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GENERALIZED SELF-INVERSIVE BICOMPLEX POLYNOMIALS WITH RESPECT TO THE j-CONJUGATION

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Abstract. In this paper, we study a kind of self-inversive polynomials in bicomplex algebra. For a bicomplex polynomial, this is the study of a relation between a kind of symmetry of its coefficients and a kind of symmetry of zeros. For our deep study, we define several new levels of self-inversivity. We prove some functional equations for standard ones, a decomposition theorem for generalized ones and a comparison theorem. Although we focus the j-conjugation in our study, our argument can be applied for other conjugations.

1. Introduction

It is well-known that for a polynomial f there is a relation between a kind of symmetry of coefficients of f and a kind of symmetry of zeros. For example, let us consider a complex polynomial $f(z) = \sum_{m=0}^{n} a_m z^m$ of degree n in one variable. A real polynomial, whose coefficients have a symmetry $a_m = \overline{a_m}$ $(m = 0, 1, \ldots, n)$ with respect to the complex conjugation, has a symmetry of zeros, that $f(c) = 0$ implies $f(\overline{c}) = 0$. Also, a self-reciprocal polynomial, whose coefficients have a symmetry $a_m = a_{n-m}$ $(m = 0, 1, \ldots, n)$, has a symmetry of zeros, that $f(c) = 0$ implies $f\left(\frac{1}{c}\right) = 0$. Combining these two symmetries, we obtain the notion of self-inversivity. A self-inversive polynomial, whose coefficients have a symmetry $\overline{a_n} a_m = a_0 \overline{a_{n-m}}$ $(m = 0, 1, \ldots, n)$, has a symmetry of zeros with respect to the unit circle, that $f(c) = 0$ implies $f\left(\frac{1}{\overline{c}}\right) = 0$. It has been important to study these polynomials in the theory of the error correcting codes and analytic number theory for a long time. See [5, 7] and Section 2.1 for more details. In this paper, we study a kind of self-inversive polynomial in bicomplex algebra.

Bicomplex algebra was introduced by Segre [8] inspired by the work of Hamilton on quaternions and defined by

$$\mathbb{BC} = \{z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\},$$

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where j is another imaginary unit commuting with the imaginary unit i of $\mathbb{C}$. In particular, $\mathbb{BC}$ is a commutative complex Clifford algebra. Since $\mathbb{BC}$ is not an integral domain, the study of zeros of a bicomplex polynomial is not easy. For example, the factorization of polynomials is not unique and the number of zeros of a monic bicomplex polynomial of degree n is n^2. See [4] and Sections 2.2 and 2.3 for more details. See [6] for recent developments in this area. In this algebra, conjugations with respect to i, j and ij are defined by $\overline{z_1} + \overline{z_2}j$, $z_1 - z_2 j$ and $\overline{z_1} - \overline{z_2}j$, respectively. Corresponding to each conjugation, we can define the notion of self-inversivity for bicomplex polynomials.

In this paper, we study self-inversive bicomplex polynomials with respect to the j-conjugation. Since the study of bicomplex polynomials is more difficult than that of the complex polynomials, we need to define several levels of self-inversivity. First, we study the relation between the symmetry of coefficients and the symmetry of zeros. Second, we prove the decomposition theorem for generalized self-inversive polynomials. Finally, we prove a comparison theorem. See Section 3 for more details. For self-inversive polynomials with respect to other conjugations, such as the i-conjugation, our argument can be applied. See Remark 3.9. Note that the ij-conjugation case was partially studied in [1].

This paper is organized as follows. In Section 2.1, we review the relation between a symmetry of coefficients and a symmetry of zeros for self-inversive complex polynomials. In Sections 2.2 and 2.3, we recall very briefly the definition and fundamental properties of bicomplex numbers and bicomplex polynomials, respectively. Section 3 is the main part of this paper. After defining several new levels of self-inversivity of bicomplex polynomials, we discuss properties of them. Finally, Section 4 contains the conclusion.

2. Preliminaries

2.1. Self-inversive complex polynomials

In this subsection, we review the relation between a symmetry of coefficients and a symmetry of zeros for self-inversive complex polynomials. In this paper, we define the notion of self-inversivity by a strong symmetry of zeros. See [5,7] for more details.

Definition. Let f be a complex polynomial of degree n in one variable with the multiset of zeros $\{c_1, \ldots, c_n\}$ in $\mathbb{C}$. We say that f is self-inversive if the equality $\{c_1, \ldots, c_n\} = \left\{\frac{1}{\overline{c_1}}, \ldots, \frac{1}{\overline{c_n}}\right\}$ holds as a multiset. Here the bar means the complex conjugation.

Note that the condition of zeros in Definition above is stronger than the property that $f(c) = 0$ implies $f(\frac{1}{\overline{c}}) = 0$. A fundamental property of self-inversive polynomials is as follows.

Theorem 2.1. *Let $f(z) = \sum_{m=0}^{n} a_m z^m$ $(a_n \neq 0)$ be a complex polynomial of degree n. The following four conditions are equivalent*:

(i) f *is self-inversive,*
(ii) $\overline{a_n}a_m = a_0\overline{a_{n-m}}$ $(m = 0, 1, 2, \ldots, n)$,
(iii) $\overline{a_n}f(z) = a_0 z^n \overline{f\left(\frac{1}{\bar{z}}\right)}$ $(z \in \mathbb{C} \setminus \{0\})$,
(iv) $\overline{a_n}(nf(z) - zf'(z)) = a_0 z^{n-1}\overline{f'\left(\frac{1}{\bar{z}}\right)}$ $(z \in \mathbb{C} \setminus \{0\})$ *and* $|a_n| = |a_0|$.

An aim of this paper is to study these conditions for bicomplex polynomials. See Theorem 3.2 below.

2.2. Bicomplex numbers

In this subsection, we recall the definition and fundamental properties of bicomplex numbers. See [3, 4] for more details.

Let $\mathbb{C}$ be the field of complex numbers with the imaginary unit i. The set of bicomplex numbers is defined by

$$\mathbb{BC} = \{Z = z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\},$$

where j is another imaginary unit independent of and commuting with i:

$$i \neq j, \quad ij = ji, \quad i^2 = j^2 = -1.$$

Defining the addition and multiplication naturally, $\mathbb{BC}$ has a structure of a commutative ring. The set of zero divisors of $\mathbb{BC}$ with 0 is described as

$$\mathfrak{S}_0 = \{Z = z_1 + z_2 j \in \mathbb{BC} \mid z_1^2 + z_2^2 = 0\},$$

that is equal to the set of non-unit elements of $\mathbb{BC}$. Setting

$$\mathbf{e} = \frac{1 + ij}{2}, \quad \mathbf{e}^\dagger = \frac{1 - ij}{2},$$

$\mathbf{e}$ and $\mathbf{e}^\dagger$ are the non-complex idempotent elements satisfying the property $\mathbf{e}\mathbf{e}^\dagger = 0$.

We define the surjective ring homomorphisms $\Phi_{\mathbf{e}} : \mathbb{BC} \longrightarrow \mathbb{C}$, $\Phi_{\mathbf{e}^\dagger} : \mathbb{BC} \longrightarrow \mathbb{C}$ by

$$\Phi_{\mathbf{e}}(Z) = z_1 - z_2 i, \quad \Phi_{\mathbf{e}^\dagger}(Z) = z_1 + z_2 i$$

for $Z = z_1 + z_2 j \in \mathbb{BC}$, respectively. Then any bicomplex number Z has the idempotent representation

$$Z = \Phi_{\mathbf{e}}(Z)\mathbf{e} + \Phi_{\mathbf{e}^\dagger}(Z)\mathbf{e}^\dagger.$$

In this paper, we denote $\Phi_{\mathbf{e}}(Z)$ and $\Phi_{\mathbf{e}^\dagger}(Z)$ by $Z_{\mathbf{e}}$ and $Z_{\mathbf{e}^\dagger}$, respectively. By the idempotent representation, we have the equality

$$\mathfrak{S}_0 = \{Z = Z_{\mathbf{e}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathbb{BC} \mid Z_{\mathbf{e}} Z_{\mathbf{e}^\dagger} = 0\} = \mathbb{C}\mathbf{e} \cup \mathbb{C}\mathbf{e}^\dagger.$$

For $Z = z_1 + z_2 j = Z_{\mathbf{e}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathbb{BC}$, we define three conjugations by

$$\bar{Z} = \overline{z_1} + \overline{z_2} j = \overline{Z_{\mathbf{e}^\dagger}}\mathbf{e} + \overline{Z_{\mathbf{e}}}\mathbf{e}^\dagger,$$

$$Z^\dagger = z_1 - z_2 j = Z_{\mathbf{e}^\dagger}\mathbf{e} + Z_{\mathbf{e}}\mathbf{e}^\dagger,$$

$$Z^* = \overline{z_1} - \overline{z_2}j = \overline{Z_{\mathbf{e}}}\mathbf{e} + \overline{Z_{\mathbf{e}^\dagger}}\mathbf{e}^\dagger$$

with respect to i, j, ij, respectively. These operations are involutive ring automorphisms.

2.3. Bicomplex polynomials

In this subsection, we recall fundamental properties of bicomplex polynomials. See [3, 4] for more details.

For a bicomplex polynomial P, $\Phi_{\mathbf{e}}(P(Z))$ and $\Phi_{\mathbf{e}^\dagger}(P(Z))$ are considered as complex polynomials in one variable, denoted by $P_{\mathbf{e}}(Z_{\mathbf{e}}), P_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})$, respectively. Then we have the idempotent representation of the polynomial P:

$$P(Z) = P_{\mathbf{e}}(Z_{\mathbf{e}})\mathbf{e} + P_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})\mathbf{e}^\dagger. \tag{2.1}$$

Note that the factorization of bicomplex polynomials is not unique but the idempotent representation is unique. By the idempotent representation, we define the two notions of zeros for bicomplex polynomials.

Definition. Let P be a bicomplex polynomial.

(i) We say that $\alpha \in \mathbb{BC}$ is a strong zero of P if $P(\alpha) = 0$ holds, equivalently $P_{\mathbf{e}}(\alpha_{\mathbf{e}}) = P_{\mathbf{e}^\dagger}(\alpha_{\mathbf{e}^\dagger}) = 0$ holds.

(ii) We say that $\alpha \in \mathbb{BC}$ is a weak zero of P if $P(\alpha) \in \mathfrak{S}_0$ holds, equivalently $P_{\mathbf{e}}(\alpha_{\mathbf{e}})P_{\mathbf{e}^\dagger}(\alpha_{\mathbf{e}^\dagger}) = 0$ holds.

Note that a strong zero of P is a weak zero of P. See Example 3.1 below for examples of bicomplex polynomials. The following lemmas play important roles in our study.

Lemma 2.2. *Let P be a bicomplex polynomial.*

(i) *If α, β are weak zeros of P satisfying $P(\alpha) \in \mathbb{C}\mathbf{e}^\dagger$ and $P(\beta) \in \mathbb{C}\mathbf{e}$, then $\alpha\mathbf{e} + \beta\mathbf{e}^\dagger$ is a strong zero of P, that is $P(\alpha\mathbf{e} + \beta\mathbf{e}^\dagger) = 0$.*

(ii) *If α is a strong zero of P, that is $P(\alpha) = 0$, then for any $\gamma \in \mathbb{BC}$, $\alpha\mathbf{e}+\gamma\mathbf{e}^\dagger, \gamma\mathbf{e}+\alpha\mathbf{e}^\dagger$ are weak zeros of P satisfying $P(\alpha\mathbf{e}+\gamma\mathbf{e}^\dagger) \in \mathbb{C}\mathbf{e}^\dagger$ and $P(\gamma\mathbf{e} + \alpha\mathbf{e}^\dagger) \in \mathbb{C}\mathbf{e}$.*

Lemma 2.3. *Let P be a bicomplex polynomial. If P has no strong zero, then either the complex polynomial $P_{\mathbf{e}}$ or $P_{\mathbf{e}^\dagger}$ is non-zero constant.*

3. j-self-inversive bicomplex polynomials

In this section, we study self-inversive bicomplex polynomials with respect to the j-conjugation. In this paper, we define the notion of self-inversivity by a symmetry of zeros. However, since in bicomplex algebra the factorization of polynomials is not unique, it is not easy to give a similar definition to the complex case. Thus we will define several levels for self-inversivity. Since it plays an important role to consider invariant elements by the operation $\frac{1}{Z^\dagger}$, we set

$$\mathcal{S}_\dagger = \{Z \in \mathbb{BC} \mid ZZ^\dagger = 1\} = \{Z \in \mathbb{BC} \mid Z_{\mathbf{e}}Z_{\mathbf{e}^\dagger} = 1\}.$$

Definition. Let P be a bicomplex polynomial.

(i) We say that P is generalized self-inversive with respect to the j-conjugation (generalized j-self-inversive) if P has at least one strong zero, and $P(\gamma) = 0$ implies $\gamma \notin \mathfrak{S}_0$ and $P\left(\frac{1}{\gamma^\dagger}\right) = 0$.

(ii) We say that P is weakly self-inversive with respect to the j-conjugation (weakly j-self-inversive) if P has at least one weak zero except $\mathfrak{S}_0$, and $P(\gamma) \in \mathfrak{S}_0$ and $\gamma \notin \mathfrak{S}_0$ imply $P\left(\frac{1}{\gamma^\dagger}\right) \in \mathfrak{S}_0$.

(iii) We say that P is strictly self-inversive with respect to the j-conjugation (strictly j-self-inversive) if there exist $\alpha \in \mathbb{BC}\setminus\mathfrak{S}_0$, $\alpha_1, \ldots, \alpha_p \in \mathcal{S}_\dagger$ and $\ell_1, \ldots, \ell_p \in \mathbb{N}$ such that P has a factorization

$$P(Z) = \alpha(Z - \alpha_1)^{\ell_1} \cdots (Z - \alpha_p)^{\ell_p}.$$

A generalized (resp. weakly) j-self-inversive polynomial has a symmetry of strong (resp. weak) zeros with respect to $\mathcal{S}_\dagger$. A strictly j-self-inversive polynomial has a symmetry of coefficients with respect to the j-conjugation, which we will see Theorem 3.2 below. In general, a strictly j-self-inversive polynomial is generalized j-self-inversive by Lemma 2.2.

Example 3.1.

(i) $P(Z) = Z^3 - \left(\frac{13}{3}\mathbf{e} + \frac{13}{2}\mathbf{e}^\dagger\right) Z^2 + \left(\frac{16}{3}\mathbf{e} + 12\mathbf{e}^\dagger\right) Z - \frac{4}{3}\mathbf{e} - \frac{9}{2}\mathbf{e}^\dagger$ is a generalized j-self-inversive polynomial. In fact, the strong zeros of P are

$$\frac{1}{3}\mathbf{e} + \frac{1}{2}\mathbf{e}^\dagger, \quad \frac{1}{3}\mathbf{e} + 3\mathbf{e}^\dagger, \quad 2\mathbf{e} + \frac{1}{2}\mathbf{e}^\dagger, \quad 2\mathbf{e} + 3\mathbf{e}^\dagger$$

(multiplicities are $1, 2, 2, 4$, respectively). P is also a weakly j-self-inversive polynomial. In fact, the weak zeros of P are

$$\frac{1}{3}\mathbf{e} + c\mathbf{e}^\dagger, \quad 2\mathbf{e} + c\mathbf{e}^\dagger, \quad c\mathbf{e} + \frac{1}{2}\mathbf{e}^\dagger, \quad c\mathbf{e} + 3\mathbf{e}^\dagger \quad (c \in \mathbb{C}).$$

(ii) $Q(Z) = Z^2 - \left(\frac{7}{3}\mathbf{e} + \frac{7}{2}\mathbf{e}^\dagger\right) Z + \left(\frac{2}{3}\mathbf{e} + \frac{3}{2}\mathbf{e}^\dagger\right)$ is a strictly j-self-inversive polynomial. In fact, Q has a factorization

$$Q(Z) = \left(Z - \left(2\mathbf{e} + \frac{1}{2}\mathbf{e}^\dagger\right)\right)\left(Z - \left(\frac{1}{3}\mathbf{e} + 3\mathbf{e}^\dagger\right)\right).$$

Note that Q has another factorization

$$Q(Z) = \left(Z - \left(2\mathbf{e} + 3\mathbf{e}^\dagger\right)\right)\left(Z - \left(\frac{1}{3}\mathbf{e} + \frac{1}{2}\mathbf{e}^\dagger\right)\right).$$

Q is also a generalized j-self-inversive polynomial. In fact, the strong zeros of Q are

$$\frac{1}{3}\mathbf{e} + \frac{1}{2}\mathbf{e}^\dagger, \quad \frac{1}{3}\mathbf{e} + 3\mathbf{e}^\dagger, \quad 2\mathbf{e} + \frac{1}{2}\mathbf{e}^\dagger, \quad 2\mathbf{e} + 3\mathbf{e}^\dagger$$

(multiplicities are $1, 1, 1, 1$, respectively).

First, we study the property of strictly j-self-inversive polynomials. Since the notion of strict j-self-inversivity corresponds to the one of complex self-inversivity, we obtain a result similar to Theorem 2.1 in the bicomplex case.

Theorem 3.2. *Let $P(Z) = \sum_{m=0}^{n} A_m Z^m$ be a bicomplex polynomial of degree $n \geq 1$ with $A_n \notin \mathfrak{S}_0$. The following four conditions are equivalent*:

(i) *P is strictly j-self-inversive,*

(ii) $A_n^\dagger A_m = A_0 A_{n-m}^\dagger$ $(m = 0, 1, 2, \dots, n)$,

(iii) $A_n^\dagger P(Z) = A_0 Z^n \left(P\left(\dfrac{1}{Z^\dagger}\right)\right)^\dagger$ $(Z \in \mathbb{BC} \setminus \mathfrak{S}_0)$,

(iv) $A_n^\dagger (nP(Z) - ZP'(Z)) = A_0 Z^{n-1} \left(P'\left(\dfrac{1}{Z^\dagger}\right)\right)^\dagger$ $(Z \in \mathbb{BC} \setminus \mathfrak{S}_0)$ *and* $A_n^\dagger A_n = A_0 A_0^\dagger$.

Proof. Since by $P(Z) = \sum_{m=0}^{n} A_m Z^m$ we have

$$A_0 Z^n \left(P\left(\frac{1}{Z^\dagger}\right)\right)^\dagger = \sum_{m=0}^{n} A_0 A_m^\dagger Z^{n-m},$$

$$A_n^\dagger (nP(Z) - ZP'(Z)) = \sum_{m=0}^{n} (n-m) A_n^\dagger A_m Z^m,$$

$$A_0 Z^{n-1} \left(P'\left(\frac{1}{Z^\dagger}\right)\right)^\dagger = \sum_{m=0}^{n} m A_0 A_m^\dagger Z^{n-m},$$

we obtain the equivalence among (ii), (iii), (iv). Note that the functional equation of (iv) is also obtained by differentiating the functional equation (iii). In fact, by differentiating the functional equation (iii) we obtain

$$A_n^\dagger P'(Z) = nA_0 Z^{n-1} \left(P\left(\frac{1}{Z^\dagger}\right)\right)^\dagger - A_0 Z^{n-2} \left(P'\left(\frac{1}{Z^\dagger}\right)\right)^\dagger.$$

By (iii) again, we obtain the functional equation of (iv).

Let us prove the equivalence between (i) and (iii). Assume that P is strictly j-self-inversive. Then there exist $\alpha \in \mathbb{BC} \setminus \mathfrak{S}_0$, $\alpha_1, \dots, \alpha_p \in \mathcal{S}_\dagger$ and $\ell_1, \dots, \ell_p \in \mathbb{N}$ such that P has a factorization

$$P(Z) = \alpha (Z - \alpha_1)^{\ell_1} \cdots (Z - \alpha_p)^{\ell_p}. \tag{3.1}$$

By (3.1) and the equalities $A_n = \alpha$, $A_0 = (-1)^n \alpha \alpha_1^{\ell_1} \cdots \alpha_p^{\ell_p}$, we have

$$\begin{aligned}
&A_0 Z^n \left(P\left(\frac{1}{Z^\dagger}\right)\right)^\dagger \\
&= A_0 Z^n \alpha^\dagger \left(\frac{1}{Z} - \alpha_1^\dagger\right)^{\ell_1} \cdots \left(\frac{1}{Z} - \alpha_p^\dagger\right)^{\ell_p} \\
&= (-1)^{\ell_1 + \cdots + \ell_p} \alpha \alpha_1^{\ell_1} \cdots \alpha_p^{\ell_p} Z^n A_n^\dagger \left(\frac{1}{Z} - \frac{1}{\alpha_1}\right)^{\ell_1} \cdots \left(\frac{1}{Z} - \frac{1}{\alpha_p}\right)^{\ell_p}
\end{aligned}$$

$$= A_n^\dagger \alpha (Z-\alpha_1)^{\ell_1} \cdots (Z-\alpha_p)^{\ell_p}$$
$$= A_n^\dagger P(Z).$$

Conversely, we assume the functional equation (iii). Comparing with the coefficient of degree n of (iii), we have $A_0 \notin \mathfrak{S}_0$. Since a strong zero of P is a divisor of $A_0 \notin \mathfrak{S}_0$, P has no strong zero on $\mathfrak{S}_0$. Since $A_n \notin \mathfrak{S}_0$ and $n \geq 1$, there exists a strong zero $\gamma \notin \mathfrak{S}_0$ of P. By (iii), $\frac{1}{\gamma^\dagger}$ is also a strong zero of P. Setting

$$\tilde{\gamma} = \gamma \mathbf{e} + \frac{1}{\gamma^\dagger} \mathbf{e}^\dagger = \gamma_{\mathbf{e}} \mathbf{e} + \frac{1}{\gamma_{\mathbf{e}}} \mathbf{e}^\dagger$$

or

$$\tilde{\gamma} = \frac{1}{\gamma^\dagger} \mathbf{e} + \gamma \mathbf{e}^\dagger = \frac{1}{\gamma_{\mathbf{e}^\dagger}} \mathbf{e} + \gamma_{\mathbf{e}^\dagger} \mathbf{e}^\dagger,$$

$\tilde{\gamma}$ is also a strong zero of P and $\tilde{\gamma} \in \mathcal{S}_\dagger$. By the factorization theorem, there exist a positive integer ℓ and a polynomial $\tilde{P}$ such that

$$P(Z) = (Z-\tilde{\gamma})^\ell \tilde{P}(Z), \quad \tilde{P}(\tilde{\gamma}) \neq 0.$$

By (iii), $\tilde{P}$ satisfies the functional equation

$$A_n^\dagger \tilde{P}(Z) = \frac{(-1)^\ell A_0}{\tilde{\gamma}^\ell} Z^{n-\ell} \left(\tilde{P}\left(\frac{1}{Z^\dagger} \right) \right)^\dagger.$$

By repeating this procedure, there exist $\alpha_1, \ldots, \alpha_p \in \mathcal{S}_\dagger$, $\ell_1, \ldots, \ell_p \in \mathbb{N}$ and a polynomial R such that P has a factorization

$$P(Z) = (Z-\alpha_1)^{\ell_1} \cdots (Z-\alpha_p)^{\ell_p} R(Z),$$

where R has no strong zero. Since the coefficient of the highest degree of R is $A_n \notin \mathfrak{S}_0$, R is constant by Lemma 2.3. Therefore P is strictly j-self-inversive. $\square$

By Theorem 3.2, for a strictly j-self-inversive polynomial P we obtain the explicit relation between $P_{\mathbf{e}}$ and $P_{\mathbf{e}^\dagger}$.

Corollary 3.3. *For any non-constant complex polynomial f, there exists a unique strictly j-self-inversive bicomplex polynomial P such that $P_{\mathbf{e}} = f$ (resp. $P_{\mathbf{e}^\dagger} = f$).*

Next, we study generalized j-self-inversive polynomials. We show that any generalized j-self-inversive polynomial P has a strictly j-self-inversive factor determined uniquely by only P.

Theorem 3.4. *Let P be a generalized j-self-inversive bicomplex polynomial.*

(i) *There exist distinct bicomplex numbers $\alpha_1, \ldots, \alpha_p \in \mathcal{S}_\dagger$, positive integers $\ell_1, \ldots, \ell_p \in \mathbb{N}$ and a polynomial R such that P has a factorization*

$$P(Z) = Q(Z) R(Z), \tag{3.2}$$

where Q is strictly j-self-inversive

$$Q(Z) = (Z-\alpha_1)^{\ell_1} \cdots (Z-\alpha_p)^{\ell_p},$$

and the quotient polynomial R has the property

$$R(\gamma) = 0 \quad \Longrightarrow \quad R\left(\frac{1}{\gamma^{\dagger}}\right) \notin \mathfrak{S}_0. \tag{3.3}$$

(ii) *The set $\alpha_1, \ldots, \alpha_p$ and $\ell_1, \ldots, \ell_p$ in* (i) *is uniquely determined by P.*
(iii) *If γ is a strong zero of R, then γ is also a strong zero of Q.*

Proof. (i) We can prove (i) similarly to the proof of Theorem 3.2. Assume that a polynomial P has strong zeros γ and $\frac{1}{\gamma^{\dagger}}$. Setting $\tilde{\gamma} = \gamma\mathbf{e} + \frac{1}{\gamma^{\dagger}}\mathbf{e}^{\dagger}$ or $\tilde{\gamma} = \frac{1}{\gamma^{\dagger}}\mathbf{e} + \gamma\mathbf{e}^{\dagger}$, $\tilde{\gamma}$ is also a strong zero of P and satisfies $\tilde{\gamma} \in \mathcal{S}_{\dagger}$. By the factorization theorem, there exist a positive number ℓ and a polynomial $\tilde{P}$ such that we have $P(Z) = (Z - \tilde{\gamma})^{\ell}\tilde{P}(Z)$ and $\tilde{P}(\tilde{\gamma}) \neq 0$. If there exists a strong zero γ' of $\tilde{P}$ such that $\frac{1}{\gamma'^{\dagger}}$ is a weak zero of $\tilde{P}$, we could repeat this procedure. Therefore we get a factorization (3.2) with the property (3.3).

(ii) Suppose that we have two factorizations

$$P(Z) = Q_1(Z)R_1(Z) = Q_2(Z)R_2(Z).$$

Here

$$Q_1(Z) = (Z - \alpha_1)^{\ell_1} \cdots (Z - \alpha_p)^{\ell_p}, \quad Q_2(Z) = (Z - \beta_1)^{m_1} \cdots (Z - \beta_q)^{m_q},$$

$\alpha_1, \ldots, \alpha_p \in \mathcal{S}_{\dagger}$ and $\beta_1, \ldots, \beta_q \in \mathcal{S}_{\dagger}$ are sets of distinct strong zeros of P, $\ell_1, \ldots, \ell_p$ and $m_1, \ldots, m_q$ are positive integers and R_1, R_2 are polynomials satisfying the property (3.3). By $P(\alpha_1) = 0$, we have $Q_2(\alpha_1)R_2(\alpha_1) = 0$.

(ii-a) In the case where $Q_2(\alpha_1) \notin \mathfrak{S}_0$, we have $R_2(\alpha_1) = 0$. By $\alpha_1 \in \mathcal{S}_{\dagger}$, we have $R_2\left(\frac{1}{\alpha_1^{\dagger}}\right) = 0$. This contradicts (3.3).

(ii-b) In the case where $Q_2(\alpha_1) \in \mathfrak{S}_0$, there exists a number r such that $\beta_{r\mathbf{e}} = \alpha_{1\mathbf{e}}$ or $\beta_{r\mathbf{e}^{\dagger}} = \alpha_{1\mathbf{e}^{\dagger}}$. By $\alpha_1, \beta_r \in \mathcal{S}_{\dagger}$, we have $r = 1$ and $\alpha_1 = \beta_r$.

We may assume $\alpha_1 = \beta_1$. By the same argument as (ii-b), we obtain $\ell_1 = m_1$. By repeating this argument, we obtain $p = q$, $\{\alpha_1, \ldots, \alpha_p\} = \{\beta_1, \ldots, \beta_q\}$ and $\ell_r = m_r$ $(r = 1, 2, \ldots, p)$.

(iii) Let γ be a strong zero of R. By (3.2), γ is a strong zero of P. Since P is generalized j-self-inversive, $\gamma \notin \mathfrak{S}_0$ and $\frac{1}{\gamma^{\dagger}}$ is also a strong zero of P. By $R\left(\frac{1}{\gamma^{\dagger}}\right) \notin \mathfrak{S}_0$ and (3.2), we have $Q\left(\frac{1}{\gamma^{\dagger}}\right) = 0$. Since Q is generalized j-self-inversive, we obtain $Q(\gamma) = 0$. □

Remark 3.5. In Theorem 3.4(iii), we also prove the following properties: $R(\gamma) \in \mathbb{C}\mathbf{e}$ implies $Q(\gamma) \in \mathbb{C}\mathbf{e}$, and $R(\gamma) \in \mathbb{C}\mathbf{e}^{\dagger}$ implies $Q(\gamma) \in \mathbb{C}\mathbf{e}^{\dagger}$, since Q is strictly j-self-inversive. We omit the detailed proof.

Example 3.6. In Example 3.1, P has a factorization

$$P(Z) = \left(Z - \left(2\mathbf{e} + \frac{1}{2}\mathbf{e}^{\dagger}\right)\right)\left(Z - \left(\frac{1}{3}\mathbf{e} + 3\mathbf{e}^{\dagger}\right)\right)\left(Z - \left(2\mathbf{e} + 3\mathbf{e}^{\dagger}\right)\right).$$

The strong zero of $R(Z) = Z - (2\mathbf{e} + 3\mathbf{e}^\dagger)$ is only $2\mathbf{e} + 3\mathbf{e}^\dagger$. Then we have

$$R\left(\frac{1}{3}\mathbf{e} + \frac{1}{2}\mathbf{e}^\dagger\right) = -\frac{5}{3}\mathbf{e} - \frac{5}{2}\mathbf{e}^\dagger \notin \mathfrak{S}_0.$$

Finally, we compare two notions of the weak and generalized j-self-inversivity.

Theorem 3.7. *Let P be a bicomplex polynomial. The following* (i) *and* (ii) *are equivalent*:

(i) *P is weakly j-self-inversive, P has a strong zero and P has no strong zero on $\mathfrak{S}_0$,*

(ii) *P is generalized j-self-inversive.*

Proof. Assume the condition (i). Since neither $P_{\mathbf{e}}$ nor $P_{\mathbf{e}^\dagger}$ are constant, we may suppose that the weak zeros of P are described as $a_r\mathbf{e} + c\mathbf{e}^\dagger$, $c\mathbf{e} + b_s\mathbf{e}^\dagger$ for some $a_r, b_s \in \mathbb{C}$ ($r = 1, 2, \ldots, p$, $s = 1, 2, \ldots, q$) and any $c \in \mathbb{C}$. Then the strong zeros of P are described as $a_r\mathbf{e} + b_s\mathbf{e}^\dagger$ ($r = 1, 2, \ldots, p$, $s = 1, 2, \ldots, q$). Since P has no strong zero on $\mathfrak{S}_0$, $a_r, b_s \neq 0$ ($r = 1, 2, \ldots, p$, $s = 1, 2, \ldots, q$). By the weak j-self-inversivity of P, $\frac{1}{c}\mathbf{e} + \frac{1}{a_r}\mathbf{e}^\dagger$ and $\frac{1}{b_s}\mathbf{e} + \frac{1}{c}\mathbf{e}^\dagger$ are also weak zeros of P for any $c \in \mathbb{C} \setminus \{0\}$. Then $\frac{1}{b_s}\mathbf{e} + \frac{1}{a_r}\mathbf{e}^\dagger$ ($r = 1, 2, \ldots, p$, $s = 1, 2, \ldots, q$) are also strong zeros of P. Therefore P is generalized j-self-inversive.

Conversely, assume the condition (ii). By the definition of the generalized j-self-inversivity, P has a strong zero and P has no strong zero on $\mathfrak{S}_0$. In particular, neither $P_{\mathbf{e}}$ nor $P_{\mathbf{e}^\dagger}$ are constant. We may suppose that the strong zeros of P are described as $a_r\mathbf{e} + b_s\mathbf{e}^\dagger$ ($r = 1, 2, \ldots, p$, $s = 1, 2, \ldots, q$). Then the weak zeros of P are described as $a_r\mathbf{e} + c\mathbf{e}^\dagger$, $c\mathbf{e} + b_s\mathbf{e}^\dagger$ for any $c \in \mathbb{C}$ ($r = 1, 2, \ldots, p$, $s = 1, 2, \ldots, q$). Since $\frac{1}{b_s}\mathbf{e} + \frac{1}{a_r}\mathbf{e}^\dagger$ is also a strong zero by the generalized j-self-inversivity, $\frac{1}{c}\mathbf{e} + \frac{1}{a_r}\mathbf{e}^\dagger$ and $\frac{1}{b_s}\mathbf{e} + \frac{1}{c}\mathbf{e}^\dagger$ are also weak zeros of P for any $c \in \mathbb{C} \setminus \{0\}$. Therefore P is weakly j-self-inversive. $\square$

Remark 3.8. For example, $Z(Z-1)$, $\mathbf{e}Z - ij$ and $\mathbf{e}$ are weakly j-self-inversive, but neither of these are generalized j-self-inversive.

Remark 3.9. In $\mathbb{BC}$, there exist eight involutive ring automorphisms $\sigma\colon \mathbb{BC} \longrightarrow \mathbb{BC}$ in total. For each σ, we can define the notion of "self-inversive bicomplex polynomials with respect to σ" by replacing the j-conjugation with the σ-action. We can obtain the same results in this section for self-inversive polynomials with respect to $\sigma(Z) = \overline{Z_{\mathbf{e}^\dagger}}\mathbf{e} + \overline{Z_{\mathbf{e}}}\mathbf{e}^\dagger$ (the i-conjugation), $\overline{Z_{\mathbf{e}^\dagger}}\mathbf{e} + Z_{\mathbf{e}}\mathbf{e}^\dagger$, $Z_{\mathbf{e}^\dagger}\mathbf{e} + \overline{Z_{\mathbf{e}}}\mathbf{e}^\dagger$ by parallel arguments. Moreover, we could obtain similar theorems for self-inversive polynomials with respect to $\sigma(Z) = Z$, $\overline{Z_{\mathbf{e}}}\mathbf{e} + \overline{Z_{\mathbf{e}^\dagger}}\mathbf{e}^\dagger$ (the ij-conjugation), $\overline{Z_{\mathbf{e}}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$, $Z_{\mathbf{e}}\mathbf{e} + \overline{Z_{\mathbf{e}^\dagger}}\mathbf{e}^\dagger$, which are proved directly by the idempotent representation (2.1) of bicomplex polynomials and properties of self-inversive and self-reciprocal complex polynomials. A result similar to Theorem 3.2 in the ij-conjugation case was obtained in [1].

4. Conclusion

We summarize this paper as follows:

- In this paper, we defined the notion of self-inversivity for bicomplex polynomials by a symmetry of zeros. However, since in bicomplex algebra the factorization of polynomials is not unique, it is not easy to give a similar definition to the complex case. We defined new several levels of self-inversivity with respect to j-conjugation: a generalized one, a weak one and a strict one.
- We proved some functional equations and relations of coefficients, which characterize strictly j-self-inversive bicomplex polynomials. This result corresponds to the similar result to the complex self-inversive polynomials.
- We proved that any generalized j-self-inversive bicomplex polynomial has a strictly j-self-inversive factor determined uniquely by it.
- We proved the relation between generalized j-self-inversive bicomplex polynomials and weakly j-self-inversive bicomplex polynomials.
- Although we focused the j-conjugation in our study, our argument can be applied for other conjugations.

For further research, we expect that our study could have good applications to the theory of error correcting codes and analytic number theory, similar to complex self-inversive polynomials in [2].

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