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A RESULT ON AN OPEN PROBLEM OF LÜ, LI AND YANG

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Abstract. In this paper we deal with the open problem posed by Lü, Li and Yang [10]. In fact, we prove the following result: Let $f(z)$ be a transcendental meromorphic function of finite order having finitely many poles, $c_1, c_2, \ldots, c_n \in \mathbb{C}\setminus\{0\}$ and $k, n \in \mathbb{N}$. Suppose $f^n(z)$, $f(z+c_1)f(z+c_2)\cdots f(z+c_n)$ share 0 CM and $f^n(z)-Q_1(z)$, $(f(z+c_1)f(z+c_2)\cdots f(z+c_n))^{(k)}-Q_2(z)$ share $(0,1)$, where $Q_1(z)$ and $Q_2(z)$ are non-zero polynomials. If $n \geq k+1$, then $(f(z+c_1)f(z+c_2)\cdots f(z+c_n))^{(k)} \equiv \frac{Q_2(z)}{Q_1(z)} f^n(z)$. Furthermore, if $Q_1(z) \equiv Q_2(z)$, then $f(z) = c\, e^{\frac{\lambda}{n} z}$, where $c, \lambda \in \mathbb{C}\setminus\{0\}$ such that $e^{\lambda(c_1+c_2+\cdots+c_n)} = 1$ and $\lambda^k = 1$. Also we exhibit some examples to show that the conditions of our result are the best possible.

1. Introduction definitions and results

By a meromorphic (resp. entire) function we shall always mean meromorphic (resp. entire) function in the whole complex plane $\mathbb{C}$. In this paper, it is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna value distribution theory of meromorphic functions. For a meromorphic function $f(z)$ in the complex plane $\mathbb{C}$, we shall use the following standard notations of the value distribution theory: $T(r,f)$, $m(r,\infty;f)$, $N(r,\infty;f)$, $\overline{N}(r,\infty;f)$,... (see, e.g., [7, 14]). We adopt the standard notation $S(r,f)$ for any quantity satisfying the relation $S(r,f) = o(T(r,f))$ as $r \to \infty$ except possibly a set of finite linear measure.

A meromorphic function $a = a(z)$ is called a small function of a meromorphic function $f(z)$ if $T(r,a) = S(r,f)$. Let us denote by $S(f)$ the class of all small functions of f. Clearly $\mathbb{C} \subset S(f)$ and if f is a transcendental function, then every polynomial is a member of $S(f)$.

Let $k \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$. We use the notation $N_{k)}(r,a;f)$ to denote the counting function of a-points of f with multiplicity not greater than k and $N_{(k+1}(r,a;f)$ to represent the counting function of a-points of f with multiplicity greater than k respectively. Similarly $\overline{N}_{k)}(r,a;f)$ and $\overline{N}_{(k+1}(r,a;f)$ are their reduced functions respectively.

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Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z) \in S(f) \cap S(g)$. If $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share $a(z)$ with CM (counting multiplicities) and if we do not consider the multiplicities, then we say that $f(z)$ and $g(z)$ share $a(z)$ with IM (ignoring multiplicities).

We now explain the notation of weighted sharing as introduced in [8].

Definition 1.1 ([8])**.** Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share a with weight k.

We write f and g share (a, k) to mean that f and g share a with weight k. Also we note that f and g share a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively.

We recall that the order $\sigma(f)$ of meromorphic function $f(z)$ is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

Furthermore, when $f(z)$ is an entire function, we have

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$

where $M(r, f) = \max\limits_{|z|=r} |f(z)|$. Let f be an entire function. We know that f can be expressed by the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We denote by

$$\mu(r, f) = \max_{\substack{n\in\mathbb{N}\\ |z|=r}} \{|a_n z^n|\} \text{ and } \nu(r, f) = \sup\{n : |a_n| r^n = \mu(r, f)\}.$$

Clearly for a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, $a_n \neq 0$, we have

$$\mu(r, P) = |a_n| r^n \ \text{ and } \ \nu(r, P) = n$$

for all r sufficiently large.

In 1996, Brück [1] discussed the possible relation between f and f' when an entire function f and its derivative f' share only one finite value CM. In this direction an interesting problem still open is the following conjecture proposed by Brück [1].

Conjecture A. *Let f be a non-constant entire function. Suppose*

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then

$$\frac{f' - a}{f - a} = c \quad \text{for some } c \in \mathbb{C} \setminus \{0\}. \tag{1.1}$$

The conjecture for the special cases (1) $a = 0$ and (2) $N(r, 0; f') = S(r, f)$ had been proved by Brück [1]. From the differential equations

$$\frac{f'(z) - a}{f(z) - a} = e^{z^n} \ (\rho_1(f) = n) \text{ and } \frac{f'(z) - a}{f(z) - a} = e^{e^z} \ (\rho_1(f) = +\infty), \tag{1.2}$$

we see that the above conjecture does not hold when $\rho_1(f) \in \mathbb{N} \cup \{\infty\}$.

The conjecture, for the case that f is of finite order, had been proved by Gundersen and Yang [6], the case that f is of infinite order with $\rho_1(f) < \frac{1}{2}$ had been proved by Chen and Shon [3]. Recently Cao [2] proved that the Brück conjecture is also true when f is of infinite order with $\rho_1(f) = \frac{1}{2}$. But the case $\rho_1(f) > \frac{1}{2}$ is still open.

For meromorphic functions the Brück conjecture fails in general. For example the meromorphic function $f(z) = \frac{2e^z+z+1}{e^z+1}$ shares the value 1 CM with $f'(z)$ while $\frac{f'-1}{f-1}$ is not a constant.

Since then, shared value problems, especially the case of f and $f^{(k)}$, where $k \in \mathbb{N}$ sharing one value or small function have undergone various extensions and improvements (see [14]).

Now it is interesting to ask what happens if f is replaced by f^n in the Brück conjecture. From the equation (1.2), we see that the conjecture does not hold when $n = 1$. Thus we only need to discuss the problem when $n \geq 2$. Yang and Zhang [15] proved that the Brück conjecture holds for the function f^n and the order restriction on f is not needed if n is relatively large. Actually they proved the following result.

Theorem A ([15]). *Let f be a non-constant entire function and $n(\geq 7)$ be an integer. If f^n and $(f^n)'$ share 1 CM, then $f^n \equiv (f^n)'$ and f assumes the form $f(z) = ce^{\frac{1}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$.*

In 2009, Zhang [16] improved and generalized Theorem A by replacing the first derivative $(f^n)'$ by the general derivative $(f^n)^{(k)}$, where $n, k \in \mathbb{N}$ and obtained the following result.

Theorem B ([16]). *Let f be a non-constant entire function, k, $n \in \mathbb{N}$ and $a(\not\equiv 0, \infty) \in S(f)$. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share 0 CM and $n > k + 4$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where $c, \lambda \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.*

In the same year, Zhang and Yang [17] further improved Theorem B by reducing the lower bound on n. Actually they proved the following result.

Theorem C ([17]). *Let f be a non-constant entire function, k, $n \in \mathbb{N}$ and $a(\not\equiv 0, \infty) \in S(f)$. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share 0 CM and $n > k + 1$. Then conclusion of Theorem B holds.*

After one year, Zhang and Yang [18] again improved their above result by reducing the lower bound of n in the following manner.

Theorem D ([18])**.** *Let f be a non-constant entire function, $k,\ n \in \mathbb{N}$. Suppose f^n and $(f^n)^{(k)}$ share 1 CM and $n \geq k+1$. Then the conclusion of Theorem B holds.*

In 2011, using the theory of normal family Lü and Yi [11] further generalized Theorem D with the idea of sharing polynomial and obtained the following result.

Theorem E ([11])**.** *Let f be a transcendental entire function, $k,\ n \in \mathbb{N}$ with $n \geq k+1$ and $Q \not\equiv 0$ be a polynomial. If $f^n - Q$ and $(f^n)^{(k)} - Q$ share 0 CM, then $f^n \equiv (f^n)^{(k)}$ and $f(z) = ce^{wz/n}$, where $c, w \in \mathbb{C} \setminus \{0\}$ such that $w^k = 1$.*

Now observing the above theorem, Lü, Li and Yang [10] asked the following question:

Question 1. What can be said "if $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share the value 0 CM"? where Q_1 and Q_2 are non-zero polynomials.

Lü, Li and Yang [10] solved the above question for $k = 1$ by giving the transcendental entire solutions of the equation

$$F' - Q_1 = Re^{\alpha}(F - Q_2), \tag{1.3}$$

where $F = f^n$, R is a rational function and α is an entire function and they obtained the following results.

Theorem F ([10])**.** *Let f be a transcendental entire function and let $F = f^n$ be a solution of equation (1.3), $n \in \mathbb{N} \setminus \{1\}$. Then $\frac{Q_1}{Q_2}$ reduces to a polynomial and $f' \equiv \frac{Q_1}{nQ_2} f$.*

Theorem G ([10])**.** *Let f be a transcendental entire function and $n \in \mathbb{N} \setminus \{1\}$. If $f^n - Q$ and $(f^n)' - Q$ share 0 CM, where $Q(\not\equiv 0)$ is a polynomial, then $f(z) = ce^{z/n}$, where $c \in \mathbb{C} \setminus \{0\}$.*

Also Lü, Li and Yang [10] proved that if $\frac{Q_1}{Q_2}$ is not a polynomial, then the differential equation (1.3) has no transcendental entire solution when $n \in \mathbb{N} \setminus \{1\}$. In [10], Lü, Li and Yang exhibited some relevant examples to show that the differential equation (1.3) has no polynomial solution and the condition $n \in \mathbb{N} \setminus \{1\}$ is sharp.

Also in the same paper Lü, Li and Yang [10] posed the following conjecture.

Conjecture 1.1. *Let f be a transcendental entire function and $n \in \mathbb{N}$. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM, where Q_1 and Q_2 are non-zero polynomials and $n \geq k+1$, then $(f^n)^{(k)} \equiv \frac{Q_2}{Q_1} f^n$. Furthermore, if $Q_1 \equiv Q_2$, then $f(z) = ce^{wz/n}$, where $c, w \in \mathbb{C} \setminus \{0\}$ and $w^k = 1$.*

Again at the end of the paper, Lü, Li and Yang [10] asked the following question.

Question 2. What can be said if the condition in Conjecture 1.1 "$(f^n)^{(k)}$" be replaced by "$\{f(z+c_1)f(z+c_2)\cdots f(z+c_n)\}^{(k)}$", where $c_j (j = 1, 2, \dots, n)$ are constants.

In the meantime, Majumder [12] fully resolved Conjecture 1.1. To the knowledge of authors Question 2 is still open. Naturally the main objective of this paper will be to give an affirmative answer of the above Question 2. To do this at first we have to check whether Question 2 is solvable or not. For the validity of Question 2, we now exhibit the following example.

Example 1.1. Let $f(z) = e^z + 1$, $Q_1(z) = 3$, $Q_2(z) = 1$ and $e^{c_1} = e^{c_2} = \frac{1}{2}$. Let $k = 1$ and $n = 2$. Note that $f^2(z) - Q_1(z) = e^{2z} + 2e^z - 2$ and $(f(z+c_1)f(z+c_2))' - Q_2(z) = \frac{1}{2}(e^{2z} + 2e^z - 2)$. Clearly $f^2(z) - Q_1(z)$ and $(f(z+c_1)f(z+c_2))' - Q_2(z)$ share 0 CM, but $(f(z+c_1)f(z+c_2))' \not\equiv \frac{Q_2(z)}{Q_1(z)} f^2(z)$.

From Example 1.1, one can easily come into conclusion that Question 2 is not solvable as its stand. As a result one may easily conclude that Question 2 is not solvable without imposing any other condition(s).

In this paper we have been able to solve Question 2 at the cost of considering the fact that $f(z)$ is a transcendental meromorphic function of finite order having finitely many poles such that $f^n(z)$ and $f(z+c_1)f(z+c_2)\cdots f(z+c_n)$ share 0 CM, where $n \in \mathbb{N}$. The following theorem is the main result of this paper.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of finite order having finitely many poles, $c_1, c_2, \dots, c_n \in \mathbb{C} \setminus \{0\}$ and $k, n \in \mathbb{N}$. Suppose $f^n(z)$, $f(z+c_1)f(z+c_2)\cdots f(z+c_n)$ share 0 CM and $f^n(z) - Q_1(z)$, $(f(z+c_1)f(z+c_2)\cdots f(z+c_n))^{(k)} - Q_2(z)$ share $(0, 1)$, where $Q_1(z)$ and $Q_2(z)$ are non-zero polynomials. If $n \geq k+1$, then $(f(z+c_1)f(z+c_2)\cdots f(z+c_n))^{(k)} \equiv \frac{Q_2(z)}{Q_1(z)} f^n(z)$. Furthermore, if $Q_1(z) \equiv Q_2(z)$, then $f(z) = c\, e^{\frac{\lambda}{n} z}$, where $c, \lambda \in \mathbb{C} \setminus \{0\}$ such that $e^{\lambda(c_1+c_2+\cdots+c_n)} = 1$ and $\lambda^k = 1$.*

Remark 1.1. It is easy to see that the condition "$n \geq k+1$" in Theorem 1.1 is sharp by the following examples.

Example 1.2. Let $f(z) = e^{2z} + 1$, $Q_1(z) = 3$, $Q_2(z) = 4$ and $c = \pi\mathrm{i}$. Let $n = k = 1$. Clearly $f(z)$ and $f(z+c)$ share 0 CM. Also $f(z) - Q_1(z)$ and $f'(z+c) - Q_2(z)$ share 0 CM, but $f'(z+c) \not\equiv \frac{Q_2(z)}{Q_1(z)} f(z)$.

Example 1.3. Let $f(z) = e^z - e^{-z}$, $Q_1(z) = 2$, $Q_2(z) = 16$ and $c_1 = c_2 = \pi\mathrm{i}$. Let $n = k = 2$. Clearly $f^2(z)$ and $f(z+c_1)f(z+c_2)$ share 0 CM. Also $f^2(z) - Q_1(z)$ and $(f(z+c_1)f(z+c_2))'' - Q_2(z)$ share 0 CM, but $(f(z+c_1)f(z+c_2))'' \not\equiv \frac{Q_2(z)}{Q_1(z)} f^2(z)$.

Remark 1.2. It is easy to see that the condition "$f^n(z)$ and $f(z+c_1)f(z+c_2)\cdots f(z+c_n)$ share 0 CM" in Theorem 1.1 is sharp by the following examples.

Example 1.4. Let $f(z)=e^{cz}+1$, $Q_1(z)=-2$, $Q_2(z)=c$ and $e^{cc_1}, e^{cc_2}, e^{cc_3}$ are the roots of the equation $6z^3-18z^2+9z-2=0$, where $c\neq 0$. Let $k=1$ and $n=3$. Clearly $f^3(z)$ and $f(z+c_1)f(z+c_2)f(z+c_3)$ have no common zeros. Also $f^3(z)-Q_1(z)$ and $(f(z+c_1)f(z+c_2)f(z+c_3))'-Q_2(z)$ share 0 CM, but $(f(z+c_1)f(z+c_2)f(z+c_3))'\not\equiv \frac{Q_2(z)}{Q_1(z)}f^3(z)$.

Example 1.5. Let $f(z)=e^{z}-e^{-z}$, $Q_1(z)=2$, $Q_2(z)=-8\mathrm{i}$, $e^{c_1}=-1$ and $e^{c_2}=\mathrm{i}$. Let $k=1$ and $n=2$. Clearly $f^2(z)$ and $f(z+c_1)f(z+c_2)$ have no common zeros. Also $f^2(z)-Q_1(z)$ and $(f(z+c_1)f(z+c_2))'-Q_2(z)$ share 0 CM, but $(f(z+c_1)f(z+c_2))'\not\equiv \frac{Q_2(z)}{Q_1(z)}f^2(z)$.

2. Lemmas

In this section we present the following lemmas which will be needed in the sequel.

Lemma 2.1 ([4])**.** *Let f be a meromorphic function of finite order σ and let $c\in\mathbb{C}\setminus\{0\}$ be fixed. Then for each $\varepsilon>0$, we have*

$$m\left(r,\infty;\frac{f(z+c)}{f(z)}\right)+m\left(r,\infty;\frac{f(z)}{f(z+c)}\right)=O\left(r^{\sigma-1+\varepsilon}\right)=S(r,f).$$

Lemma 2.2 ([4])**.** *Let f be a meromorphic function of finite order σ and let $c\in\mathbb{C}\setminus\{0\}$ be fixed. Then for each $\varepsilon>0$, we have*

$$T(r,f(z+c))=T(r,f(z))+O(r^{\sigma-1+\varepsilon})+O(\log r).$$

Lemma 2.3 ([5])**.** *Suppose that f is a transcendental meromorphic function and that $f^nP(f)=Q(f)$, where $P(f)$ and $Q(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients and the degree of $Q(f)$ is at most n. Then $m(r,P(f))=S(r,f)$.*

Lemma 2.4 ([13])**.** *Let f be a non-constant meromorphic function and let $a_n(\not\equiv 0), a_{n-1},\ldots,a_0\in S(f)$. Then $T(r,a_nf^n+a_{n-1}f^{n-1}+\cdots+a_1f+a_0)=nT(r,f)+S(r,f)$.*

Lemma 2.5 ([7])**.** *Let f be a non-constant meromorphic function and let $a_1,a_2\in S(f)$. Then*

$$T(r,f)\leq \overline{N}(r,\infty;f)+\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f)+S(r,f).$$

Lemma 2.6 ([7], Lemma 3.5)**.** *Suppose that F is meromorphic in a domain D and set $f=\frac{F'}{F}$. Then for $n\in\mathbb{N}$, we have*

$$\frac{F^{(n)}}{F}=f^n+\frac{n(n-1)}{2}f^{n-2}f'+a_nf^{n-3}f''+b_nf^{n-4}(f')^2+P_{n-3}(f),$$

where $a_n=\frac{1}{6}n(n-1)(n-2)$, $b_n=\frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n\leq 3$ and has degree $n-3$ when $n>3$.

Lemma 2.7 ([9], Corollary 2.3.4). *Let f be a transcendental meromorphic function and $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f)$$

and if f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 2.8 ([9], Lemma 1.3.1). *Let $P(z) = \sum_{i=0}^{n} a_i z^i$, where $a_n \neq 0$. Then $\forall$ $\varepsilon > 0$, there exists $r_0 > 0$ such that $\forall$ $r = |z| > r_0$ the inequalities $(1-\varepsilon)|a_n|r^n \leq |P(z)| \leq (1+\varepsilon)|a_n|r^n$ hold.*

Lemma 2.9 ([9], Theorem 3.2). *Let f be a transcendental entire function, $\nu(r,f)$ be the central index of f. Then there exists a set $E \subset (1,+\infty)$ with finite logarithmic measure, we choose z satisfying $|z| = r \notin [0,1] \cup E$ and $|f(z)| = M(r,f)$ such that*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^j (1+o(1)) \ \ \textit{for } j \in \mathbb{N}.$$

Lemma 2.10 ([9], Theorem 3.1). *If f is an entire function of order $\sigma(f)$, then*

$$\sigma(f) = \limsup_{r \longrightarrow \infty} \frac{\log \nu(r,f)}{\log r}.$$

3. Proof of the theorem

Proof of Theorem 1.1. Let

$$F(z) = f^n(z) \quad \text{and} \quad G(z) = (f(z+c_1)f(z+c_2)\cdots f(z+c_n))^{(k)}. \tag{3.1}$$

Using Lemma 2.4, we can conclude that $S(r,F) = S(r,f)$. Also from Lemma 2.5, we see that

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}(r,Q_1;F) + S(r,f) \\ &= \overline{N}(r,0;f) + \overline{N}(r,Q_1;F) + S(r,f). \end{aligned}$$

Since $n \geq k+1$, it follows that $\overline{N}(r,Q_1;F) \neq S(r,f)$. As $F - Q_1$ and $G - Q_2$ share $(0,1)$, it follows that $\overline{N}(r,Q_2;G) \neq S(r,f)$. Let

$$F_1(z) = \frac{f^n(z)}{Q_1(z)} \ \text{ and } \ G_1(z) = \frac{(f(z+c_1)f(z+c_2)\cdots f(z+c_n))^{(k)}}{Q_2(z)}.$$

Clearly F_1 and G_1 share $(1,1)$ except for the zeros of $Q_i(z)$, where $i = 1,2$. Again since $f^n(z)$ and $f(z+c_1)f(z+c_2)\cdots f(z+c_n)$ share 0 CM and $f(z)$ has finitely many poles, it follows that

$$f(z+c_1)f(z+c_2)\cdots f(z+c_n) = f^n(z)\alpha(z)e^{\eta(z)}, \tag{3.2}$$

where $\alpha(z)$ is a rational function and $\eta(z)$ is a polynomial. Let $\psi(z) = \alpha(z)\, e^{\eta(z)}$. Clearly ψ is of finite order. Now by Lemma 2.1, we have

$$\begin{aligned} m(r,\infty;\psi) &= m\left(r,\infty;\frac{f^n\alpha e^{\eta}}{f^n}\right) \\ &= m\left(r,\infty;\frac{f(z+c_1)f(z+c_2)\cdots f(z+c_n)}{f^n(z)}\right) \\ &\leq \sum_{i=1}^{n} m\left(r,\infty;\frac{f(z+c_i)}{f(z)}\right) = S(r,f). \end{aligned}$$

Therefore $T(r,\psi) = N(r,\infty;\psi) + m(r,\infty;\psi) = S(r,f)$ and so $T(r,\psi^{(j)}) = S(r,f)$ for $j = 1,2,\ldots$. Again by Lemma 2.2, we have

$$T\left(r,\prod_{i=1}^{n} f(z+c_i)\right) \leq \sum_{i=1}^{n} T\left(r,f(z+c_i)\right) = nT(r,f) + S(r,f).$$

Also by Lemmas 2.1 and 2.4, we have

$$\begin{aligned} nT(r,f) = T(r,f^n) + S(r,f) &= m(r,\infty;f^n) + S(r,f) \\ &\leq m\left(r,\infty;\prod_{i=1}^{n}\frac{f(z)}{f(z+c_i)}\right) \\ &\quad + m\left(r,\infty;\prod_{i=1}^{n} f(z+c_i)\right) + S(r,f) \\ &\leq T\left(r,\prod_{i=1}^{n} f(z+c_i)\right) + S(r,f). \end{aligned}$$

Therefore $T\left(r,\prod_{i=1}^{n} f(z+c_i)\right) = nT(r,f) + S(r,f)$ and so $S\left(r,\prod_{i=1}^{n} f(z+c_i)\right) = S(r,f)$.

On the other hand using Lemmas 2.1 and 2.7, we have $m\left(r,\infty;\frac{G}{F}\right) = S(r,f)$. Set

$$\Phi = \frac{F_1'(F_1-G_1)}{F_1(F_1-1)} = \frac{F_1'}{F_1-1}\left(1-\frac{Q_1}{Q_2}\frac{G}{F}\right). \tag{3.3}$$

We now consider the following two cases.

Case 1. Suppose $\Phi \not\equiv 0$. Then by Lemma 2.7, it is clear that $m(r,\infty;\Phi) = S(r,f)$. Let z_1 be a zero of f with multiplicity p such that $Q_i(z_1) \neq 0$, where $i = 1,2$. Then z_1 will be a zero of F_1 with multiplicity np. Since $f^n(z)$ and $f(z+c_1)f(z+c_2)\cdots f(z+c_n)$ share 0 CM, it follows that z_1 is a zero of $f(z+c_1)f(z+c_2)\cdots f(z+c_n)$ with multiplicity np and so z_1 is a zero of G_1 with multiplicity $np-k$. Now from (3.3), we get

$$\Phi(z) = O\left((z-z_1)^{np-k-1}\right). \tag{3.4}$$

Since $n \geq k+1$, it follows that Φ is holomorphic at z_1.

Let z_2 be a common zero of $F_1 - 1$ and $G_1 - 1$ such that $Q_i(z_2) \neq 0$, where $i = 1, 2$. Suppose z_2 is a zero of $F_1 - 1$ of multiplicity q. Since F_1 and G_1 share $(1, 1)$ except for the zeros of $Q_1(z)$ and $Q_2(z)$ respectively, it follows that z_2 must be a zero of $G_1 - 1$ of multiplicity r. Then in some neighbourhood of z_2, we get by Taylor's expansion

$$F_1(z) - 1 = b_q(z - z_2)^q + b_{q+1}(z - z_2)^{q+1} + \cdots, b_q \neq 0 \text{ and}$$
$$G_1(z) - 1 = c_r(z - z_2)^r + c_{r+1}(z - z_2)^{r+1} + \cdots, c_r \neq 0.$$

Clearly

$$F_1'(z) = qb_q(z - z_2)^{q-1} + (q+1)b_{q+1}(z - z_2)^q + \cdots.$$

Note that

$$F_1(z) - G_1(z) = \begin{cases} b_q(z - z_2)^q + \cdots, & \text{if } q < r, \\ -c_r(z - z_2)^r - \cdots, & \text{if } q > r, \\ (b_q - c_q)(z - z_2)^q + \cdots, & \text{if } q = r. \end{cases}$$

Therefore from (3.3), we get

$$\Phi(z) = O\left((z - z_2)^{t-1}\right), \tag{3.5}$$

where $t \geq \min\{q, r\}$. Now from (3.5), it follows that Φ is holomorphic at z_2. Therefore we conclude that the poles of Φ may come from the poles of f or the zeros of Q_i for $i = 1, 2$. Since f has finitely many poles, it follows that Φ may have finitely many poles. Consequently $N(r, \infty; \Phi) = O(\log r)$ and so $T(r, \Phi) = S(r, f)$. On the other hand from (3.5), we see that $\overline{N}_{(2}(r, 1; F_1) \leq N(r, 0; \Phi) \leq T(r, \Phi) + O(1) = S(r, f)$, i.e., $\overline{N}_{(2}(r, 1; F_1) = S(r, f)$. Since F_1 and G_1 share $(1, 1)$ except for the zeros of $Q_1(z)$ and $Q_2(z)$ respectively, it follows that $\overline{N}_{(2}(r, 1; G_1) = S(r, f)$. Consequently we have $\overline{N}_{(2}(r, Q_1; F) = S(r, f)$ and $\overline{N}_{(2}(r, Q_2; G) = S(r, f)$. Again from (3.3), we get

$$\frac{1}{F_1} = \frac{1}{\Phi} \frac{F_1'}{F_1(F_1 - 1)} \left(1 - \frac{Q_1}{Q_2} \frac{G}{F}\right).$$

Therefore by Lemma 2.7, we have $m(r, \infty; \frac{1}{F_1}) = S(r, f)$ and so

$$m\left(r, \infty; \frac{1}{f}\right) = S(r, f). \tag{3.6}$$

Now we consider following two sub-cases.

Sub-case 1.1. Suppose $n > k+1$. Then from (3.4), we see that

$$N(r, 0; f) \leq N(r, 0; \Phi) \leq T\left(r, \frac{1}{\Phi}\right) \leq T(r, \Phi) + O(1) = S(r, f). \tag{3.7}$$

Now from (3.6) and (3.7), we conclude that $T(r, f) = S(r, f)$, which is a contradiction.

Sub-case 1.2. Suppose $n = k+1$. Then from (3.4), we have $N_{(2}(r,0;f) \leq N(r,0;\Phi) \leq T(r,\Phi) + O(1) = S(r,f)$ and so using (3.6), we conclude that

$$T(r,f) = N_{1)}(r,0;f) + S(r,f). \tag{3.8}$$

Let

$$\beta = \frac{G - Q_2}{F - Q_1}, \text{ i.e., } G - Q_2 = \beta(F - Q_1). \tag{3.9}$$

Since Q_2 is a polynomial and G is a transcendental meromorphic function, it follows that $G \not\equiv Q_2$. Similarly we can prove that $F \not\equiv Q_1$. Consequently $\beta \not\equiv 0$.

First suppose $\beta \in S(f)$. From (3.9), we have $G - \beta F \equiv Q_2 - \beta Q_1$, i.e.,

$$(f(z+c_1)f(z+c_2)\cdots f(z+c_{k+1}))^{(k)} - \beta(z)f^{k+1}(z) \equiv Q_2(z) - \beta(z)Q_1(z).$$

Since $f(z+c_1)f(z+c_2)\cdots f(z+c_{k+1})$ and $f^{k+1}(z)$ share 0 CM, it follows that $N(r,0;f) = S(r,f)$ and so from (3.8), we arrive at a contradiction.

Next suppose $\beta \notin S(f)$. Since F and G are of finite order, from (3.9) we conclude that

$$\sigma(\beta) \leq \max\{\sigma(G - Q_2), \sigma(F - Q_1)\} = \max\{\sigma(G), \sigma(F)\} < +\infty,$$

i.e., β is of finite order. As $F - Q_1$ and $G - Q_2$ share $(0,1)$, it follows that β has a zero at z_3 if z_3 is a zero of $F - Q_1$ and $G - Q_2$ with multiplicities $p_3(\geq 2)$ and $q_3(\geq 2)$ respectively such that $p_3 < q_3$ and β has a pole at z_3 if $q_3 < p_3$. Since F and G have finitely many poles, it follows that $N(r,\infty;F) = O(\log r) = N(r,\infty;G)$. Since $\overline{N}_{(2}(r,Q_1;F) = S(r,f)$ and $\overline{N}_{(2}(r,Q_2;G) = S(r,f)$, from (3.9), we have

$$\begin{aligned}
&\overline{N}(r,0;\beta) \leq \overline{N}_{(2}(r,Q_2;G) + O(\log r) = S(r,f) \quad \text{and}\\
&\overline{N}(r,\infty;\beta) \leq \overline{N}_{(2}(r,Q_1;F) + O(\log r) = S(r,f).
\end{aligned}$$

Let $\xi = \frac{\beta'}{\beta}$. Using Lemma 2.2, we have

$$\begin{aligned}
T(r,\beta) &\leq T(r,G) + T(r,F) + S(r,f)\\
&\leq nT(r,f) + (k+1)T\left(r, \prod_{i=1}^{n} f(z+c_i)\right) + S(r,f)\\
&= n(k+2)T(r,f) + S(r,f)
\end{aligned}$$

which implies that $T(r,\beta) = O(T(r,f))$ and so $S(r,\beta)$ can be replaced by $S(r,f)$. Consequently

$$\begin{aligned}
T(r,\xi) &= N\left(r,\infty;\frac{\beta'}{\beta}\right) + m\left(r,\infty;\frac{\beta'}{\beta}\right)\\
&= \overline{N}(r,0;\beta) + \overline{N}(r,\infty;\beta) + S(r,\beta) \leq S(r,f),
\end{aligned}$$

i.e., $T(r,\xi) = S(r,f)$. Now differentiating (3.9) once, we get

$$G' - Q_2' = \beta'(F - Q_1) + \beta(F' - Q_1'). \tag{3.10}$$

Now combining (3.9) and (3.10), we get

$$G'F - \frac{\beta'}{\beta}GF - GF' = Q_1G' - \Big(\frac{\beta'}{\beta}Q_1 + Q_1'\Big)G - Q_2F' + \Big(Q_2' - \frac{\beta'}{\beta}Q_2\Big)F + \frac{\beta'}{\beta}Q_1Q_2 + Q_2Q_1' - Q_1Q_2',$$

i.e.,

$$G'F - \xi GF - GF' = Q_1G' - \Big(\xi Q_1 + Q_1'\Big)G - Q_2F' + \Big(Q_2' - \xi Q_2\Big)F + \xi Q_1Q_2 + Q_2Q_1' - Q_1Q_2'. \tag{3.11}$$

By induction, we have

$$(f^{k+1})' = (k+1)f^kf', \quad (f^{k+1})'' = (k+1)kf^{k-1}(f')^2 + (k+1)f^kf'',$$

$$(f^{k+1})''' = (k+1)k(k-1)f^{k-2}(f')^3 + 3(k+1)kf^{k-1}f'f'' + (k+1)f^kf'''$$

and so on. Thus

$$(f^{k+1})^{(k)} = (k+1)!f(f')^k + \frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f''+ \cdots + (k+1)f^kf^{(k)}.$$

Also

$$(f^{k+1})^{(k-1)} = \{(k+1)k(k-1)\cdots 3\}f^2(f')^{k-1} + \cdots = \frac{1}{2}(k+1)!f^2(f')^{k-1} + \cdots .$$

Clearly each term of $(f^{k+1})^{(k-i)}$ $(1 \le i \le k-1)$ contains f^m $(2 \le m \le k)$ as a factor. By Leibnitz's rule for differentiating a product, we have

$$\begin{aligned} G = (f^{k+1}\psi)^{(k)} &= \sum_{i=0}^{k}\binom{k}{i}(f^{k+1})^{(k-i)}\psi^{(i)} \\ &= (f^{k+1})^{(k)}\psi + k(f^{k+1})^{(k-1)}\psi' + \cdots + f^{k+1}\psi^{(k)} \\ &= (k+1)!f(f')^k\psi + \frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f''\psi \\ &\quad + \frac{k}{2}(k+1)!f^2(f')^{k-1}\psi' + T_1(f), \end{aligned} \tag{3.12}$$

where $T_1(f)$ is a differential polynomial in f such that each term of $T_1(f)$ contains $f^m(3 \le m \le k+1)$ as a factor. Therefore we have

$$\frac{f'}{f}G = (k+1)!(f')^{k+1}\psi + \frac{k(k-1)}{4}(k+1)!f(f')^{k-1}f''\psi + \frac{k}{2}(k+1)!f(f')^k\psi' + T_2(f), \tag{3.13}$$

where $T_2(f)$ is a differential polynomial in f such that each term of $T_2(f)$ contains $f^m(2 \leq m \leq k)$ as a factor. Again from (3.12), we get

$$G' = (k+1)!(f')^{k+1}\psi + \frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f''\psi + (k+1)(k+1)!f(f')^k\psi' + T_3(f), \tag{3.14}$$

where $T_3(f)$ is a differential polynomial in f such that each term of $T_3(f)$ contains $f^m(2 \leq m \leq k+1)$ as a factor. Also from (3.14), we have

$$G'' = \frac{(k+1)(k+2)}{2}(k+1)!(f')^k f''\psi + (k+2)(k+1)!(f')^{k+1}\psi' + T_4(f), \tag{3.15}$$

where $T_4(f)$ is a differential polynomial in f such that each term of $T_4(f)$ contains $f^m(1 \leq m \leq k+1)$ as a factor. Substituting (3.1), (3.12), (3.13) and (3.14) into (3.11), we have

$$f^{k+1}P(f) = Q(f), \tag{3.16}$$

where

$$\begin{aligned} P(f) &= G' - \xi G - (k+1)\frac{f'}{f}G \\ &= -k(k+1)!(f')^{k+1}\psi + \Big[\frac{(k+1)(2-k)}{2}\psi' - \xi\psi\Big](k+1)!f(f')^k \\ &\quad + \frac{k(k+1)(3-k)(k+1)!}{4}f(f')^{k-1}f''\psi + \cdots \\ &= A_0(f')^{k+1} + B_0 f(f')^k \\ &\quad + \frac{k(k+1)(3-k)(k+1)!}{4}f(f')^{k-1}f''\psi + R_1(f) \end{aligned} \tag{3.17}$$

is a differential polynomial in f of the degree $k+1$,

$$A_0 = -k(k+1)!\psi, \;\; B_0 = \left(\frac{(k+1)(2-k)}{2}\psi' - \xi\psi\right)(k+1)!$$

and $R_1(f)$ is a differential polynomial in f such that each terms of $R_1(f)$ contains $f^m(2 \leq m \leq k+1)$ as a factor and

$$\begin{aligned} Q(f) &= Q_1G' - \big(\xi Q_1 + Q_1'\big)G - Q_2F' + \big(Q_2' - \xi Q_2\big)F \\ &\quad + \xi Q_1Q_2 + Q_2Q_1' - Q_1Q_2' \end{aligned} \tag{3.18}$$

is a differential polynomial in f of degree $k+1$. Now we consider following two sub-cases.

Sub-case 1.2.1. Suppose $P(f) \equiv 0$. Then from (3.16), we have $Q(f) \equiv 0$ and so from (3.11), we have $G'F - \frac{\beta'}{\beta}GF - GF' \equiv 0$, i.e., $\frac{G'}{G} \equiv \frac{\beta'}{\beta} + \frac{F'}{F}$. By integration, we have $G = d_0\beta F$, i.e.,

$$(f(z+c_1)f(z+c_2)\cdots f(z+c_{k+1}))^{(k)} = d_0\beta(z)f^{k+1}(z), \text{ where } d_0 \in \mathbb{C}\setminus\{0\}.$$

By the given condition, we have $f(z+c_1)f(z+c_2)\cdots f(z+c_{k+1})$ and $f^{k+1}(z)$ share 0 CM. Since $\overline{N}(r,\infty;\beta)=S(r,f)$, it follows that $\overline{N}(r,0;f)=S(r,f)$. Then from (3.8), we arrive at a contradiction.

Sub-case 1.2.2. Suppose $P(f)\not\equiv 0$. Clearly $P(f)$ is of finite order. Using Lemma 2.3, we conclude that $m(r,P(f))=S(r,f)$. Since $N(r,\infty;P(f))=O(\log r)$, we have

$$T(r,P(f))=S(r,f) \text{ and } T(r,P'(f))=S(r,f). \tag{3.19}$$

Now differentiating (3.17) once, we get

$$P'(f)=A_1(f')^kf''+B_1(f')^{k+1}+S_1(f), \tag{3.20}$$

where

$$A_1=-\frac{1}{4}k(k+1)^2(k+1)!\psi,\ \ B_1=\left(\frac{(1-k)(k+2)}{2}\psi'-\xi\psi\right)(k+1)!$$

and $S_1(f)$ is a differential polynomial in f such that each term of $S_1(f)$ contains $f^m(1\leq m\leq k+1)$ as a factor. Let z_4 be a simple zero of $f(z)$. Then from (3.17) and (3.20), we have respectively

$$\begin{aligned}&P(f(z_4))=A_0(z_4)\big(f'(z_4)\big)^{k+1} \quad\text{and}\\&P'(f(z_4))=A_1(z_4)\big(f'(z_4)\big)^kf''(z_4)+B_1(z_4)\big(f'(z_4)\big)^{k+1}.\end{aligned}$$

This shows that z_4 is a zero of

$$P(f)f''-(K_1P'(f)-K_2P(f))\,f',$$

where

$$\begin{aligned}&K_1=\frac{A_0}{A_1}=\frac{4}{(k+1)^2}\quad\text{and}\\&K_2=\frac{B_1}{A_1}=-\frac{4}{k(k+1)^2}\left(\frac{(1-k)(k+2)}{2}\frac{\psi'}{\psi}-\xi\right)\in S(f).\end{aligned}$$

Let

$$\Phi_1=\frac{P(f)f''-(K_1P'(f)-K_2P(f))\,f'}{f}. \tag{3.21}$$

Suppose $\Phi_1\not\equiv 0$. Then by Lemma 2.7, we have $m(r,\infty;\Phi_1)=S(r,f)$. Since $N(r,\infty;\Phi_1)=S(r,f)$, it follows that $\Phi_1\in S(f)$. From (3.21), we see that

$$f''=\alpha_1f+\beta_1f', \tag{3.22}$$

where

$$\alpha_1=\frac{\Phi_1}{P(f)}\ \text{ and }\ \beta_1=K_1\frac{P'(f)}{P(f)}-K_2. \tag{3.23}$$

Clearly $\alpha_1,\ \beta_1\in S(f)$. Note that (3.22) is also true even when $\Phi_1\equiv 0$. Actually in that case $\alpha_1\equiv 0$. Also (3.23) yields

$$P'(f)=\Big(\frac{\beta_1}{K_1}+\frac{K_2}{K_1}\Big)P(f) \tag{3.24}$$

and

$$\beta_1 = K_1\frac{P'(f)}{P(f)} - K_2 = \frac{A_0}{A_1}\frac{P'(f)}{P(f)} - \frac{B_1}{A_1},$$

i.e.,

$$A_1\beta_1 + B_1 - A_0\frac{P'(f)}{P(f)} \equiv 0. \tag{3.25}$$

Now from (3.18), we have

$$Q(f) = Q_1G' - \big(\xi Q_1 + Q_1'\big)G - Q_2F' + \big(Q_2' - \xi Q_2\big)F + \gamma, \tag{3.26}$$

where $\gamma = \xi Q_1Q_2 + Q_2Q_1' - Q_1Q_2'$. Obviously γ is of finite order. Suppose $\gamma \equiv 0$. By integration, we have $\beta = d_1\frac{Q_2}{Q_1}$, where $d_1 \in \mathbb{C}\setminus\{0\}$ and so $\beta \in S(f)$, which is a contradiction. Consequently $\gamma \not\equiv 0$. Also $\gamma \in S(f)$. Similarly we have $\xi Q_1 + Q_1' \not\equiv 0$. Differentiating (3.26) once, we get

$$\begin{aligned} Q'(f) = {} & Q_1'G' + Q_1G'' - (\xi Q_1 + Q_1')G' - (\xi Q_1 + Q_1')'G \\ & - Q_2'F' - Q_2F'' + (Q_2' - \xi Q_2)'F + (Q_2' - \xi)F' + \gamma'. \end{aligned} \tag{3.27}$$

Now we consider following two sub-cases.

Sub-case 1.2.2.1. Suppose $k = 1$. Using (3.1), (3.12) and (3.14) into (3.26), we have

$$\begin{aligned} Q(f) = {} & Q_1\Big\{2(f')^2\psi + 4ff'\psi' + 2ff''\psi + T_3(f)\Big\} \\ & - \big(\xi Q_1 + Q_1'\big)\big(2ff'\psi + f^2\psi'\big) \\ & - 2Q_2ff' + \big(Q_2' - \xi Q_2\big)f^2 + \gamma. \end{aligned} \tag{3.28}$$

Again using (3.1), (3.12), (3.14) and (3.15) into (3.27), we have

$$\begin{aligned} & Q'(f) \\ & = Q_1'\Big\{2(f')^2\psi + 4ff'\psi' + 2ff''\psi + T_3(f)\Big\} \\ & \quad + Q_1\Big\{6(f')^2\psi' + 6f'f''\psi + T_4(f)\Big\} \\ & \quad - \big(\xi Q_1 + Q_1'\big)\Big\{2(f')^2\psi + 4ff'\psi' + 2ff''\psi + T_3(f)\Big\} \\ & \quad - \big(\xi Q_1 + Q_1'\big)'\{2ff'\psi + f^2\psi'\} - 2Q_2'ff' - Q_2\Big\{2(f')^2 + 2ff''\Big\} \\ & \quad + \big(Q_2' - \xi Q_2\big)'f^2 + 2\big(Q_2' - \xi\big)ff' + \gamma'. \end{aligned} \tag{3.29}$$

Let z_5 be a simple zero of $f(z)$ such that $P(f(z_5)) \neq 0, \infty$. Clearly from (3.16), we conclude that z_5 must be a zero of $Q(f)$. Consequently from (3.28) and (3.29), we conclude that

$$\gamma(z_5) = A_2(z_5)(f'(z_5))^2 \text{ and } \gamma'(z_5) = A_3(z_5)f'(z_5)f''(z_5) + B_3(z_5)(f'(z_5))^2,$$

where

$$A_2 = -2Q_1\psi,\ A_3 = -6Q_1\psi \text{ and } B_3 = 2Q_1\psi\xi + 2Q_2 - 6Q_1\psi'.$$

This shows that z_5 is a zero of

$$\gamma f'' - (K_3\gamma' - K_4\gamma) f',$$

where

$$K_3 = \frac{A_2}{A_3} = \frac{1}{3} \text{ and } K_4 = \frac{B_3}{A_3} = -\left(\frac{1}{3}\xi + \frac{1}{3}\frac{Q_2}{Q_1\psi} - \frac{\psi'}{\psi}\right) \in S(f).$$

Let

$$\Phi_2 = \frac{\gamma f'' - (K_3\gamma' - K_4\gamma) f'}{f}. \tag{3.30}$$

Suppose $\Phi_2 \not\equiv 0$. Then by Lemma 2.7, we have $m(r,\infty;\Phi_2) = S(r,f)$. Since $N(r,\infty;\Phi_2) = S(r,f)$, it follows that $\Phi_2 \in S(f)$. From (3.30), we see that

$$f'' = \varrho_1 f + \phi_1 f', \tag{3.31}$$

where

$$\varrho_1 = \frac{\Phi_2}{\gamma} \text{ and } \phi_1 = K_3\frac{\gamma'}{\gamma} - K_4. \tag{3.32}$$

Clearly $\varrho_1,\ \phi_1 \in S(f)$. Note that (3.31) is also true even when $\Phi_2 \equiv 0$. Actually in that case $\varrho_1 \equiv 0$. Now we claim that $\phi_1 \not\equiv \beta_1$. If $\phi_1 \equiv \beta_1$, then from (3.23) and (3.32), we have

$$\frac{1}{3}\frac{\gamma'}{\gamma} + \frac{1}{3}\xi + \frac{1}{3}\frac{Q_2}{Q_1\psi} - \frac{\psi'}{\psi} = \frac{P'(f)}{P(f)} - \xi$$

i.e.,

$$3\frac{P'(f)}{P(f)} - \frac{\gamma'}{\gamma} - 4\frac{\beta'}{\beta} + 3\frac{\psi'}{\psi} = Re^{-\eta}, \tag{3.33}$$

where $R = \frac{Q_2}{Q_1\alpha}$. Let us consider following two sub-cases.

Sub-case 1.2.2.1.1. Suppose $\eta \in \mathbb{C}$. For the sake of simplicity we assume that $e^{\eta} = 1$. Using Lemma 2.7, we see that

$$\begin{aligned} m\left(r,\infty;\frac{G}{F}\right) &= m\left(r,\infty;\frac{(f(z+c_1)f(z+c_2))'}{f^2(z)}\right) \\ &\le m\left(r,\infty;\frac{(f(z+c_1)f(z+c_2))'}{f(z+c_1)f(z+c_2)}\right) \\ &\quad + m\left(r,\infty;\frac{f(z+c_1)f(z+c_2)}{f^2(z)}\right) \\ &= O(\log r) + m(r,\infty;\alpha) = O(\log r) + O(\log r) \\ &= O(\log r), \end{aligned}$$

i.e.,

$$m\left(r,\infty;\frac{G}{F}\right) = O(\log r).$$

Finally by using Lemma 2.7, we conclude that $m(r,\infty;\Phi)=O(\log r)$. Since $N(r,\infty;\Phi)=O(\log r)$, we have $T(r,\Phi)=O(\log r)$ and so Φ is a rational function.

Now from (3.5), we see that $\overline{N}_{(2}(r,1;F_1)\leq N(r,0;\Phi)\leq T(r,\Phi)+O(1)=O(\log r)$, i.e., $\overline{N}_{(2}(r,1;F_1)=O(\log r)$. Since F_1 and G_1 share $(1,1)$ except for the zeros of $Q_1(z)$ and $Q_2(z)$ respectively, it follows that $\overline{N}_{(2}(r,1;G_1)=O(\log r)$. Clearly $N_{(2}(r,Q_1;F)=O(\log r)$ and $N_{(2}(r,Q_2;G)=O(\log r)$. Since $F-Q_1$ and $G-Q_2$ share $(0,1)$, from (3.9) it follows that β has finitely many zeros and poles. Since β is of finite order, by Hadamard factorization theorem, we can take

$$\beta=\gamma_1 e^{\eta_1}, \tag{3.34}$$

where γ_1 is a rational function and η_1 is a polynomial. We claim that η_1 is a non-constant polynomial. If not, suppose η_1 is constant. Then from (3.34), we see that $\beta\in S(f)$, which is a contradiction. Hence η_1 is a non-constant polynomial and so we let $\deg(\eta_1)=m\geq 1$. Let $\eta_1(z)=c_mz^m+c_{m-1}z^{m-1}+\cdots+c_0$, where $c_i\in\mathbb{C}$ for $i=0,1,\ldots,m$ and $c_m\neq 0$.

Since $f(z)$ has finitely many poles, by Hadamard factorization theorem, we can take $f(z)=\frac{g(z)}{d(z)}$, where $g(z)$ is a transcendental entire function and $d(z)$ is a non-zero polynomial.

Let $g_1(z)=g^2(z)$. Then we have $f^2(z)=\frac{g_1(z)}{d^2(z)}$ and so from (3.2), we have $f(z+c_1)f(z+c_2)=\alpha(z)\frac{g^2(z)}{d^2(z)}=\frac{U_1(z)}{V_1(z)}g^2(z)=\frac{U_1(z)}{V_1(z)}g_1(z)$, where U_1, V_1 are non-zero polynomials. Therefore

$$G=\left(\frac{U_1}{V_1}g_1\right)'=\left(\frac{U_1}{V_1}\right)'g_1+\frac{U_1}{V_1}g_1'=\frac{V_1U_1'-U_1V_1'}{V_1^2}g_1+\frac{U_1}{V_1}g_1'. \tag{3.35}$$

Now from (3.9), (3.34) and (3.35), we have

$$\gamma_1e^{\eta_1}=\frac{\frac{V_1U_1'-U_1V_1'}{V_1^2}g_1+\frac{U_1}{V_1}g_1'-Q_2}{\frac{g_1}{d^2}-Q_1},$$

i.e.,

$$\frac{V_1}{U_1d^2}\gamma_1e^{\eta_1}=\frac{\frac{U_1'}{U_1}-\frac{V_1'}{V_1}+\frac{g_1'}{g_1}-\frac{V_1}{U_1}\frac{Q_2}{g_1}}{1-\frac{d^2Q_1}{g_1}},$$

i.e.,

$$\gamma_2e^{\eta_1}=\frac{\frac{U_1'}{U_1}-\frac{V_1'}{V_1}+\frac{g_1'}{g_1}-\frac{V_1}{U_1}\frac{Q_2}{g_1}}{1-\frac{d^2Q_1}{g_1}},$$

where $\gamma_2=\frac{V_1}{U_1d^2}\gamma_1$ is rational function. Therefore

$$\eta_1=\log\frac{1}{\gamma_2}\frac{\frac{U_1'}{U_1}-\frac{V_1'}{V_1}+\frac{g_1'}{g_1}-\frac{V_1}{U_1}\frac{Q_2}{g_1}}{1-\frac{d^2Q_1}{g_1}},$$

where $\log h$ means the principle branch of logarithm. Now by Lemma 2.8, we have $\forall\ \varepsilon > 0$, there exists $r_0 > 0$ such that $\forall\ r = |z| > r_0$ the inequalities

$$(1-\varepsilon)|c_m||z|^m \leq |\eta_1(z)| \leq (1+\varepsilon)|c_m||z|^m$$

hold. Therefore

$$(1-\varepsilon) \leq \frac{|\eta_1(z)|}{|c_m||z|^m} \leq (1+\varepsilon)\ \ \forall\ |z| > r_0,$$

i.e., $\lim\limits_{|z|\to\infty} \dfrac{|\eta_1(z)|}{|c_m||z|^m} = 1$, i.e., $|\eta_1(z)| = |c_m||z|^m(1+o(1))$, i.e.,

$$|\eta_1(z)| = |c_m|r^m(1+o(1)).$$

Consequently we have

$$\begin{aligned} |c_m|r^m(1+o(1)) &= |\eta_1(z)| \\ &= \left| \log \frac{1}{\gamma_2(z)} \frac{\frac{U_1'(z)}{U_1(z)} - \frac{V_1'(z)}{V_1(z)} + \frac{g_1'(z)}{g_1(z)} - \frac{V_1(z)}{U_1(z)}\frac{Q_2(z)}{g_1(z)}}{1 - \frac{d^2(z)Q_1(z)}{g_1(z)}} \right|. \end{aligned} \tag{3.36}$$

Since g_1 is a transcendental entire function, it follows that $M(r, g_1) \to \infty$ as $r \to \infty$. Again we let

$$M(r, g_1) = |g_1(z_r)|\,, \ \text{ where } \ z_r = re^{i\theta} \ \text{ and } \ \theta \in [0, 2\pi). \tag{3.37}$$

Then from (3.37) and Lemma 2.9, there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure such that for some point $z_r = re^{i\theta}(\theta \in [0, 2\pi))$ satisfying $|z_r| = r \notin E$ and $M(r, g_1) = |g_1(z_r)|$, we have

$$\frac{g_1'(z_r)}{g_1(z_r)} = \left(\frac{\nu(r, g_1)}{z_r}\right)(1+o(1)) \text{ as } r \to \infty. \tag{3.38}$$

Let $a_{m_1}z^{m_1}$ and $b_{n_1}z^{n_1}$ denote the leading terms in the polynomials $Q_2(z)V_1(z)$ and $U_1(z)$ respectively. Taking $\varepsilon = \frac{1}{2}$, we get from Lemma 2.8 that

$$\frac{1}{2}|a_{m_1}|r^{m_1} \leq |Q_2(z_r)V_1(z_r)| \leq \frac{3}{2}|a_{m_1}|r^{m_1}$$

and

$$\frac{1}{2}|b_{n_1}|r^{n_1} \leq |U_1(z_r)| \leq \frac{3}{2}|b_{n_1}|r^{n_1}.$$

Therefore

$$\left|\frac{Q_2(z_r)V_1(z_r)}{U_1(z_r)}\right| \leq \frac{|a_{m_1}|r^{m_1}}{|b_{n_1}|r^{n_1}}.$$

Since F is a transcendental entire function, we know that $M(r, F)$ increases faster than the maximum modulus of any polynomial and hence faster than any power of r. Now from (3.37), we have

$$\lim_{r\to+\infty}\left|\frac{V_1(z_r)}{U_1(z_r)}\frac{Q_2(z_r)}{g_1(z_r)}\right| \leq \lim_{r\to+\infty} \frac{|a_{m_1}|r^{m_1}}{|b_{n_1}|r^{n_1}M(r, g_1(z_r))} = 0. \tag{3.39}$$

Similarly we have

$$\lim_{r\to+\infty}\left|\frac{d^2(z_r)Q_1(z_r)}{g_1(z_r)}\right| = 0. \tag{3.40}$$

Also

$$\lim_{r\to+\infty}\left|\frac{U_1'(z_r)}{U_1(z_r)}\right| = 0 \text{ and } \lim_{r\to+\infty}\left|\frac{V_1'(z_r)}{V_1(z_r)}\right| = 0. \tag{3.41}$$

Since g_1 is of finite order, from Lemma 2.10, we have

$$\log\nu(r,g_1) = O(\log r). \tag{3.42}$$

Therefore from (3.36)-(3.42), we have

$$|c_m|r^m(1+o(1)) = |\eta_1(z_r)| = O(\log r)$$

for $|z_r| = r \notin E$, which is impossible.

Sub-case 1.2.2.1.2. Suppose $\eta \notin \mathbb{C}$. If the equation (3.33) has no meromorphic solution in $\mathbb{C}$, then we arrive at a contradiction. Next we suppose the equation (3.33) has a meromorphic solution in $\mathbb{C}$. By integration, we have

$$P^3(f)\psi^3\gamma^{-1}\beta^{-4} = d_2e^{\varphi}, \tag{3.43}$$

where $\varphi(z) = \int_0^z R(t)e^{-\eta(t)}dt$ and $d_2 \in \mathbb{C}\setminus\{0\}$. Since $\sigma(g) = \sigma(g')$, it follows that

$$\sigma(\varphi) = \sigma(\varphi') = \sigma(Re^{-\eta}) = \sigma(e^{-\eta}) = \deg(\eta) \geq 1$$

and so e^{φ} is of infinite order. Since $P(f)$, ψ and β are of finite order, it follows that $P^3(f)\psi^3\gamma^{-1}\beta^{-4}$ is of finite order. Therefore from (3.43), we arrive at a contradiction.

Consequently both Sub-cases 1.2.2.1.1 and 1.2.2.1.2 lead to a contradiction. Hence $\phi_1(z) \not\equiv \beta_1(z)$. Now from (3.17), (3.20) and (3.31), we have respectively

$$P(f) = A_0(f')^2 + T_{11}ff' + T_{12}f^2 \quad \text{and} \tag{3.44}$$

$$P'(f) = (A_1\phi_1 + B_1)(f')^2 + S_{11}ff' + S_{12}f^2, \tag{3.45}$$

where $T_{1j}, S_{1j} \in S(f)$ for $j = 1, 2$. Multiplying (3.44) by $P'(f)$ and (3.45) by $P(f)$ and then subtracting, we get

$$u_0(f')^2 + u_1ff' + u_2f^2 \equiv 0, \tag{3.46}$$

where

$$u_0 = P(f)\left(A_1\phi_1 + B_1 - A_0\frac{P'(f)}{P(f)}\right) \in S(f) \quad \text{and} \tag{3.47}$$

$$u_j = P(f)S_{1j} - P'(f)T_{1j} \in S(f) \quad \text{for } j = 1, 2.$$

Since $\beta_1 \not\equiv \phi_1$ and $P(f) \not\equiv 0$, from (3.25) and (3.47) we have $u_0 \not\equiv 0$. Let $S_1 = \{i : u_i \not\equiv 0, i = 1, 2\}$. Now from (3.46), we see that a simple zero of f

must be either a zero of u_0 or a pole of at least one of u_i, where $i \in S_1$. Therefore

$$N_{1)}(r,0;f) \leq N(r,0;u_0) + \sum_{i \in S_1} N(r,\infty;u_i) + S(r,f) \leq S(r,f)$$

and so from (3.8), we arrive at a contradiction.

Sub-case 1.2.2.2. Suppose $k \geq 2$. Then using (3.1), (3.12) and (3.14) into (3.26), we have

$$\begin{aligned}(3.48)\quad & Q(f)\\ &= Q_1\Big\{(k+1)!(f')^{k+1}\psi + \frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f''\psi\\ &\quad + (k+1)(k+1)!f(f')^k\psi' + T_3(f)\Big\} - \big(\xi Q_1 + Q_1'\big)\Big\{(k+1)!f(f')^k\psi\\ &\quad + \frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f''\psi + \frac{k}{2}(k+1)!f^2(f')^{k-1}\psi' + T_1(f)\Big\}\\ &\quad - (k+1)Q_2f^kf' + \big(Q_2' - \xi Q_2\big)f^{k+1} + \gamma.\end{aligned}$$

Again using (3.1), (3.12), (3.14) and (3.15) into (3.27), we have

$$\begin{aligned}(3.49)\quad & Q'(f)\\ &= Q_1'\Big\{(k+1)!(f')^{k+1}\psi + \frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f''\psi\\ &\quad + (k+1)(k+1)!f(f')^k\psi' + T_3(f)\Big\}\\ &\quad + Q_1\Big\{\frac{(k+1)(k+2)}{2}(k+1)!(f')^kf''\psi\\ &\quad + (k+2)(k+1)!(f')^{k+1}\psi' + T_4(f)\Big\} - \big(\xi Q_1 + Q_1'\big)\Big\{(k+1)!(f')^{k+1}\psi\\ &\quad + \frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f''\psi + (k+1)(k+1)!f(f')^k\psi' + T_3(f)\Big\}\\ &\quad - \big(\xi Q_1 + Q_1'\big)'\Big\{(k+1)!f(f')^k\psi + \frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f''\psi\\ &\quad + \frac{k}{2}(k+1)!f^2(f')^{k-1}\psi' + T_1(f)\Big\} - (k+1)Q_2'f^kf'\\ &\quad - Q_2\Big\{k(k+1)f^{k-1}(f')^2 + (k+1)f^kf''\Big\} + \big(Q_2' - \xi Q_2\big)'f^{k+1}\\ &\quad + (k+1)\big(Q_2' - \xi\big)f^kf' + \gamma'.\end{aligned}$$

Let z_6 be a simple zero of $f(z)$ such that $P(f(z_6)) \neq 0, \infty$. Then from (3.16), we conclude that z_6 must be a zero of $Q(f)$. Now from (3.48) and (3.49), we have respectively

$$\begin{aligned}&\gamma(z_6) = A_4(z_6)(f'(z_6))^{k+1} \quad \text{and}\\ &\gamma'(z_6) = A_5(z_6)(f'(z_6))^k f''(z_6) + B_5(z_6)(f'(z_6))^{k+1},\end{aligned}$$

where $A_4 = -(k+1)!Q_1\psi$, $A_5 = -\frac{k^2+3k+2}{2}(k+1)!Q_1\psi$ and $B_5 = (k+1)!\,(\xi Q_1\psi - (k+2)Q_1\psi')$. This shows that z_6 is a zero of $\gamma f'' - (K_5\gamma' - K_6\gamma)\, f'$, where

$$K_5 = \frac{A_4}{A_5} = \frac{2}{k^2+3k+2} \quad \text{and}$$
$$K_6 = \frac{B_5}{A_5} = -\frac{2}{k^2+3k+2}\left(\xi - (k+2)\frac{\psi'}{\psi}\right) \in S(f).$$

Let

$$\Phi_3 = \frac{\gamma f'' - (K_5\gamma' - K_6\gamma)\, f'}{f}. \tag{3.50}$$

Suppose $\Phi_3 \not\equiv 0$. Then by Lemma 2.7, we have $m(r,\infty;\Phi_3) = S(r,f)$. Since $N(r,\infty;\Phi_3) = S(r,f)$, it follows that $\Phi_3 \in S(f)$. From (3.50), we see that

$$f'' = \zeta_1 f + \delta_1 f', \tag{3.51}$$

where

$$\zeta_1 = \frac{\Phi_3}{\gamma} \ \text{ and } \ \delta_1 = K_5\frac{\gamma'}{\gamma} - K_6. \tag{3.52}$$

Clearly $\zeta_1,\ \delta_1 \in S(f)$. Note that (3.51) is also true even when $\Phi_3 \equiv 0$. Actually in that case $\zeta_1 \equiv 0$. Now we claim that $\delta_1 \not\equiv \beta_1$. If $\delta_1 \equiv \beta_1$, then from (3.23) and (3.52), we have

$$\frac{2}{(k+1)(k+2)}\frac{\gamma'}{\gamma} - \frac{2}{k+1}\frac{\psi'}{\psi} + \frac{2}{(k+1)(k+2)}\xi$$
$$\equiv \frac{4}{(k+1)^2}\frac{P'}{P} - \frac{4}{k(k+1)^2}\xi + \frac{2(1-k)(k+2)}{k(k+1)^2}\frac{\psi'}{\psi},$$

i.e.,

$$(k^2+3k+4)\frac{\beta'}{\beta} \equiv 2k(k+2)\frac{P'}{P} - k(k+1)\frac{\gamma'}{\gamma} + 2(k+2)\frac{\psi'}{\psi}.$$

By integration, we have

$$\beta^{(k^2+3k+4)} \equiv \frac{d_3 P^{2k(k+2)}\psi^{2(k+2)}}{\gamma^{k(k+1)}},$$

where $d_3 \in \mathbb{C}\setminus\{0\}$ and so from (3.19), we have $\beta \in S(f)$, which is a contradiction. Hence $\delta_1(z) \not\equiv \beta_1(z)$. Now differentiating (3.51) and using it repeatedly we have

$$f^{(i)} = \zeta_{i-1} f + \delta_{i-1} f', \tag{3.53}$$

where $\zeta_{i-1}, \delta_{i-1} \in S(f)$ for $i \geq 2$. Also from (3.17), (3.20) and (3.53), we have respectively

$$P(f) = A_0(f')^{k+1} + \sum_{j=1}^{k+1} g_{2j} f^j (f')^{k+1-j} \quad \text{and} \tag{3.54}$$

$$P'(f) = (A_1\delta_1 + B_1)(f')^{k+1} + \sum_{j=1}^{k+1} h_{2j} f^j (f')^{k+1-j}, \tag{3.55}$$

where $g_{2j}, h_{2j} \in S(f)$. Multiplying (3.54) by $P'(f)$ and (3.55) by $P(f)$ and then subtracting we get

$$H_0(f')^{k+1} + H_1 f(f')^k + \cdots + H_{k+1} f^{k+1} \equiv 0, \tag{3.56}$$

where

$$H_0 = P(f)\left(A_1\delta_1 + B_1 - A_0\frac{P'(f)}{P(f)}\right) \in S(f) \text{ and} \tag{3.57}$$

$$H_j = P(f)h_{2j} - P'(f)g_{2j} \in S(f) \text{ for } j = 1, 2, \ldots, k+1. \tag{3.58}$$

Since $\beta_1 \not\equiv \delta_1$ and $P(f) \not\equiv 0$, from (3.25) and (3.57), we have $H_0 \not\equiv 0$. Then from (3.56), one can easily conclude that $N_{1)}(r, 0; f) = S(r, f)$ and so we arrive at a contradiction from (3.8).

Case 2. Suppose $\Phi \equiv 0$. Since f is a transcendental meromorphic function, it follows that $F_1' \not\equiv 0$. Then from (3.3), we have $F_1 \equiv G_1$, i.e., $(f(z+c_1)f(z+c_2)\cdots f(z+c_n))^{(k)} \equiv \frac{Q_2(z)}{Q_1(z)} f^n(z)$. Furthermore if $Q_1(z) \equiv Q_2(z)$, then

$$f^n(z) \equiv (f(z+c_1)f(z+c_2)\cdots f(z+c_n))^{(k)}. \tag{3.59}$$

Let z_7 be a zero of $f(z)$ with multiplicity p_1. Then z_7 is a zero of $f^n(z)$ with multiplicity np_1. Since $f^n(z)$ and $f(z+c_1)f(z+c_2)\cdots f(z+c_n)$ share 0 CM, it follows that z_7 must be a zero of $f(z+c_1)f(z+c_2)\cdots f(z+c_n)$ with multiplicity np_1. Consequently z_7 will be a zero of $(f(z+c_1)f(z+c_2)\cdots f(z+c_n))^{(k)}$ with multiplicity $np_1 - k$. Therefore from (3.59), we arrive at a contradiction. As a result we have $f(z) \neq 0$ and $(f(z+c_1)f(z+c_2)\cdots f(z+c_n))^{(k)} \neq 0$.

Let $G_2(z) = f(z+c_1)f(z+c_2)\cdots f(z+c_n)$. Then $(G_2(z))^{(k)} \neq 0$.

Since $f(z)$ is a transcendental meromorphic function with finitely many poles and $f(z) \neq 0$, $f(z)$ must take the form $f(z) = \frac{1}{P_1(z)} e^{P_2(z)}$, where $P_1(z)$ is a non-zero polynomial and $P_2(z)$ is a non-constant polynomial. Therefore $G_2(z) = \frac{1}{P_3(z)} e^{P_4(z)}$, where $P_3(z) = P_1(z+c_1)P_1(z+c_2)\cdots P_1(z+c_n)$ and $P_4(z) = \sum_{i=1}^n P_2(z+c_i)$. Clearly $G_2(z) \neq 0$. Let

$$g(z) = \frac{G_2'(z)}{G_2(z)} = P_4'(z) - \frac{P_3'(z)}{P_3(z)}.$$

Clearly g is a non-zero rational function, otherwise $P_3(z) = d_3 e^{P_4(z)}$, which is impossible. Therefore by Lemma 2.6, we have

$$\frac{G_2^{(k)}(z)}{G_2(z)} = g^k(z) + Q_{k-1}^1(g(z)),$$

where $Q_0^1(g) \equiv 0$ and $Q_{i-1}^1(g)$ $(i = 1, 2, \ldots, k)$ is a differential polynomial of degree $i-1$ in g and its derivatives. Clearly $\frac{G_2^{(k)}}{G_2} \not\equiv 0$, otherwise G_2 will be a polynomial of degree at most $k-1$, which contradicts the fact that G_2 is a transcendental meromorphic function.

We claim that P_4' is a constant. If not, suppose P_4' is non-constant. Then g must be a non-constant rational function. Therefore we see that

$$\frac{G_2^{(k)}(z)}{G_2(z)} \sim g^k(z) \sim (P_4'(z))^k \to \infty \text{ as } z \to \infty.$$

We know that every non-constant rational function assumes every value in the closed complex plane. Consequently $G_2^{(k)} = 0$ somewhere in the open complex plane. But since $f(z) \neq 0$, from (3.59), we see that $G_2^{(k)}$ has no zeros. Therefore we arrive at a contradiction. Hence P_4' is a constant. Let $P_4' = \lambda$. Since P_4 is a non-constant, it follows that $\lambda \neq 0$.

If $g(z)$ is non-constant, then we see that $g(z) = \lambda$, $g'(z) = g''(z) = \cdots = 0$ at ∞. Also we observe that $\frac{G_2^{(k)}}{G_2} = \lambda^k$ at ∞. Again $\frac{G_2^{(k)}}{G_2}$ must have a zero in the open complex plane. Therefore we again arrive at a contradiction. Consequently g is constant. Since g and P_4' are constants, from $\frac{P_3'}{P_3} = P_4' - g$, we conclude that $P_3' \equiv 0$, i.e., P_3 is a non-zero constant, i.e., P_1 is a non-zero constant. Therefore we must have $P_4' = \lambda = g$ and so $G_2(z) = e^{\lambda z + d}$, where $d \in \mathbb{C}$. Finally $f(z)$ assumes the form $f(z) = c\ e^{\frac{\lambda}{n} z}$, where $c \in \mathbb{C} \setminus \{0\}$, $e^{\lambda(c_1+c_2+\cdots+c_n)} = 1$ and $\lambda^k = 1$. This completes the proof. □

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