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# $S$-NOETHERIAN IN BI-AMALGAMATIONS

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Abstract. This paper establishes necessary and sufficient conditions for a bi-amalgamation to inherit the $S$-Noetherian property. The new results compare to previous works carried on various settings of duplications and amalgamations, and capitalize on recent results on bi-amalgamations. Our results allow us to construct new and original examples of rings satisfying the $S$-Noetherian property.

## 1. Introduction

All rings considered below are assumed to be commutative with identity and all modules are assumed to be unital. Let $A$ be a ring. A nonempty subset $S$ of $A$ is said to be a multiplicative subset if $1 \in S$ and for each $a, b \in S$ we have $ab \in S$. Throughout this note, $\text{Spec}(A)$ will denote the set of all prime ideals of $A$. Any concept and notation not defined here can be found in [1, 14]. In [1], Anderson and Dumitrescu introduced the concept of $S$-Noetherian rings as a generalization of Noetherian rings. Let $A$ be a ring, $S$ be a multiplicative set of $A$, and $E$ be an $A$-module. According to [1], we say that $E$ is *$S$-finite* if there exist a finitely generated submodule $F$ of $E$ and $s \in S$ such that $sE \subseteq F$. Also, we say that $E$ is *$S$-Noetherian* if each submodule of $E$ is $S$-finite. A ring $A$ is said to be *$S$-Noetherian* if it is $S$-Noetherian as an $A$-module (i.e., if each ideal of $A$ is $S$-finite). In [1], Anderson and Dumitrescu gave a number of $S$-variants of well-known results for Noetherian rings: $S$-versions of Cohen's result, the Eakin-Nagata theorem, the Hilbert Basis theorem, and under certain supplementary hypothesis, they studied the transfer of the $S$-Noetherian property to the ring of polynomials and the ring of formal power series. In [17], Liu studied when the ring of generalized power series is $S$-Noetherian. Finally, it is worthwhile recalling that, Lim and Oh investigated necessary and sufficient conditions for an amalgamated algebra to inherit the $S$-Noetherian property (see [16]). Recently $S$-Noetherian and $S$-version of many special rings, ideals and modules draw attention. See, for examples [12, 17, 20].

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Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let $J$ and $J'$ be two ideals of $B$ and $C$, respectively, such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamation (or bi-amalgamated algebra) of $A$ with $(B, C)$ along $(J, J')$ with respect to $(f, g)$ is the subring of $B \times C$ given by:

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}.$$

This construction is a natural generalization of duplications [4, 5, 9, 10] and amalgamations [6–8]. In [14], the authors introduce and provide original examples of bi-amalgamations and, in particular, show that Boisen–Sheldon's CPI-extensions [3] can be viewed as bi-amalgamations. They also show how every bi-amalgamation can arise as a natural pullback (or even as a conductor square) and then characterize pullbacks that can arise as bi-amalgamations. This allowed them to characterize Traverso's glueings of prime ideals [19, 21, 22, 24], which can be viewed as special bi-amalgamations. Then the last two sections deal, respectively, with the transfer of some basic ring theoretic properties to bi-amalgamations and the study of their prime ideal structures. All their results recover known results on duplications and amalgamations.

In [11], the authors investigate the transfer of the property of coherence in the bi-amalgamation. On the other hand, in [15], the authors establish necessary and sufficient conditions for a bi-amalgamation to inherit the arithmetical property, with applications on the weak global dimension and transfer of the semihereditary property. Also, in [18], the authors investigate the transfer of the notions of Gaussian and Prüfer rings to the bi-amalgamation and theirs results recover all well known results on amalgamations and generate new original examples of rings satisfying these properties. Suitable background on bi-amalgamated algebra is [11, 14, 15, 18].

In this paper, we pursue the investigation on the structure of the ring of the form $A \bowtie^{f,g} (J, J')$, with a particular attention to the $S$-Notherian ring property introduced and studied in [1]. More precisely, we give a necessary and sufficient condition for the bi-amalgamations to be $S$-Noetherian. Our results generalize the well-known results provided in [16] and [14]. Our main goal is to provide new classes of rings which satisfy this property.

For a set $U$, denote by $\langle U \rangle$ the ideal generated by $U$. We refer the reader to [23] for the undefined terminology and notation.

## 2. Main results

The first main result establishes necessary and sufficient conditions for a bi-amalgamation to inherit the $S$-Noetherian property. For this purpose, let us adopt the following notation: Let $S$ be a multiplicative subset of $A$. Clearly $f(S)$ and $g(S)$ are multiplicative subsets of $f(A)+J$ and $g(A)+J'$ respectively. Set $S' := \{(f(s), g(s)) \mid s \in S\}$, which is a multiplicative subset of $A \bowtie^{(f,g)} (J, J')$. For brevity, we say that a ring homomorphism $f : A \to B$ is said to be *$S$-finite* if $B$ is an $S$-finite $A$-module. According to [1], a multiplicative set $S$ of a ring $A$ is called *anti-archimedean* if $(\bigcap_{n \geq 1} s^n A) \cap S \neq \emptyset$ for every $s \in S$.

**Proposition 2.1.** *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two $S$-finite homomorphisms. Consider the following two conditions*:

(1) *$B$ is an $f(S)$-Noetherian ring and $C$ is a $g(S)$-Noetherian ring.*
(2) *$A \bowtie^{f,g} (J, J')$ is an $S'$-Noetherian ring.*

*Then* (1) *implies* (2) *and the converse is true if $S'$ is anti-archimedean.*

*Proof.* (1) ⇒ (2) By [1, Corollary 7], it suffices to show that $B \times C$ is $S'$-finite as an $A \bowtie^{f,g} (J, J')$-module. Since $B$ and $C$ are $S$-finite $A$-modules, there exist elements $s_1, s_2 \in S$ such that $f(s_1)B \subseteq f(A)b_1 + \cdots + f(A)b_n \subseteq B$ and $g(s_2)C \subseteq g(A)c_1 + \cdots + g(A)c_m \subseteq C$, where each $b_i \in B$ and $c_j \in C$. Put $s = s_1s_2$. For each $(b, c) \in B \times C$, there exist $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ in $A$ such that

$$\begin{aligned}(f(s), g(s))(b, c) &= (f(s)b, g(s)c)\\ &= (f(s_1s_2)b, 0) + (0, g(s_1s_2)c)\\ &= (f(s_2), g(s_2))(f(s_1)b, 0) + (f(s_1), g(s_1))(0, g(s_2)c)\\ &= \sum_{i=1}^{n} \big(f(\alpha_i), g(\alpha_i))(f(s_2), g(s_2)\big)(b_i, 0)\\ &\quad + \sum_{j=1}^{m} \big(f(\beta_j), g(\beta_j))(f(s_1), g(s_1)\big)(0, c_j)\\ &\in \big\langle \{(b_i, 0), (0, c_j) \mid 1 \le i \le n, 1 \le j \le m\}\big\rangle.\end{aligned}$$

Therefore we obtain

$$(f(s), g(s))B \times C \subseteq \big\langle \{(b_i, 0), (0, c_j) \mid 1 \le i \le n, 1 \le j \le m\}\big\rangle \subseteq B \times C.$$

So $B \times C$ is an $S'$-finite $A \bowtie^{f,g} (J, J')$-module.

(2) ⇒ (1) Under the additional hypothesis that $S'$ is anti-archimedean, the result follows immediately from [1, Proposition 9] and the fact that $B \times C = A \bowtie^{f,g} (J, J')[w_1, \ldots, w_n]$, where $w_1, \ldots, w_n \in B \times C$. □

Our next goal is to describe more precisely the $S$-Noetherianity of $A \bowtie^{(f,g)} (J, J')$ by using its pullback structure. In fact, $A \bowtie^{f,g} (J, J')$ is the pullback $\alpha \times_{A/I} \beta$, where $I := f^{-1}(J) = g^{-1}(J')$, $\alpha : f(A) + J \longrightarrow A/I$ and $\beta : g(A) + J' \longrightarrow A/I$ (see [14, Section 3]).

**Theorem 2.2.** *Under the above notation. The following conditions are equivalent.*

(1) *$A \bowtie^{(f,g)} (J, J')$ is an $S'$-Noetherian ring.*

(2) *$f(A) + J$ is an $f(S)$-Noetherian ring and $J'$ is an $S'$-Noetherian $A \bowtie^{(f,g)} (J, J')$-module* (*with the $A \bowtie^{(f,g)} (J, J')$-module structure naturally induced by the surjective canonical homomorphism $A \bowtie^{(f,g)} (J, J') \twoheadrightarrow g(A) + J'$*).

(3) *$f(A) + J$ is an $f(S)$-Noetherian ring and $g(A) + J'$ is a $g(S)$-Noetherian ring.*

The proof of the above theorem involves the following lemma, which is of independent interest.

Let $D$ be a pullback of $\lambda$ and $\mu$ as in [16, Definition 2.2]. We denote the restriction to $D$ of the projection of $A \times B$ onto $A$ (resp., $B$) by $p_A$ (resp., $p_B$).

**Lemma 2.3** ([16, Proposition 2.3]). *Let $D$ be the pullback of $\lambda$ and $\mu$ as in* [16, Definition 2.2]*, where $\mu$ is surjective. If $S$ is a multiplicative subset of $D$, then the following*

(1) *$D$ is an $S$-Noetherian ring.*

(2) *$A$ is a $p_A(S)$-Noetherian ring and $Ker(\mu)$ is an $S$-Noetherian $D$-module.*

*Proof of Theorem 2.2.* (1) $\Leftrightarrow$ (2) This follows immediately from Lemma 2.3.

(1) $\Rightarrow$ (3) Assume that $A \bowtie^{(f,g)} (J, J')$ is $S'$-Noetherian. By [14, Proposition 4.1(2)], $\frac{A\bowtie^{(f,g)}(J,J')}{0\times J'} \simeq f(A) + J$ and $\frac{A\bowtie^{(f,g)}(J,J')}{J\times 0} \simeq g(A) + J'$. The result follows immediately from [17, Lemma 2.2] since $p_1(S') = f(S)$ and $p_2(S') = g(S)$, where $p_1$ and $p_2$ denote the surjective canonical homomorphisms $A \bowtie^{(f,g)} (J, J') \twoheadrightarrow f(A) + J$ and $A \bowtie^{(f,g)} (J, J') \twoheadrightarrow g(A) + J'$ respectively.

(3) $\Rightarrow$ (2) Let $K$ be an $A \bowtie^{(f,g)} (J, J')$-submodule of $J'$. Clearly $K$ is an ideal of $g(A) + J'$. Since $g(A) + J'$ is $g(S)$-Noetherian, there exist $s \in S$ and $k_1, \ldots, k_n \in K$ such that $g(s)K \subseteq (g(A) + J')k_1 + \cdots + (g(A) + J')k_n \subseteq K$. Then $(f(s), g(s))K \subseteq (A \bowtie^{(f,g)} (J, J'))k_1 + \cdots + (A \bowtie^{(f,g)} (J, J'))k_n \subseteq K$. Therefore $K$ is an $S'$-finite $A \bowtie^{(f,g)} (J, J')$-module. Finally $J'$ is an $S'$-Noetherian $A \bowtie^{(f,g)} (J, J')$-module. $\square$

When $S$ consists of units of $A$, we reobtain [14, Proposition 3.2].

**Corollary 2.4.** *Under the above notation, we have: $A \bowtie^{f,g} (J, J')$ is Noetherian if and only if $f(A) + J$ and $g(A) + J'$ are Noetherian.*

Recall that $A \bowtie^{(i,f)} (I, J) = A \bowtie^f J$, where $i = id_A$ and $I = f^{-1}(J)$. Put $S' := \{(s, f(s)) \mid s \in S\}$. Then $S'$ is a multiplicative subset of $A \bowtie^f J$. In this case, we reobtain the special case of amalgamated algebras, as recorded in the next corollary.

**Corollary 2.5** ([16, Theorem 3.2]). *Let $f : A \longrightarrow B$ be a ring homomorphism, $J$ an ideal of $B$, and $S$ a multiplicative subset of $A$. Then the following statements are equivalent.*

(1) *$A \bowtie^f J$ is an $S'$-Noetherian ring.*

(2) *$A$ is an $S$-Noetherian ring and $J$ is an $S'$-Noetherian $A \bowtie^f J$-module.*

(3) *$A$ is an $S$-Noetherian ring and $f(A) + J$ is an $f(S)$-Noetherian ring.*

By using [1, Corollary 5], we give a more useful criterion for the $S'$-Noetherian property of $A \bowtie^{f,g} (J, J')$. To do this, we need to recall the form of the prime ideals of the bi-amalgamations. For this, let us adopt the following notations: For $L \in \mathrm{Spec}(f(A) + J)$ and $L' \in \mathrm{Spec}(g(A) + J')$, consider the prime ideals of $A \bowtie^{f,g} (J, J')$ given by:

$\bar{L} := (L \times (g(A) + J')) \cap A \bowtie^{f,g} (J, J')$

$$= \{(f(a)+j, g(a)+j') \mid a \in A, (j,j') \in J \times J' \text{ and } f(a)+j \in L\};$$

$$\bar{L}' := ((f(A)+J) \times L') \cap A \bowtie^{f,g} (J,J')$$
$$= \{(f(a)+j, g(a)+j') \mid a \in A, (j,j') \in J \times J' \text{ and } g(a)+j' \in L'\}.$$

**Lemma 2.6** ([14, Proposition 5.3]). *Under the above notation, let $\mathcal{P}$ be a prime ideal of $A \bowtie^{f,g} (J,J')$. Then*

(1) *$J \times J' \subseteq \mathcal{P}$ if and only if there exists a unique $P \in \operatorname{Spec}(A)$ with $P \supseteq I$ such that $\mathcal{P} = P \bowtie^{f,g} (J,J')$, where $I := f^{-1}(J) = g^{-1}(J')$. In this case, there exist $L \in \operatorname{Spec}(f(A)+J)$ with $L \supseteq J$ and $L' \in \operatorname{Spec}(g(A)+J')$ with $L' \supseteq J'$ such that $\mathcal{P} = \bar{L} = \bar{L}'$.*

(2) *$J \times J' \nsubseteq \mathcal{P}$ if and only if there exists a unique $L$ in $\operatorname{Spec}(f(A)+J)$ (or $L \in \operatorname{Spec}(g(A)+J)$) such that $J \nsubseteq L$ (or $J' \nsubseteq L$) and $\mathcal{P} = \bar{L}$.*

*Consequently, we have*:

$$\operatorname{Spec}(A \bowtie^{f,g} (J,J') = \{\bar{L} \mid L \in \operatorname{Spec}(f(A)+J) \cup \operatorname{Spec}(g(A)+J)\}.$$

**Theorem 2.7.** *Under the above notation, assume that $A$ is an $S$-Noetherian ring and $B$ (resp., $C$) is an $S$-finite $A$-module (with the $A$-module structure induced by $f$ (resp., by $g$)). Then $A \bowtie^{f,g} (J,J')$ is an $S'$-Noetherian ring.*

*Proof.* Let $\mathcal{P}$ be a prime ideal of $A \bowtie^{f,g} (J,J')$ disjoint from $S'$. By Lemma 2.6, we have two cases:

**Case 1.** $\mathcal{P}$ has the form $P \bowtie^{f,g} (J,J')$, where $P$ is a prime ideal of $A$ disjoint from $S$. Since $A$ is $S$-Noetherian, there exist $s_1 \in S$ and $a_1, a_2, \ldots, a_n \in P$ such that $s_1 P \subseteq \langle \{a_1, a_2, \ldots, a_n\} \rangle \subseteq P$. Since $B$ is an $S$-finite $A$-module, there exist an element $s_2 \in S$ and $j_1, \ldots, j_m \in J$ such that $s_2 J \subseteq A.j_1 + \cdots + A.j_m \subseteq J$ (or $f(s_2)J \subseteq f(A)j_1 + \cdots + f(A)j_m \subseteq J$). Also $C$ is an $S$-finite $A$-module, there exist $s_3 \in S$ and $j'_1, \ldots, j'_p \in J'$ such that $s_3 J' \subseteq A.j'_1 + \cdots + A.j'_p \subseteq J'$ (or $g(s_3)J' \subseteq g(A)j'_1 + \cdots + g(A)j'_p \subseteq J'$). Put $s = s_1 s_2 s_3$. Let $(f(p)+j, g(p)+j') \in P \bowtie^{f,g} (J,J')$. So $p \in P$, and thus there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ such that $sp = \sum_{i=1}^{n} \alpha_i a_i$. Also $j \in J$ and $j' \in J'$ implies that there exist $\beta_1, \beta_2, \ldots, \beta_m$ and $\gamma_1, \gamma_2, \ldots, \gamma_p$ in $A$ such that $jf(s) = \sum_{k=1}^{m} f(\beta_k) j_k$ and $j' g(s) = \sum_{l=1}^{q} g(\gamma_l) j'_l$. Then

$$\begin{aligned}
&(f(s), g(s))(f(p)+j, g(p)+j') \\
&= (f(sp) + jf(s), g(sp) + j'g(p)) \\
&= \big(\sum_{i=1}^{n} f(\alpha_i) f(a_i) + \sum_{k=1}^{m} f(\beta_k) j_k,\ \sum_{i=1}^{n} g(\alpha_i) g(a_i) + \sum_{l=1}^{q} g(\gamma_l) j'_l\big) \\
&= \sum_{i=1}^{n} (f(\alpha_i), g(\alpha_i))(f(a_i), g(a_i)) + \sum_{k=1}^{m} (f(\beta_k), g(\beta_k))(j_k, 0) \\
&\quad + \sum_{l=1}^{q} (f(\gamma_l), g(\gamma_l))(0, j'_l) \\
&\in \big\langle \{(f(a_i), g(a_i)), (j_k, 0), (0, j'_l) \mid 1 \le i \le n,\ 1 \le k \le m,\ 1 \le l \le q\} \big\rangle.
\end{aligned}$$

**Case 2.** $\mathcal{P} = \bar{L}$ for some prime ideal $L$ of $f(A) + J$ or $\mathcal{P} = \bar{L}'$ for some prime ideal $L'$ of $g(A) + J'$. Without loss of generality we may assume that $\mathcal{P} = \bar{L}$. Note that $p_B(\bar{L}) = \{f(a) + j \mid (f(a) + j, g(a) + j') \in \bar{L}\}$ is an $A$-submodule of $B$. As $B$ is an $S$-finite $A$-module and $A$ is an $S$-Noetherian ring, there exist $s_1 \in S$ and $f(a_1) + j_1, f(a_2) + j_2, \ldots, f(a_n) + j_n \in p_B(\bar{L})$ such that

$$f(s_1)p_B(\bar{L}) \subseteq f(A)(f(a_1) + j_1) + \cdots + f(A)(f(a_n) + j_n) \subseteq p_B(\bar{L}).$$

Let $(f(a) + j, g(a) + j') \in \bar{L}$. Then

$$(f(s_1), g(s_1))(f(a) + j, g(a) + j') = \big(\sum_{i=1}^{n} f(\alpha_i)(f(a_i) + j_i), g(s_1)g(a) + g(s_1)j'\big)$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$. Note that $g(s_1)g(a) - \sum_{i=1}^{n} g(\alpha_i)g(a_1) \in J'$, and hence $s_1 a - \sum_{i=1}^{n} \alpha_i a_1 \in g^{-1}(J')$. As $g^{-1}(J')$ is an ideal of $A$ and $A$ is an $S$-Noetherian ring, there exist an element $s_2 \in S$ and $b_1, b_2, \ldots, b_m \in A$ such that

$$s_2 g^{-1}(J') \subseteq \langle \{b_1, b_2, \ldots, b_m\} \rangle \subseteq g^{-1}(J').$$

Hence $s_2(s_1 a - \sum_{i=1}^{n} \alpha_i a_1) = \sum_{k=1}^{m} \beta_k b_k$ for some $\beta_1, \beta_2, \ldots, \beta_m \in A$. So $s_2 s_1 a = s_2 \sum_{i=1}^{n} \alpha_i a_i + \sum_{k=1}^{m} \beta_k b_k$. Moreover, as $C$ is an $S$-finite $A$-module, there exists $s_3 \in S$ such that $g(s_3)J' \subseteq g(A)j'_1 + g(A)j'_2 + \cdots + g(A)j'_p \subseteq J'$. Putting $s = s_1 s_2 s_3$, we have:

$$\begin{aligned}
&(f(s), g(s))(f(a) + j, g(a) + j') \\
&= \big(f(s_2 s_3)(\sum_{i=1}^{n} f(\alpha_i)(f(a_i) + j_i)), g(s_1 s_2 s_3)g(a) + g(s_1 s_2 s_3)j'\big) \\
&= \big(\sum_{i=1}^{n} f(s_2 s_3)f(\alpha_i)f(a_i) + \sum_{i=1}^{n} f(s_2 s_3)f(\alpha_i)j_i, \\
&\quad g(s_2 s_3)(\sum_{i=1}^{n} g(\alpha_i)g(a_i)) + g(s_3)(\sum_{k=1}^{m} g(\beta_k)g(b_k) + g(s_1 s_2)(\sum_{l=1}^{p} g(\eta_l)j'_l)\big) \\
&= \sum_{i=1}^{n} \big(f(s_2 s_3)f(\alpha_i), g(s_2 s_3)g(\alpha_i)\big)(f(a_i), g(a_i)) \\
&\quad + \sum_{i=1}^{n} \big(f(s_2 s_3)f(\alpha_i), g(s_2 s_3)g(\alpha_i)\big)(j_i, 0) \\
&\quad + \sum_{i=k}^{m} \big(f(s_3)f(\beta_k), g(s_3)g(\beta_k)\big)(0, g(b_k)) \\
&\quad + \sum_{l=1}^{p} \big(f(s_1 s_2)f(\eta_l), g(s_1 s_2)g(\eta_l)\big)(0, j'_l) \\
&\in \big\langle \{(f(a_i), g(a_i)), (j_i, 0), (0, g(b_k)), (0, j'_l) \mid 1 \le i \le n,\ 1 \le k \le m, \text{ and } 1 \le l \le p\} \big\rangle.
\end{aligned}$$

Then it follows from all cases that $\mathcal{P}$ is $S'$-finite, which implies that $A \bowtie^{f,g} (J, J')$ is an $S'$-Noetherian ring. $\square$

We close this note by the following examples.

**Example 2.8.** Let $A$ be an $S$-Noetherian ring and $I, K$ two proper ideals of $A$ such that $I \subseteq K$. Set $B := A/I$ and $C := A \times A$. Let $f : A \to B$ be the canonical ring homomorphism and $g : A \to C$ be the canonical embedding on the first component. Consider $J := K/I$ and $J' := K \times 0$ the ideals of $B$ and $C$ respectively. Note that $B$ and $C$ are $S$-finite as $A$-modules. Then by Theorem 2.7, $A \bowtie^{f,g} (J, J')$ is an $S'$-Noetherian ring.

The following example uses the $A \propto E$ construction introduced and studied in [13]. Let $A$ be a ring and $E$ an $A$-module. Then $A \propto E = A \times E$ is a ring with identity $(1, 0)$ under addition defined by $(a, e) + (b, f) = (a + b, e + f)$ and multiplication defined by $(a, e)(b, f) = (ab, af + be)$. See for instance [2, 10].

**Example 2.9.** Let $A$ be an $S$-Noetherian ring and $E$ and $E'$ two $A$-modules such that $E \neq E'$ which are $S$-finite. Consider the natural injective ring homomorphisms $f : A \to A \propto E$ and $g : A \to A \propto E'$ and set $J := I \propto E$ and $J' := I \propto E'$ to be two ideals of $A \propto E$ and $A \propto E'$ respectively for some ideal $I$ of $A$. We claim that the bi-amalgamation $A \bowtie^{f,g} (J, J')$ is an $S'$-Noetherian ring. Indeed, notice first that $f^{-1}(J) = g^{-1}(J') = I$, $f(A) + J = A \propto E$, and $g(A) + J = A \propto E'$. Furthermore, $f(A) + J$ is an $f(S)$-Noetherian ring and $g(A) + J'$ is a $g(S)$-Noetherian ring by [16, Theorem 3.8.]. Now the assertion follows immediately from Theorem 2.2(3).

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