(z)$ and two variables $F_{l:m;n}^{p:q;k}$ given below.

A generalized hypergeometric function ${}_pF_q(z)$ is defined for complex $a_i, b_j \in \mathbb{C}, b_j \neq 0, -1, \ldots, (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$ by the generalized hypergeometric series [18, 19]

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=1}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{21}$$

${}_pF_q$ is absolutely convergent for all values of $z \in \mathbb{C}$ if $p \leq q$ and also entire function of z. Also the series form of Kampé de Fériet function by means of the generalized hypergeometric series of two variables is given as [18, 19]

$$\begin{aligned} &F_{l':m';n'}^{\mathfrak{p}:\mathfrak{q};k}\left[\begin{array}{c} (a_{\mathrm{p}}) : (b_{\mathfrak{q}}); (c_k); \\ (\alpha_{\mathfrak{p}}) : (\beta_{\mathfrak{q}}); (\gamma_k); \end{array} \mathfrak{X}, \mathfrak{Y}\right] \\ &= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{\mathfrak{p}} (a_j)_{r+s} \prod_{j=1}^{\mathfrak{q}} (b_j)_r \prod_{j=1}^{k'} (c_j)_s}{\prod_{j=1}^{l'} (\alpha_j)_{r+s} \prod_{j=1}^{m'} (\beta_j)_r \prod_{j=1}^{n'} (\gamma_j)_s} \frac{\mathfrak{X}^r}{r!} \frac{\mathfrak{Y}^s}{s!}. \end{aligned} \tag{22}$$

This double hypergeometric series is convergent (absolutely) for all values of $\mathfrak{X}$ and $\mathfrak{Y}$, if $\mathfrak{p}+\mathfrak{q} < l'+m'+1$ and $\mathfrak{p}+k' < l'+n'+1$. Further, if $\mathfrak{p}+\mathfrak{q} = l'+m'+1$ and $\mathfrak{p}+k = l'+n'+1$, along with the following any one sets of conditions:

(i) $\mathfrak{p} \leq l$, $\max\{|\mathfrak{X}|, |\mathfrak{Y}|\} < 1$;

(ii) $\mathfrak{p} > l$, $|\mathfrak{X}|^{\frac{1}{(\mathfrak{p}-l)}} + |\mathfrak{Y}|^{\frac{1}{(\mathfrak{p}-l)}} < 1$.

2. Fractional integration of $\tilde{I}_{\tau}^{k,s}(z)$

The main results in this section are based on preliminary assertions including composition formulas of the Saigo's fractional integrals (1) and (2) with the power function. The left-hand sided and right-hand sided generalized integrations (1) and (2) of a power function are given by the following results.

Lemma 2.1. *Let* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c} \in \mathbb{C}$.

(a) *Let* $\Re(\mathfrak{a}) > 0$ *and* $\Re(\mathfrak{h}) > \max\{0, \Re(\mathfrak{b} - \mathfrak{c})\}$. *Then*

$$(I_{0+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}} t^{\mathfrak{h}-1})(x) = \frac{\Gamma(\mathfrak{h})\Gamma(\mathfrak{h} + \mathfrak{c} - \mathfrak{b})}{\Gamma(\mathfrak{h} - \mathfrak{b})\Gamma(\mathfrak{h} + \mathfrak{a} + \mathfrak{c})} x^{\mathfrak{h}-\mathfrak{b}-1}. \tag{23}$$

In its special case, for $x > 0$, *we have*

$$(I_{0+}^{\mathfrak{a}} t^{\mathfrak{h}-1})(x) = \frac{\Gamma(\mathfrak{h})}{\Gamma(\mathfrak{h} + \mathfrak{a})} x^{\mathfrak{h}+\mathfrak{a}-1}, \quad (\min\{\Re(\mathfrak{a}), \Re(\mathfrak{h})\} > 0), \tag{24}$$

and

$$(I_{\mathfrak{c},\mathfrak{a}}^{+}t^{\mathfrak{h}-1})(x) = \frac{\Gamma(\mathfrak{h}+\mathfrak{c})}{\Gamma(\mathfrak{h}+\mathfrak{a}+\mathfrak{c})}x^{\mathfrak{h}-1}, \quad (\Re(\mathfrak{a})>0,\ \Re(\mathfrak{h})>-\Re(\mathfrak{c})). \tag{25}$$

(b) *If* $\Re(\mathfrak{a})>0$ *and* $\Re(\mathfrak{h})<1+\min\{\Re(\mathfrak{b}),\Re(\mathfrak{c})\}$, *then*

$$(I_{-}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}t^{\mathfrak{h}-1})(x) = \frac{\Gamma(\mathfrak{b}-\mathfrak{h}+1)\Gamma(\mathfrak{c}-\mathfrak{h}+1)}{\Gamma(1-\mathfrak{h})\Gamma(\mathfrak{a}+\mathfrak{b}+\mathfrak{c}-\mathfrak{h}+1)}x^{\mathfrak{h}-\mathfrak{b}-1}. \tag{26}$$

In particular, for $x>0$, *we have*

$$(I_{-}^{\mathfrak{a}}t^{\mathfrak{h}-1})(x) = \frac{\Gamma(1-\mathfrak{a}-\mathfrak{h})}{\Gamma(1-\mathfrak{h})}x^{\mathfrak{h}+\mathfrak{a}-1}, \quad (0<\Re(\mathfrak{a})<1-\Re(\mathfrak{h})) \tag{27}$$

and

$$(K_{\mathfrak{c},\mathfrak{a}}^{-}t^{\mathfrak{h}-1})(x) = \frac{\Gamma(\mathfrak{c}-\mathfrak{h}+1)}{\Gamma(\mathfrak{a}+\mathfrak{c}-\mathfrak{h}+1)}x^{\mathfrak{h}-1}, \quad (\Re(\mathfrak{h})<1+\Re(\mathfrak{h})). \tag{28}$$

2.1. Left-sided fractional integration

Here we consider the Saigo's left-hand sided fractional integration (1) of the generalized modified Bessel function of the second kind (15).

Theorem 2.2. *Let* $n\in\mathbb{N}$, $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{h},\tau_j\in\mathbb{C}$, $s\in\{1,2,\ldots,k\}$, $\tau=\{0,\pm1,\pm2,\ldots\}$, $k=\{1,2,3,\ldots\}$ *and* $a_j,\rho_j\in\mathbb{R}_+\,(j=1,2,\ldots,n)$ *such that* $\Re(\mathfrak{a})>0,\Re(\tau_j)>-1,\Re(\mathfrak{h}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j)>\max[0,\Re(\mathfrak{b}-\mathfrak{c})]$. *Then there holds the formula*:

$$\left(I_{0^+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}(\omega_j t^{\rho_j})\right]\right)(x) \tag{29}$$
$$= x^{\mathfrak{h}-\mathfrak{b}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j x^{\rho_j}}{2}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-}times}}\right)$$
$$\times\frac{\Gamma(\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(\mathfrak{h}-\mathfrak{b}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}\frac{\Gamma(\mathfrak{h}+\mathfrak{c}-\mathfrak{b}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}$$
$$\times F_{2:2;\cdots;2}^{2:1;\cdots;1}\left[\begin{matrix}[U:\Lambda],[V:\Lambda]: & \overbrace{[1:1]}^{n\text{-}times}\\ [W:\Lambda],[Z:\Lambda]: & [\tau_1+1:1],\ldots,[\tau_n+1:1]:\underbrace{[k-s+1:k]}_{n\text{-}times}\end{matrix}\ \middle|\ \left(\frac{\omega_1 x^{\rho_1}}{2}\right)^{k+1},\ldots,\left(\frac{\omega_n x^{\rho_n}}{2}\right)^{k+1}\right],$$

where $\Lambda=(k+1)\rho_1,\ldots,(k+1)\rho_n, U=\mathfrak{h}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, V=\mathfrak{h}+\mathfrak{c}-\mathfrak{b}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, W=\mathfrak{h}-\mathfrak{b}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, Z=\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j$ *and* $F_{2:2;\cdots;2}^{2:1;\cdots;1}(\cdot)$ *is given by* (16).

Proof. Here, we note that Δ_i in (18) is given by $\Delta_i=1+nk>0$ $(i=1,\ldots,n\in\mathbb{N},\ k=\{1,2,3,\ldots\})$, and therefore $F_{2:2;\cdots;2}^{2:1;\cdots;1}(\cdot)$ in the right side of equation (29), is defined. To prove (29), we first apply (15) and use (1) and then change the order of integration and summation. Then, we find

$$\left(I_{0^+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}(\omega_j t^{\rho_j})\right]\right)(x)$$
$$=\left(I_{0^+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\left(\sum_{r_j=0}^{\infty}\frac{\left(\frac{\omega_j t^{\rho_j}}{2}\right)^{\tau_j+k-s+(k+1)r_j}}{\Gamma(\tau_j+r_j+1)\,\Gamma(k-s+1+kr_j)}\right)\right]\right)(x)$$
$$=\sum_{r_1,\ldots,r_n=0}^{\infty}\frac{\left(\frac{\omega_1}{2}\right)^{\tau_1+k-s+(k+1)r_1}}{\Gamma(\tau_1+r_1+1)\,\Gamma(k-s+1+kr_1)}\cdots\frac{\left(\frac{\omega_n}{2}\right)^{\tau_n+k-s+(k+1)r_n}}{\Gamma(\tau_n+r_n+1)\,\Gamma(k-s+1+kr_n)}$$
$$\times\left(I_{0^+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}t^{\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i-1}\right)(x).$$

By given condition in Theorem 2.2 statement for any $r_j\in\mathbb{N}_0$ $(j=1,\ldots,n)$ $\Re(\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i)\geq\Re(\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)\geq\max[0,\Re(\mathfrak{b}-\mathfrak{c})]$. Applying (23), we obtain

$$\left(I_{0^+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}(\omega_j t^{\rho_j})\right]\right)(x)$$
$$=\sum_{r_1,\ldots,r_n=0}^{\infty}\frac{\left(\frac{\omega_1}{2}\right)^{\tau_1+k-s+(k+1)r_1}}{\Gamma(\tau_1+r_1+1)\,\Gamma(k-s+1+kr_1)}\cdots\frac{\left(\frac{\omega_n}{2}\right)^{\tau_n+k-s+(k+1)r_n}}{\Gamma(\tau_n+r_n+1)\,\Gamma(k-s+1+kr_n)}$$
$$\times\frac{\Gamma(\mathfrak{h}+\sum_{i=1}^{n}\{(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i\})}{\Gamma(\mathfrak{h}-\mathfrak{b}+\sum_{i=1}^{n}\{(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i\})}$$
$$\times\frac{\Gamma(\mathfrak{h}+\mathfrak{c}-\mathfrak{b}+\sum_{i=1}^{n}\{(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i\})}{\Gamma(\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{i=1}^{n}\{(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i\})}$$
$$\times x^{\mathfrak{h}-\mathfrak{b}+\sum_{i=1}^{n}\{(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i\}-1}.$$

Now using the definition of Pochhammer symbol (17), we have

$$\left(I_{0^+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}(\omega_j t^{\rho_j})\right]\right)(x)$$
$$=x^{\mathfrak{h}-\mathfrak{b}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j x^{\rho_j}}{2}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-times}}}\right)$$
$$\times\frac{\Gamma(\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(\mathfrak{h}-\mathfrak{b}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}\frac{\Gamma(\mathfrak{h}+\mathfrak{c}-\mathfrak{b}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}$$

$$\times \sum_{r_1,\ldots,r_n=0}^{\infty} \frac{(1)_{r_1}\cdots(1)_{r_n}}{(\tau_1+1)_{r_1}\cdots(\tau_n+1)_{r_n}(k-s+1)_{kr_1}\cdots(k-s+1)_{kr_n}}$$
$$\times \frac{(\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)_{(k+1)r_1\rho_1+\cdots+(k+1)r_n\rho_n}}{(\mathfrak{h}-\mathfrak{b}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)_{(k+1)r_1\rho_1+\cdots+(k+1)r_n\rho_n}}$$
$$\times \frac{(\mathfrak{h}+\mathfrak{c}-\mathfrak{b}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)_{(k+1)r_1\rho_1+\cdots+(k+1)r_n\rho_n}}{(\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)_{(k+1)r_1\rho_1+\cdots+(k+1)r_n\rho_n}}$$
$$\times \frac{\left(\frac{\omega_1 x_1^{\rho}}{2}\right)^{(k+1)r_1}\cdots\left(\frac{\omega_n x_n^{\rho}}{2}\right)^{(k+1)r_n}}{r_1!\cdots r_n!}.$$

Expressing the last summation in terms of the generalized Lauricella function or Srivastava and Daoust hypergeometric function (16), we immediately establish the required result (29), which completes the proof of Theorem 2.2. □

Substituting $\mathfrak{b}=-\mathfrak{a}$ in Theorem 2.2 yields the following result for the familiar Riemann-Liouville (R-L) fractional integral (5).

Corollary 2.3. *Let* $n\in\mathbb{N}$, $\mathfrak{a},\mathfrak{h},\tau_j\in\mathbb{C}$, $s\in\{1,2,\ldots,k\}$, $\tau=\{0,\pm1,\pm2,\ldots\}$, $k=\{1,2,3,\ldots\}$ *and* $\omega_j,\rho_j\in\mathbb{R}_+$ $(j=1,2,\ldots,n)$ *such that* $\Re(\mathfrak{a})>0,\Re(\tau_j)>-1,\Re(\mathfrak{h}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j)>0$. *Then there holds the formula*:

$$(30)\qquad \left(I_{0+}^{\mathfrak{a}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}(\omega_j t^{\rho_j})\right]\right)(x)$$
$$= x^{\mathfrak{h}+\mathfrak{a}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j x^{\rho_j}}{2}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-}times}}\right)$$
$$\times \frac{\Gamma(\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(\mathfrak{h}+\mathfrak{a}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}$$
$$\times\ F_{1:2;\cdots;2}^{1:1;\cdots;1}\left[\begin{matrix}[U':\Lambda]: & \overbrace{[1:1]}^{n\text{-}times} \\ [W':\Lambda]: & [\tau_1+1:1],\ldots,[\tau_n+1:1]:\underbrace{[k-s+1:k]}_{n\text{-}times}\end{matrix}\ \middle|\ \left(\frac{\omega_1x^{\rho_1}}{2}\right)^{k+1},\ldots,\left(\frac{\omega_nx^{\rho_n}}{2}\right)^{k+1}\right],$$

where $\Lambda=(k+1)\rho_1,\ldots,(k+1)\rho_n$, $U'=\mathfrak{h}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j$, $W'=\mathfrak{h}+\mathfrak{a}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j$ *and* $F_{1:2;\cdots;2}^{1:1;\cdots;1}(\cdot)$ *is given by* (16).

Substituting $\mathfrak{b}=0$ in Theorem 2.2 yields the following result for the so-called Erdélyi-Kober fractional integral (9).

Corollary 2.4. *Let* $n\in\mathbb{N}$, $\mathfrak{a},\mathfrak{c},\mathfrak{h},\tau_j\in\mathbb{C}$, $s\in\{1,2,\ldots,k\}$, $\tau=\{0,\pm1,\pm2,\ldots\}$, $k=\{1,2,3,\ldots\}$ *and* $\omega_j,\rho_j\in\mathbb{R}_+(j=1,2,\ldots,n)$ *such that* $\Re(\mathfrak{a})>0,\Re(\tau_j)>$

$-1, \Re(\mathfrak{h} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j) > \Re(-\mathfrak{c})$. *Then there holds the formula*:

$$\left(I_{\mathfrak{c},\mathfrak{a}}^{+}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}(\omega_j t^{\rho_j})\right]\right)(x) \tag{31}$$

$$= x^{\mathfrak{h}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j x^{\rho_j}}{2}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-times}}}\right)$$

$$\times \frac{\Gamma(\mathfrak{h} + \sum_{i=1}^{n}(\tau_i + k - s)\rho_i)}{\Gamma(\mathfrak{h} + \mathfrak{a} + \mathfrak{c} + \sum_{i=1}^{n}(\tau_i + k - s)\rho_i)}$$

$$\times F_{1:2;\cdots;2}^{1:1;\cdots;1}\left[\begin{matrix}[V'':\Lambda]: & \overbrace{[1:1]}^{n\text{-times}} \\ [Z'':\Lambda]: & [\tau_1+1:1],\ldots,[\tau_n+1:1]:\underbrace{[k-s+1:k]}_{n\text{-times}}\end{matrix} \,\Big|\, \left(\frac{\omega_1 x^{\rho_1}}{2}\right)^{k+1},\ldots,\left(\frac{\omega_n x^{\rho_n}}{2}\right)^{k+1}\right],$$

where $\Lambda = (k+1)\rho_1, \ldots, (k+1)\rho_n, V'' = \mathfrak{h} + \mathfrak{c} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j, Z'' = \mathfrak{h} + \mathfrak{a} + \mathfrak{c} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j$ *and* $F_{1:2;\cdots;2}^{1:1;\cdots;1}(\cdot)$ *is given by* (16).

2.2. Right-sided fractional integration

The following result yields the generalized right-hand sided fractional integration of the product of generalized modified Bessel functions.

Theorem 2.5. *Let* $n \in \mathbb{N}$, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{h}, \tau_j \in \mathbb{C}$, $s \in \{1, 2, \ldots, k\}$, $\tau = \{0, \pm 1, \pm 2, \ldots\}$, $k = \{1, 2, 3, \ldots\}$ *and* $\omega_j, \rho_j \in \mathbb{R}_+ (j = 1, 2, \ldots, n)$ *such that* $\Re(\mathfrak{a}) > 0, \Re(\tau_j) > -1, \Re(\mathfrak{h} - \sum_{j=1}^{n}(\tau_j + k - s)\rho_j) < 1 + \min[\Re(\mathfrak{b}), \Re(\mathfrak{c})]$. *Then there holds the formula*:

$$\left(I_{-}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}\left(\frac{\omega_j}{t^{\rho_j}}\right)\right]\right)(x) \tag{32}$$

$$= x^{\mathfrak{h}-\mathfrak{b}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j}{2x^{\rho_j}}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-times}}}\right)$$

$$\times \frac{\Gamma(1 + \mathfrak{b} - \mathfrak{h} + \sum_{i=1}^{n}(\tau_i + k - s)\rho_i)}{\Gamma(1 - \mathfrak{h} + \sum_{i=1}^{n}(\tau_i + k - s)\rho_i)} \frac{\Gamma(1 + \mathfrak{c} - \mathfrak{h} + \sum_{i=1}^{n}(\tau_i + k - s)\rho_i)}{\Gamma(1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} - \mathfrak{h} + \sum_{i=1}^{n}(\tau_i + k - s)\rho_i)}$$

$$\times F_{2:2;\cdots;2}^{2:1;\cdots;1}\left[\begin{matrix}[P:\Lambda],[Q:\Lambda]: & \overbrace{[1:1]}^{n\text{-times}} \\ [R:\Lambda],[S:\Lambda]: & [\tau_1+1:1],\ldots,[\tau_n+1:1]:\underbrace{[k-s+1:k]}_{n\text{-times}}\end{matrix} \,\Big|\, \left(\frac{\omega_1}{2x^{\rho_1}}\right)^{k+1},\ldots,\left(\frac{\omega_n}{2x^{\rho_n}}\right)^{k+1}\right],$$

where $\Lambda = (k+1)\rho_1, \ldots, (k+1)\rho_n,\ P = 1 + \mathfrak{b} - \mathfrak{h} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j, Q = 1 + \mathfrak{c} - \mathfrak{h} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j, R = 1 - \mathfrak{h} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j, S = 1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} - \mathfrak{h} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j$ *and* $F_{2:2;\cdots;2}^{2:1;\cdots;1}(\cdot)$ *is given by* (16).

Proof. Here, we note that Δ_i in (18) is given by $\Delta_i = 1+nk > 0$ ($i = 1, \ldots, n \in \mathbb{N}$, $k = \{1, 2, 3, \ldots\}$), and therefore $F^{2:1;\cdots;1}_{2:2;\cdots;2}(\cdot)$ in the right side of equation (32) is defined. To prove (30), we apply (15) and use (2) and change the order of integration and summation. Then, we find

$$\begin{aligned}
&\left(I_{-}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}\left(\frac{\omega_j}{t^{\rho_j}}\right)\right]\right)(x)\\
&= \left(I_{-}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\left(\sum_{r_j=0}^{\infty}\frac{\left(\frac{\omega_j}{2t^{\rho_j}}\right)^{\tau_j+k-s+(k+1)r_j}}{\Gamma(\tau_j+r_j+1)\,\Gamma(k-s+1+kr_j)}\right)\right]\right)(x)\\
&= \sum_{r_1,\ldots,r_n=0}^{\infty}\frac{\left(\frac{\omega_1}{2}\right)^{\tau_1+k-s+(k+1)r_1}}{\Gamma(\tau_1+r_1+1)\,\Gamma(k-s+1+kr_1)}\cdot\frac{\left(\frac{\omega_n}{2}\right)^{\tau_n+k-s+(k+1)r_n}}{\Gamma(\tau_n+r_n+1)\,\Gamma(k-s+1+kr_n)}\\
&\quad\times\left(I_{0^+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}t^{\mathfrak{h}-\sum_{i=1}^{n}(\tau_i+k-s)\rho_i-(k+1)r_i\rho_i-1}\right)(x).
\end{aligned}$$

By given condition for any $r_j \in \mathbb{N}_0$ ($j = 1, \ldots, n$) $\Re(\mathfrak{h} - \sum_{i=1}^{n}(\tau_i + k - s)\rho_i - (k+1)r_i\rho_i) \leq \Re(\mathfrak{h} - \sum_{i=1}^{n}(\tau_i + k - s)\rho_i) < 1 + \min[\Re(\mathfrak{b}), \Re(\mathfrak{c})]$. Applying (26), we obtain

$$\begin{aligned}
&\left(I_{-}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}\left(\frac{\omega_j}{t^{\rho_j}}\right)\right]\right)(x)\\
&= \sum_{r_1,\ldots,r_n=0}^{\infty}\frac{\left(\frac{\omega_1}{2}\right)^{\tau_1+k-s+(k+1)r_1}}{\Gamma(\tau_1+r_1+1)\,\Gamma(k-s+1+kr_1)}\cdots\frac{\left(\frac{\omega_n}{2}\right)^{\tau_n+k-s+(k+1)r_n}}{\Gamma(\tau_n+r_n+1)\,\Gamma(k-s+1+kr_n)}\\
&\qquad\times\frac{\Gamma(1+\mathfrak{b}-\mathfrak{h}+\sum_{i=1}^{n}\{(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i\})}{\Gamma(1-\mathfrak{h}+\sum_{i=1}^{n}\{(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i\})}\\
&\qquad\times\frac{\Gamma(1-\mathfrak{h}+\mathfrak{c}+\sum_{i=1}^{n}\{(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i\})}{\Gamma(1-\mathfrak{h}+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\sum_{i=1}^{n}\{(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i\})}\\
&\qquad\times x^{\mathfrak{h}-\mathfrak{b}-\sum_{i=1}^{n}\{(\tau_i+k-s)\rho_i+(k+1)r_i\rho_i\}-1}.
\end{aligned}$$

Now using the definition of Pochhammer symbol (17), we have

$$\begin{aligned}
&\left(I_{-}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}\left(\frac{\omega_j}{t^{\rho_j}}\right)\right]\right)(x)\\
&= x^{\mathfrak{h}-\mathfrak{b}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j}{2x^{\rho_j}}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-times}}}\right)\\
&\quad\times\frac{\Gamma(1+\mathfrak{b}-\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(1-\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}\frac{\Gamma(1-\mathfrak{h}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(1-\mathfrak{h}+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}
\end{aligned}$$

$$\times \sum_{r_1,\ldots,r_n=0}^{\infty} \frac{(1)_{r_1}\cdots(1)_{r_n}}{(\tau_1+1)_{r_1}\cdots(\tau_n+1)_{r_n}(k-s+1)_{kr_1}\cdots(k-s+1)_{kr_n}}$$
$$\times \frac{(1+\mathfrak{b}-\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)_{(k+1)r_1\rho_1+\cdots(k+1)r_n\rho_n}}{(1-\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)_{(k+1)r_1\rho_1+\cdots(k+1)r_n\rho_n}}$$
$$\times \frac{(1-\mathfrak{h}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)_{(k+1)r_1\rho_1+\cdots(k+1)r_n\rho_n}}{(1-\mathfrak{h}+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)_{(k+1)r_1\rho_1+\cdots(k+1)r_n\rho_n}}$$
$$\times \frac{\left(\frac{\omega_1}{2x_1^{\rho}}\right)^{(k+1)r_1}\cdots\left(\frac{\omega_n}{2x_n^{\rho}}\right)^{(k+1)r_n}}{r_1!\cdots r_n!}.$$

Expressing the last summation in terms of the generalized Lauricella function or Srivastava and Daoust hypergeometric function (16), we immediately establish the required result (32), which completes the proof of Theorem 2.5. □

Substituting $\mathfrak{b}=-\mathfrak{a}$ in Theorem 2.5 yields the following result for the familiar Riemann-Liouville fractional integral (6).

Corollary 2.6. *Let* $n\in\mathbb{N}$*,* $\mathfrak{a},\mathfrak{h},\tau_j\in\mathbb{C}$*,* $s\in\{1,2,\ldots,k\}$*,* $\tau=\{0,\pm1,\pm2,\ldots\}$*,* $k=\{1,2,3,\ldots\}$ *and* $\omega_j,\rho_j\in\mathbb{R}_+$ $(j=1,2,\ldots,n)$ *such that* $\Re(\tau_j)>-1, 0<\Re(\mathfrak{a})<1-\Re(\mathfrak{h}-\sum_{j=1}^{n}(\tau_j+k-s)\rho_j)$*. Then there holds the formula*:

$$\left(I_{-}^{\mathfrak{a}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}\left(\frac{\omega_j}{t^{\rho_j}}\right)\right]\right)(x) \tag{33}$$
$$= x^{\mathfrak{h}+\mathfrak{a}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j}{2x^{\rho_j}}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-times}}}\right)$$
$$\times \frac{\Gamma(1-\mathfrak{a}-\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(1-\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}$$
$$\times F_{1:2;\cdots;2}^{1:1;\cdots;1}\left[\begin{matrix}[P':\Lambda]: & \overbrace{[1:1]}^{n\text{-times}} \\ [R':\Lambda]: & [\tau_1+1:1],\ldots,[\tau_n+1:1]:\underbrace{[k-s+1:k]}_{n\text{-times}}\end{matrix} \mid \left(\frac{\omega_1}{2x^{\rho_1}}\right)^{k+1},\ldots,\left(\frac{\omega_n}{2x^{\rho_n}}\right)^{k+1}\right],$$

where $\Lambda=(k+1)\rho_1,\ldots,(k+1)\rho_n$*,* $P'=1-\mathfrak{a}-\mathfrak{h}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, R'=1-\mathfrak{h}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j$ *and* $F_{1:2;\cdots;2}^{1:1;\cdots;1}(\cdot)$ *is given by* (16).

Substituting $\mathfrak{b}=0$ in Theorem 2.5 yields the following result for the so-called Erdélyi-Kober fractional integral (10).

Corollary 2.7. *Let* $n\in\mathbb{N}$*,* $\mathfrak{a},\mathfrak{c},\mathfrak{h},\tau_j\in\mathbb{C}$*,* $s\in\{1,2,\ldots,k\}$*,* $\tau=\{0,\pm1,\pm2,\ldots\}$*,* $k=\{1,2,3,\ldots\}$ *and* $\omega_j,\rho_j\in\mathbb{R}_+$ $(j=1,2,\ldots,n)$ *such that* $\Re(\mathfrak{a})>0,\Re(\tau_j)>$

$-1, \Re(\mathfrak{h} - \sum_{j=1}^{n}(\tau_j + k - s)\rho_j) < 1 + \Re(\mathfrak{c})$. *Then there holds the formula*:

$$\left(I_{-}^{\mathfrak{c},\mathfrak{a}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}_{\tau_j}^{k,s}\left(\frac{\omega_j}{t^{\rho_j}}\right)\right]\right)(x) \tag{34}$$

$$= x^{\mathfrak{h}-\mathfrak{b}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j}{2x^{\rho_j}}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-}times}}\right)$$

$$\times \frac{\Gamma(1+\mathfrak{c}-\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(1+\mathfrak{a}+\mathfrak{c}-\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}$$

$$\times\ F_{1:2;\cdots;2}^{1:1;\cdots;1}\left[\begin{matrix}[Q'':\Lambda]: & \overbrace{[1:1]}^{n\text{-}times} \\ [S'':\Lambda]: & [\tau_1+1:1],\ldots,[\tau_n+1:1]:\underbrace{[k-s+1:k]}_{n\text{-}times}\end{matrix}\ \Big|\ \left(\frac{\omega_1}{2x^{\rho_1}}\right)^{k+1},\ldots,\left(\frac{\omega_n}{2x^{\rho_n}}\right)^{k+1}\right],$$

$\Lambda = (k+1)\rho_1, \ldots, (k+1)\rho_n$ *where* $Q'' = 1 + \mathfrak{c} - \mathfrak{h} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j, S'' = 1 + \mathfrak{a} + \mathfrak{c} - \mathfrak{h} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j$ *and* $F_{1:2;\cdots;2}^{1:1;\cdots;1}(\cdot)$ *is given by* (16).

3. Fractional differentiation of $\tilde{I}_{\tau}^{k,s}(z)$

Our main results in this section are based on preliminary assertions giving composition formulas of generalized fractional differentiation (3) and (4) with the power function. Since the proof of the theorems in this section is based on similar techniques used in Section 2, so here, we give the results without proof. The left-hand sided and right-hand sided generalized differentiation (3) and (4) of a power function are given by the following results.

Lemma 3.1. *Let* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c} \in \mathbb{C}$. *Then*

(a) *If* $\Re(\mathfrak{a}) > 0$ *and* $\Re(\mathfrak{h}) > -\min\{0, \Re(\mathfrak{a}+\mathfrak{b}+\mathfrak{c})\}$, *then*

$$(D_{0+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}t^{\mathfrak{h}-1})(x) = \frac{\Gamma(\mathfrak{h})\Gamma(\mathfrak{h}+\mathfrak{a}+\mathfrak{b}+\mathfrak{c})}{\Gamma(\mathfrak{h}+\mathfrak{b})\Gamma(\mathfrak{h}+\mathfrak{c})}x^{\mathfrak{h}+\mathfrak{b}-1}. \tag{35}$$

In particular, for $x > 0$, *we have*

$$(D_{0+}^{\mathfrak{a}}t^{\mathfrak{h}-1})(x) = \frac{\Gamma(\mathfrak{h})}{\Gamma(\mathfrak{h}-\mathfrak{a})}x^{\mathfrak{h}-\mathfrak{a}-1}, \quad (\Re(\mathfrak{a}) > 0,\ \Re(\mathfrak{h}) > 0) \tag{36}$$

and

$$(D_{\mathfrak{c},\mathfrak{a}}^{+}t^{\mathfrak{h}-1})(x) = \frac{\Gamma(\mathfrak{h}+\mathfrak{a}+\mathfrak{c})}{\Gamma(\mathfrak{h}+\mathfrak{c})}x^{\mathfrak{h}-1}, \quad (\Re(\mathfrak{a}) > 0,\ \Re(\mathfrak{h}) > -\Re(\mathfrak{a}+\mathfrak{c})). \tag{37}$$

(b) *If* $\Re(\mathfrak{a}) > 0, \Re(\mathfrak{h}) < 1 + \min\{\Re(-\mathfrak{b}-n), \Re(\mathfrak{a}+\mathfrak{c})\}$ *and* $n = [\Re(\mathfrak{a})] + 1$, *then*

$$(D_{-}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}t^{\mathfrak{h}-1})(x) = \frac{\Gamma(1-\mathfrak{h}-\mathfrak{b})\Gamma(1-\mathfrak{h}+\mathfrak{a}+\mathfrak{c})}{\Gamma(1-\mathfrak{h})\Gamma(1-\mathfrak{h}+\mathfrak{c}-\mathfrak{b})}x^{\mathfrak{h}+\mathfrak{b}-1}. \tag{38}$$

In particular, for $x > 0$, we have

(39)
$$(D_{-}^{\mathfrak{a}} t^{\mathfrak{h}-1})(x) = \frac{\Gamma(1-\mathfrak{h}+\mathfrak{a})}{\Gamma(1-\mathfrak{h})} x^{\mathfrak{h}-\mathfrak{a}-1}, \quad (\Re(\mathfrak{a}) > 0,\, \Re(\mathfrak{h}) < 1 + \Re(\mathfrak{a}) - n)$$

and

(40)
$$(D_{\mathfrak{c},\mathfrak{a}}^{-} t^{\mathfrak{h}-1})(x) = \frac{\Gamma(1-\mathfrak{h}+\mathfrak{a}+\mathfrak{c})}{\Gamma(1-\mathfrak{h}-\mathfrak{c})} x^{\mathfrak{h}-1}, \ (\Re(\mathfrak{a}) > 0,\, \Re(\mathfrak{h}) < 1 + \Re(\mathfrak{a}+\mathfrak{c}) - n).$$

3.1. Left-sided fractional differentiation

First we consider the generalized left-hand sided fractional differentiation (3) of the generalized modified Bessel function of the second kind (15).

Theorem 3.2. *Let $n \in \mathbb{N}$, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{h}, \tau_j \in \mathbb{C}$, $s \in \{1, 2, \ldots, k\}$, $\tau = \{0, \pm 1, \pm 2, \ldots\}$, $k = \{1, 2, 3, \ldots\}$ and $\omega_j, \rho_j \in \mathbb{R}_+$ $(j = 1, 2, \ldots, n)$ such that $\Re(\mathfrak{a}) > 0, \Re(\tau_j) > -1, \Re(\mathfrak{h} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j) > -\min\{0, \Re(\mathfrak{a}+\mathfrak{b}+\mathfrak{c})\}$. Then there holds the formula*:

(41)
$$\left(D_{0+}^{\mathfrak{a},\mathfrak{b},\mathfrak{c}} \left[t^{\mathfrak{h}-1} \prod_{j=1}^{n} \tilde{I}_{\tau_j}^{k,s}(\omega_j t^{\rho_j}) \right] \right)(x)$$
$$= x^{\mathfrak{h}-\mathfrak{b}-1} \left(\prod_{j=1}^{n} \frac{\left(\frac{\omega_j x^{\rho_j}}{2}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-times}}} \right)$$
$$\times \frac{\Gamma(\mathfrak{h} + \sum_{i=1}^{n}(\tau_i + k - s)\rho_i)}{\Gamma(\mathfrak{h} + \mathfrak{b} + \sum_{i=1}^{n}(\tau_i + k - s)\rho_i)} \frac{\Gamma(\mathfrak{h} + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \sum_{i=1}^{n}(\tau_i + k - s)\rho_i)}{\Gamma(\mathfrak{h} + \mathfrak{c} + \sum_{i=1}^{n}(\tau_i + k - s)\rho_i)}$$
$$\times F_{2:2;\cdots;2}^{2:1;\cdots;1} \left[\begin{matrix} [U:\Lambda], [V:\Lambda]: & \overbrace{[1:1]}^{n\text{-times}} \\ [W:\Lambda], [Z:\Lambda]: & [\tau_1+1:1], \ldots, [\tau_n+1:1] : \underbrace{[k-s+1:k]}_{n\text{-times}} \end{matrix} \,\middle|\, \left(\frac{\omega_1 x^{\rho_1}}{2}\right)^{k+1}, \ldots, \left(\frac{\omega_n x^{\rho_n}}{2}\right)^{k+1} \right],$$

where $\Lambda = (k+1)\rho_1, \ldots, (k+1)\rho_n, U = \mathfrak{h} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j, V = \mathfrak{h} + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j, W = \mathfrak{h} + \mathfrak{b} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j, Z = \mathfrak{h} + \mathfrak{c} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j$ *and* $F_{2:2;\cdots;2}^{2:1;\cdots;1}(\cdot)$ *is given by* (16).

Corollary 3.3. *Let $n \in \mathbb{N}$, $\mathfrak{a}, \mathfrak{h}, \tau_j \in \mathbb{C}$, $s \in \{1, 2, \ldots, k\}$, $\tau = \{0, \pm 1, \pm 2, \ldots\}$, $k = \{1, 2, 3, \ldots\}$ and $\omega_j, \rho_j \in \mathbb{R}_+$ $(j = 1, 2, \ldots, n)$ such that $\Re(\mathfrak{a}) > 0, \Re(\tau_j) > -1, \Re(\mathfrak{h} + \sum_{j=1}^{n}(\tau_j + k - s)\rho_j) > -\Re(\mathfrak{c})$. Then there holds the formula*:

(42)
$$\left(D_{0+}^{\mathfrak{a}} \left[t^{\mathfrak{h}-1} \prod_{j=1}^{n} \tilde{I}_{\tau_j}^{k,s}(\omega_j t^{\rho_j}) \right] \right)(x)$$

$$= x^{\mathfrak{h}+\mathfrak{a}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j x^{\rho_j}}{2}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-}times}}\right)$$

$$\times \frac{\Gamma(\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(\mathfrak{h}-\mathfrak{a}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}$$

$$\times\ F^{1:1;\cdots;1}_{1:2;\cdots;2}\left[\begin{matrix}[U':\Lambda]: & \overbrace{[1:1]}^{n\text{-}times} \\ [W':\Lambda]: & [\tau_1+1:1],\ldots,[\tau_n+1:1]:\underbrace{[k-s+1:k]}_{n\text{-}times}\end{matrix}\ \middle|\ \left(\frac{\omega_1 x^{\rho_1}}{2}\right)^{k+1},\ldots,\left(\frac{\omega_n x^{\rho_n}}{2}\right)^{k+1}\right],$$

where $\Lambda = (k+1)\rho_1,\ldots,(k+1)\rho_n, U' = \mathfrak{h}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, W = \mathfrak{h}-\mathfrak{a}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j$ *and* $F^{1:1;\cdots;1}_{1:2;\cdots;2}(\cdot)$ *is given by* (16).

Corollary 3.4. *Let* $n\in\mathbb{N}$, $\mathfrak{a},\mathfrak{c},\mathfrak{h},\tau_j\in\mathbb{C}$, $s\in\{1,2,\ldots,k\}$, $\tau=\{0,\pm1,\pm2,\ldots\}$, $k=\{1,2,3,\ldots\}$ *and* $\omega_j,\rho_j\in\mathbb{R}_+$ $(j=1,2,\ldots,n)$ *such that* $\Re(\mathfrak{a})>0, \Re(\tau_j)>-1, \Re(\mathfrak{h}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j)>-\Re(\mathfrak{a}+\mathfrak{c})$. *Then there holds the formula*:

$$\left(D^{+}_{\mathfrak{c},\mathfrak{a}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}^{k,s}_{\tau_j}(\omega_j t^{\rho_j})\right]\right)(x) \tag{43}$$

$$= x^{\mathfrak{h}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j x^{\rho_j}}{2}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-}times}}\right)$$

$$\times \frac{\Gamma(\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(\mathfrak{h}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}$$

$$\times\ F^{1:1;\cdots;1}_{1:2;\cdots;2}\left[\begin{matrix}[V':\Lambda]: & \overbrace{[1:1]}^{n\text{-}times} \\ [Z':\Lambda]: & [\tau_1+1:1],\ldots,[\tau_n+1:1]:\underbrace{[k-s+1:k]}_{n\text{-}times}\end{matrix}\ \middle|\ \left(\frac{\omega_1 x^{\rho_1}}{2}\right)^{k+1},\ldots,\left(\frac{\omega_n x^{\rho_n}}{2}\right)^{k+1}\right],$$

where $\Lambda=(k+1)\rho_1,\ldots,(k+1)\rho_n,\ V'=\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, Z'=\mathfrak{h}+\mathfrak{c}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j$ *and* $F^{1:1;\cdots;1}_{1:2;\cdots;2}(\cdot)$ *is given by* (16).

3.2. Right-sided fractional differentiation

The following result yields the generalized right-hand sided fractional differentiation of the product of generalized modified Bessel functions.

Theorem 3.5. *Let* $n\in\mathbb{N}$, $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{h},\tau_j\in\mathbb{C}$, $s\in\{1,2,\ldots,k\}$, $\tau=\{0,\pm1,\pm2,\ldots\}$, $k=\{1,2,3,\ldots\}$ *and* $\omega_j,\rho_j\in\mathbb{R}_+$ $(j=1,2,\ldots,n)$ *such that* $\Re(\mathfrak{a})>0, \Re(\tau_j)>-1, \Re(\mathfrak{h}-\sum_{j=1}^{n}(\tau_j+k-s)\rho_j)<1+\min[\Re(\mathfrak{b}),\Re(\mathfrak{c})]$. *Then there holds the formula*:

$$\left(D^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}_{-}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}^{k,s}_{\tau_j}\left(\frac{\omega_j}{t^{\rho_j}}\right)\right]\right)(x) \tag{44}$$

$$= x^{\mathfrak{h}-\mathfrak{b}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j}{2x^{\rho_j}}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-times}}}\right)$$
$$\times\frac{\Gamma(1-\mathfrak{h}-\mathfrak{b}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(1-\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}\frac{\Gamma(1-\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(1-\mathfrak{h}+\mathfrak{c}-\mathfrak{b}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}$$
$$\times\ F^{2:1;\cdots;1}_{2:2;\cdots;2}\left[\begin{matrix}[P:\Lambda],[Q:\Lambda]: & \overbrace{[1:1]}^{n\text{-times}}\\ [R:\Lambda],[S:\Lambda]: & [\tau_1+1:1],\ldots,[\tau_n+1:1]:\underbrace{[k-s+1:k]}_{n\text{-times}}\end{matrix}\ \Big|\ \left(\frac{\omega_1}{2x^{\rho_1}}\right)^{k+1},\ldots,\left(\frac{\omega_n}{2x^{\rho_n}}\right)^{k+1}\right],$$

where $\Lambda=(k+1)\rho_1,\ldots,(k+1)\rho_n$, $P=1-\mathfrak{h}-\mathfrak{b}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, Q=1-\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, R=1-\mathfrak{h}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, S=1-\mathfrak{h}+\mathfrak{c}-\mathfrak{b}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j$ *and* $F^{2:1;\cdots;1}_{2:2;\cdots;2}(\cdot)$ *is given by* (16).

Theorem 3.6. *Let* $n\in\mathbb{N}$, $\mathfrak{a},\mathfrak{h},\tau_j\in\mathbb{C}$, $s\in\{1,2,\ldots,k\}$, $\tau=\{0,\pm1,\pm2,\ldots\}$, $k=\{1,2,3,\ldots\}$ *and* $\omega_j,\rho_j\in\mathbb{R}_+$ $(j=1,2,\ldots,n)$ *such that* $\Re(\mathfrak{a})>0,\Re(\tau_j)>-1,\Re(\mathfrak{h}-\sum_{j=1}^{n}(\tau_j+k-s)\rho_j)<1+\min[\Re(\mathfrak{b}),\Re(\mathfrak{c})]$. *Then there holds the formula*:

$$(45)\qquad \left(D^{\mathfrak{a}}_{-}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}^{k,s}_{\tau_j}\left(\frac{\omega_j}{t^{\rho_j}}\right)\right]\right)(x)$$
$$= x^{\mathfrak{h}+\mathfrak{a}-1}\left(\prod_{j=1}^{n}\frac{\left(\frac{\omega_j}{2x^{\rho_j}}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1)\underbrace{\Gamma(k-s+1)}_{n\text{-times}}}\right)$$
$$\times\frac{\Gamma(1-\mathfrak{h}+\mathfrak{a}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(1-\mathfrak{h}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}$$
$$\times\ F^{2:1;\cdots;1}_{2:2;\cdots;2}\left[\begin{matrix}[P':\Lambda]: & \overbrace{[1:1]}^{n\text{-times}}\\ [R':\Lambda]: & [\tau_1+1:1],\ldots,[\tau_n+1:1]:\underbrace{[k-s+1:k]}_{n\text{-times}}\end{matrix}\ \Big|\ \left(\frac{\omega_1}{2x^{\rho_1}}\right)^{k+1},\ldots,\left(\frac{\omega_n}{2x^{\rho_n}}\right)^{k+1}\right],$$

where $\Lambda=(k+1)\rho_1,\ldots,(k+1)\rho_n$, $P'=1-\mathfrak{h}+\mathfrak{a}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, R'=1-\mathfrak{h}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j$, *and* $F^{2:1;\cdots;1}_{2:2;\cdots;2}(\cdot)$ *is given by* (16).

Theorem 3.7. *Let* $n\in\mathbb{N}$, $\mathfrak{a},\mathfrak{c},\mathfrak{h},\tau_j\in\mathbb{C}$, $s\in\{1,2,\ldots,k\}$, $\tau=\{0,\pm1,\pm2,\ldots\}$, $k=\{1,2,3,\ldots\}$ *and* $\omega_j,\rho_j\in\mathbb{R}_+$ $(j=1,2,\ldots,n)$ *such that* $\Re(\mathfrak{a})>0,\Re(\tau_j)>-1,\Re(\mathfrak{h}-\sum_{j=1}^{n}(\tau_j+k-s)\rho_j)<1+\min[\Re(\mathfrak{b}),\Re(\mathfrak{c})]$. *Then there holds the formula*:

$$(46)\qquad \left(D^{-}_{\mathfrak{c},\mathfrak{a}}\left[t^{\mathfrak{h}-1}\prod_{j=1}^{n}\tilde{I}^{k,s}_{\tau_j}\left(\frac{\omega_j}{t^{\rho_j}}\right)\right]\right)(x)$$

$$= x^{\mathfrak{h}-1} \left(\prod_{j=1}^{n} \frac{\left(\frac{\omega_j}{2x^{\rho_j}}\right)^{\tau_j+k-s}}{\Gamma(\tau_j+1) \underbrace{\Gamma(k-s+1)}_{n\text{-}times}} \right)$$

$$\times \frac{\Gamma(1-\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}{\Gamma(1-\mathfrak{h}+\mathfrak{c}+\sum_{i=1}^{n}(\tau_i+k-s)\rho_i)}$$

$$\times \ F^{1:1;\cdots;1}_{1:2;\cdots;2} \left[\begin{array}{l} [Q':\Lambda]: \qquad \overbrace{[1:1]}^{n\text{-}times} \\ [S':\Lambda]: \ [\tau_1+1:1],\ldots,[\tau_n+1:1]: \underbrace{[k-s+1:k]}_{n\text{-}times} \end{array} \Big| \left(\frac{\omega_1}{2x^{\rho_1}}\right)^{k+1}, \ldots, \left(\frac{\omega_n}{2x^{\rho_n}}\right)^{k+1} \right],$$

where $\Lambda = (k+1)\rho_1, \ldots, (k+1)\rho_n$, $Q' = 1-\mathfrak{h}+\mathfrak{a}+\mathfrak{c}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j, S = 1-\mathfrak{h}+\mathfrak{c}+\sum_{j=1}^{n}(\tau_j+k-s)\rho_j$ *and* $F^{1:1;\cdots;1}_{1:2;\cdots;2}(\cdot)$ *is given by* (16).

Concluding remarks and observations

Our present study is based mainly on the generalized modified Bessel function of the second type introduced by Griffiths *et al.* [2]

$$\tilde{I}^{k,s}_{\tau}(z) = \left(\frac{z}{2}\right)^{\tau+k-s} \sum_{r=0}^{\infty} \frac{1}{(k(r+1)-s)! \quad \Gamma(\tau+r+1)} \left(\frac{z}{2}\right)^{r(k+1)},$$

where $s \in \{1, 2, \ldots, k\}$, $\tau = \{0, \pm 1, \pm 2, \ldots\}$, $z \in \mathbb{C}$ and $k = \{1, 2, 3, \ldots\}$. The motivation for introducing $\tilde{I}^{k,s}_{\tau}(z)$ by Griffiths *et al.* [2] was George Luchak's papers [10, 11] in finding the continuous time solution of the single-channel queueing equations characterized by a time-dependent Poisson-distributed arrival rate and introduced modified Bessel function of the first type

$$\tilde{I}^{k}_{\tau}(z) = \left(\frac{z}{2}\right)^{\tau} \sum_{r=0}^{\infty} \frac{1}{r! \quad \Gamma(\tau+rk+1)} \left(\frac{z}{2}\right)^{r(k+1)}.$$

The special case for $k = s = 1$ and $k = 1$ in $\tilde{I}^{k,s}_{\tau}(z)$ and $\tilde{I}^{k}_{\tau}(z)$, respectively reduces to modified bessel function $I_{\tau}(z)$ (see, for details, [2, 10, 11]):

$$I_{\tau}(z) = \left(\frac{z}{2}\right)^{\tau} \sum_{r=0}^{\infty} \frac{1}{r! \quad \Gamma(\tau+r+1)} \left(\frac{z}{2}\right)^{2r}.$$

The generalized modified Bessel function of the second type can be expressed in terms of Fox-Wright function ${}_p\Psi_q(z)$ [18] in slightly corrected version given in Griffiths *et al.* [2, p. 273]:

$$\tilde{I}^{k,s}_{\tau}(z) = \left(\frac{z}{2}\right)^{\tau+k-s} {}_1\Psi_2 \left[\begin{array}{c} (1,1); \\ (\tau+1, r), (k-s+1, r); \end{array} \frac{z^{k+1}}{2^{k+1}} \right]. \tag{47}$$

In our present investigation, we have established formulas (images) for compositions of generalized fractional integrals and differential containing the Gauss' hypergeometric ${}_2F_1(x)$ function in the kernels with the n-times product of generalized modified Bessel function of the second type given by (15) in terms of generalized Lauricella function or Srivastava-Daoust hypergeometric function. Special cases for the findings are obtained for the classical Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential operators. Various other related papers on operators can be found as other aspects in diverse areas of mathematical, physical, engineering and statistical sciences by some authors (see, [1, 3, 5–7, 13, 14, 17] and the references therein).

Furthermore, another compositions of generalized fractional integrals and differential for the product of classical Bessel function $J_\nu(z)$ of the first kind

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)}\, {}_0F_1\left(\text{---}\,; \nu+1; -\frac{1}{4}z^2\right).$$

has been developed by Kilbas and Sebastian [7] and Sebastian and Gorenflo [17].

Moreover, the results established in this paper are presumably new and different those of obtained for classical Bessel function $J_\nu(z)$ in [7, 17].

Acknowledgements. The authors express their deepest thanks to the worthy Editor(s) and referee for his/her valuable comments and suggestions that helped to improve this paper in its present form. This work is supported by the Competitive Research Scheme (CRS) project funded by the TEQIP-III (ATU) Rajasthan Technical University Kota under grant number TEQIP-III/RTU(ATU)CRS/2019-20/50.

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