S-NOETHERIAN IN BI-AMALGAMATIONS

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Abstract. This paper establishes necessary and sufficient conditions for a bi-amalgamation to inherit the S-Noetherian property. The new results compare to previous works carried on various settings of duplications and amalgamations, and capitalize on recent results on bi-amalgamations. Our results allow us to construct new and original examples of rings satisfying the S-Noetherian property.

1. Introduction

All rings considered below are assumed to be commutative with identity and all modules are assumed to be unital. Let $A$ be a ring. A nonempty subset $S$ of $A$ is said to be a multiplicative subset if $1 \in S$ and for each $a, b \in S$ we have $ab \in S$. Throughout this note, $\text{Spec}(A)$ will denote the set of all prime ideals of $A$. Any concept and notation not defined here can be found in [1,14].

In [1], Anderson and Dumitrescu introduced the concept of S-Noetherian rings as a generalization of Noetherian rings. Let $A$ be a ring, $S$ be a multiplicative set of $A$, and $E$ be an $A$-module. According to [1], we say that $E$ is $S$-finite if there exist a finitely generated submodule $F$ of $E$ and $s \in S$ such that $sE \subseteq F$. Also, we say that $E$ is $S$-Noetherian if each submodule of $E$ is $S$-finite. A ring $A$ is said to be $S$-Noetherian if it is $S$-Noetherian as an $A$-module (i.e., if each ideal of $A$ is $S$-finite). In [1], Anderson and Dumitrescu gave a number of $S$-variants of well-known results for Noetherian rings: $S$-versions of Cohen’s result, the Eakin-Nagata theorem, the Hilbert Basis theorem, and under certain supplementary hypothesis, they studied the transfer of the $S$-Noetherian property to the ring of polynomials and the ring of formal power series. In [17], Liu studied when the ring of generalized power series is $S$-Noetherian. Finally, it is worthwhile recalling that, Lim and Oh investigated necessary and sufficient conditions for an amalgamated algebra to inherit the $S$-Noetherian property (see [16]). Recently $S$-Noetherian and $S$-version of many special rings, ideals and modules draw attention. See, for examples [12,17,20].

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Let \( f : A \to B \) and \( g : A \to C \) be two ring homomorphisms and let \( J \) and \( J' \) be two ideals of \( B \) and \( C \), respectively, such that \( f^{-1}(J) = g^{-1}(J') \). The bi-amalgamation (or bi-amalgamated algebra) of \( A \) with \((B,C)\) along \((J,J')\) with respect to \((f,g)\) is the subring of \( B \times C \) given by:

\[
A \triangledown f,g (J,J') := \{ (f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J' \}.
\]

This construction is a natural generalization of duplications \([4, 5, 9, 10]\) and amalgamations \([6–8]\). In \([14]\), the authors introduce and provide original examples of bi-amalgamations and, in particular, show that Boisen–Sheldon’s CPI-extensions \([3]\) can be viewed as bi-amalgamations. They also show how every bi-amalgamation can arise as a natural pullback (or even as a conductor square) and then characterize pullbacks that can arise as bi-amalgamations. This allowed them to characterize Traverso’s glueings of prime ideals \([19, 21, 22, 24]\), which can be viewed as special bi-amalgamations. Then the last two sections deal, respectively, with the transfer of some basic ring theoretic properties to bi-amalgamations and the study of their prime ideal structures. All their results recover known results on duplications and amalgamations.

In \([11]\), the authors investigate the transfer of the property of coherence in the bi-amalgamation. On the other hand, in \([15]\), the authors establish necessary and sufficient conditions for a bi-amalgamation to inherit the arithmetical property, with applications on the weak global dimension and transfer of the semiheredity. Also, in \([18]\), the authors investigate the transfer of the notions of Gaussian and Pr"{u}fer rings to the bi-amalgamation and theirs results recover all well known results on amalgamations and generate new original examples of rings satisfying these properties. Suitable background on bi-amalgamated algebra is \([11, 14, 15, 18]\).

In this paper, we pursue the investigation on the structure of the ring of the form \( A \triangledown f,g (J,J') \), with a particular attention to the \( S \)-Notherian ring property introduced and studied in \([1]\). More precisely, we give a necessary and sufficient condition for the bi-amalgamations to be \( S \)-Noetherian. Our results generalize the well-known results provided in \([16]\) and \([14]\). Our main goal is to provide new classes of rings which satisfy this property.

For a set \( U \), denote by \( \langle U \rangle \) the ideal generated by \( U \). We refer the reader to \([23]\) for the undefined terminology and notation.

2. Main results

The first main result establishes necessary and sufficient conditions for a bi-amalgamation to inherit the \( S \)-Noetherian property. For this purpose, let us adopt the following notation: Let \( S \) be a multiplicative subset of \( A \). Clearly \( f(S) \) and \( g(S) \) are multiplicative subsets of \( f(A) + J \) and \( g(A) + J' \) respectively. Set \( S' := \{ (f(s), g(s)) \mid s \in S \} \), which is a multiplicative subset of \( A \triangledown f,g (J,J') \). For brevity, we say that a ring homomorphism \( f : A \to B \) is said to be \( S \)-finite if \( B \) is an \( S \)-finite \( A \)-module. According to \([1]\), a multiplicative set \( S \) of a ring \( A \) is called anti-archimedean if \( \bigcap_{n \geq 1} s^n A \cap S \neq \emptyset \) for every \( s \in S \).
Proposition 2.1. Let \( f : A \to B \) and \( g : A \to C \) be two \( S \)-finite homomorphisms. Consider the following two conditions:

1. \( B \) is an \( f(S) \)-Noetherian ring and \( C \) is a \( g(S) \)-Noetherian ring.
2. \( A \cong_{f,g} (J,J') \) is an \( S' \)-Noetherian ring.

Then (1) implies (2) and the converse is true if \( S' \) is anti-archimedean.

Proof. (1) \(\Rightarrow\) (2) By [1, Corollary 7], it suffices to show that \( B \times C \) is \( S' \)-finite as an \( A \cong_{f,g} (J,J') \)-module. Since \( B \) and \( C \) are \( S \)-finite \( A \)-modules, there exist elements \( s_1, s_2 \in S \) such that \( f(s_1)B \subseteq f(A)b_1 + \cdots + f(A)b_n \subseteq B \) and \( g(s_2)C \subseteq g(A)c_1 + \cdots + g(A)c_m \subseteq C \), where each \( b_i \in B \) and \( c_j \in C \). Put \( s = s_1s_2 \). For each \((b,c) \in B \times C\), there exist \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_m \in A \) such that

\[
(f(s), g(s)) \in (f(s)b, g(s)c) \subseteq (f(s_1s_2)b_0, 0) + (0, g(s_1s_2)c) = (f(s_2), g(s_2))((f(s_1)b, 0) + (f(s_1), g(s_1))(0, g(s_2)c) = \sum_{i=1}^{n} (f(\alpha_i), g(\alpha_i))(f(s_2), g(s_2))(b_i, 0) + \sum_{j=1}^{m} (f(\beta_j), g(\beta_j))(f(s_1), g(s_1))(0, c_j) \in \langle \{(b_i, 0), (0, c_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\} \rangle.
\]

Therefore we obtain

\[
(f(s), g(s))B \times C \subseteq \langle \{(b_i, 0), (0, c_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\} \rangle \subseteq B \times C.
\]

So \( B \times C \) is an \( S' \)-finite \( A \cong_{f,g} (J,J') \)-module.

(2) \(\Rightarrow\) (1) Under the additional hypothesis that \( S' \) is anti-archimedean, the result follows immediately from [1, Proposition 9] and the fact that \( B \times C = A \cong_{f,g} (J,J')[w_1, \ldots, w_n] \), where \( w_1, \ldots, w_n \in B \times C \). \(\square\)

Our next goal is to describe more precisely the \( S \)-Noetherianity of \( A \cong_{f,g} (J,J') \) by using its pullback structure. In fact, \( A \cong_{f,g} (J,J') \) is the pullback \( \alpha \times_{A/I} \beta \), where \( I := f^{-1}(J) = g^{-1}(J') \), \( \alpha : f(A) + J \to A/I \) and \( \beta : g(A) + J' \to A/I \) (see [14, Section 3]).

Theorem 2.2. Under the above notation. The following conditions are equivalent.

1. \( A \cong_{f,g} (J,J') \) is an \( S' \)-Noetherian ring.
2. \( f(A) + J \) is an \( f(S) \)-Noetherian ring and \( J' \) is an \( S' \)-Noetherian \( A \cong_{f,g} (J,J') \)-module (with the \( A \cong_{f,g} (J,J') \)-module structure naturally induced by the surjective canonical homomorphism \( A \cong_{f,g} (J,J') \to g(A) + J' \)).
3. \( f(A) + J \) is an \( f(S) \)-Noetherian ring and \( g(A) + J' \) is a \( g(S) \)-Noetherian ring.
The proof of the above theorem involves the following lemma, which is of independent interest. Let $D$ be a pullback of $\lambda$ and $\mu$ as in [16, Definition 2.2]. We denote the restriction to $D$ of the projection of $A \times B$ onto $A$ (resp., $B$) by $p_A$ (resp., $p_B$).

**Lemma 2.3** ([16, Proposition 2.3]). Let $D$ be the pullback of $\lambda$ and $\mu$ as in [16, Definition 2.2], where $\mu$ is surjective. If $S$ is a multiplicative subset of $D$, then the following

1. $D$ is an $S$-Noetherian ring.
2. $A$ is a $p_A(S)$-Noetherian ring and $\text{Ker}(\mu)$ is an $S$-Noetherian $D$-module.

**Proof of Theorem 2.2.** (1) $\iff$ (2) This follows immediately from Lemma 2.3.

(1) $\implies$ (3) Assume that $A \bowtie f,g(J,J')$ is $S$-Noetherian. By [14, Proposition 4.1(2)],

\[ \frac{A \bowtie f,g(J,J')}{\text{Ann}(J,J')} \simeq f(A) + J \quad \text{and} \quad \frac{A \bowtie f,g(J,J')}{\text{Ann}(J,J')} \simeq g(A) + J'. \]

The result follows immediately from [17, Lemma 2.2] since $p_1(S') = f(S)$ and $p_2(S') = g(S)$, where $p_1$ and $p_2$ denote the surjective canonical homomorphisms $A \bowtie f,g(J,J') \twoheadrightarrow f(A) + J$ and $A \bowtie f,g(J,J') \twoheadrightarrow g(A) + J'$ respectively.

(3) $\implies$ (2) Let $K$ be an $A \bowtie f,g(J,J')$-submodule of $J'$. Clearly $K$ is an ideal of $g(A) + J'$. Since $g(A) + J'$ is $g(S)$-Noetherian, there exist $s \in S$ and $k_1, \ldots, k_n \in K$ such that $g(s)K \subseteq (g(A) + J')k_1 + \cdots + (g(A) + J')k_n \subseteq K$. Then $(f(s), g(s))K \subseteq (A \bowtie f,g(J,J'))k_1 + \cdots + (A \bowtie f,g(J,J'))k_n \subseteq K$. Therefore $K$ is an $S'$-finite $A \bowtie f,g(J,J')$-module. Finally $J'$ is an $S'$-Noetherian $A \bowtie f,g(J,J')$-module.

When $S$ consists of units of $A$, we reobtain [14, Proposition 3.2].

**Corollary 2.4.** Under the above notation, we have $A \bowtie f,g(J,J')$ is Noetherian if and only if $f(A) + J$ and $g(A) + J'$ are Noetherian.

Recall that $A \bowtie f(J,J) = A \bowtie f J$, where $i = \text{id}_A$ and $I = f^{-1}(J)$. Put $S' := \{(s, f(s)) | s \in S\}$. Then $S'$ is a multiplicative subset of $A \bowtie f J$. In this case, we reobtain the special case of amalgamated algebras, as recorded in the next corollary.

**Corollary 2.5** ([16, Theorem 3.2]). Let $f : A \rightarrow B$ be a ring homomorphism, $J$ an ideal of $B$, and $S$ a multiplicative subset of $A$. Then the following statements are equivalent.

1. $A \bowtie f J$ is an $S'$-Noetherian ring.
2. $A$ is an $S$-Noetherian ring and $J$ is an $S'$-Noetherian $A \bowtie f J$-module.
3. $A$ is an $S$-Noetherian ring and $f(A) + J$ is an $f(S)$-Noetherian ring.

By using [1, Corollary 5], we give a more useful criterion for the $S'$-Noetherian property of $A \bowtie f,J(J,J')$. To do this, we need to recall the form of the prime ideals of the bi-amalgamations. For this, let us adopt the following notations: For $L \in \text{Spec}(f(A) + J)$ and $L' \in \text{Spec}(g(A) + J')$, consider the prime ideals of $A \bowtie f,J(J,J')$ given by:

\[ \bar{L} := (L \times (g(A) + J')) \cap A \bowtie f,J(J,J') \]
= \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J' \text{ and } f(a) + j \in L\};
\hat{L}' := ((f(A) + J) \times \hat{L}') \cap A \bowtie J ( J, J')
= \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J' \text{ and } g(a) + j' \in \hat{L}'\}.

Lemma 2.6 ([14, Proposition 5.3]). Under the above notation, let \(\mathcal{P}\) be a prime ideal of \(A \bowtie J ( J, J')\). Then
(1) \(J \times J' \subseteq \mathcal{P}\) if and only if there exists a unique \(P \in \text{Spec}(A)\) with \(P \supseteq I\) such that \(\mathcal{P} = P \bowtie J ( J, J')\), where \(I := f^{-1}(J) = g^{-1}(J')\). In this case, there exist \(L \in \text{Spec}(f(A) + J)\) with \(L \supseteq J\) and \(L' \in \text{Spec}(g(A) + J')\) with \(L' \supseteq J'\) such that \(\mathcal{P} = L = L'\).
(2) \(J \times J' \not\subseteq \mathcal{P}\) if and only if there exists a unique \(L \in \text{Spec}(f(A) + J)\) (or \(L \in \text{Spec}(g(A) + J)\)) such that \(J \not\subseteq L\) (or \(J' \not\subseteq L\)) and \(\mathcal{P} = \hat{L}\).

Consequently, we have:
\[\text{Spec}(A \bowtie J ( J, J')) = \{\hat{L} \mid L \in \text{Spec}(f(A) + J) \cup \text{Spec}(g(A) + J)\}\].

Theorem 2.7. Under the above notation, assume that \(A\) is an \(S\)-Noetherian ring and \(B\) (resp., \(C\)) is an \(S\)-finite \(A\)-module (with the \(A\)-module structure induced by \(f\) (resp., by \(g\))). Then \(A \bowtie J ( J, J')\) is an \(S'\)-Noetherian ring.

Proof. Let \(\mathcal{P}\) be a prime ideal of \(A \bowtie J ( J, J')\) disjoint from \(S'\). By Lemma 2.6, we have two cases:

Case 1. \(\mathcal{P}\) has the form \(P \bowtie J ( J, J')\), where \(P\) is a prime ideal of \(A\) disjoint from \(S\). Since \(A\) is \(S\)-Noetherian, there exist \(s_1 \in S\) and \(a_1, a_2, \ldots, a_n \in P\) such that \(s_1 P \subseteq \{(a_1, a_2, \ldots, a_n)\} \subseteq P\). Since \(B\) is an \(S\)-finite \(A\)-module, there exist an element \(s_2 \in S\) and \(j_1, \ldots, j_m \in J\) such that \(s_2 J \subseteq A, j_1 + \cdots + A, j_m \subseteq J\) (or \(f(s_2)J \subseteq f(A)j_1 + \cdots + f(A)j_m \subseteq J\)). Also \(C\) is an \(S\)-finite \(A\)-module, there exist \(s_3 \in S\) and \(j'_1, \ldots, j'_p \in J'\) such that \(s_3 J' \subseteq A, j'_1 + \cdots + A, j'_p \subseteq J'\) (or \(g(s_3)J' \subseteq g(A)j'_1 + \cdots + g(A)j'_p \subseteq J'\)). Put \(s = s_1 s_2 s_3\). Let \((f(p) + j, g(p) + j') \in P \bowtie J ( J, J')\). So \(p \in P\), and thus there exist \(\alpha_1, \alpha_2, \ldots, \alpha_n \in A\) such that \(sp = \sum_{i=1}^n \alpha_i a_i\). Also \(j \in J\) and \(j' \in J'\) implies that there exist \(\beta_1, \beta_2, \ldots, \beta_m\) and \(\gamma_1, \gamma_2, \ldots, \gamma_p\) in \(A\) such that \(jf(s) = \sum_{k=1}^m f(\beta_k)j_k\) and \(j'g(s) = \sum_{l=1}^q g(\gamma_l)j'_l\).
Then
\[(f(s), g(s))(f(p) + j, g(p) + j') = (f(sp) + jf(s), g(sp) + j'g(p))\]
\[= \sum_{i=1}^n f(\alpha_i)f(a_i) + \sum_{k=1}^m f(\beta_k)j_k, \sum_{i=1}^n g(\alpha_i)g(a_i) + \sum_{l=1}^q g(\gamma_l)j'_l\]
\[\sum_{i=1}^n (f(\alpha_i), g(\alpha_i))(f(a_i), g(a_i)) + \sum_{k=1}^m (f(\beta_k), g(\beta_k))(j_k, 0)\]
\[+ \sum_{l=1}^q (f(\gamma_l), g(\gamma_l))(0, j'_l)\]
\[\in \{(f(a_i), g(a_i)), (j_k, 0), (0, j'_l) \mid 1 \leq i \leq n, 1 \leq k \leq m, 1 \leq l \leq q\}\].
Case 2. \( \mathcal{P} = L \) for some prime ideal \( L \) of \( f(A) + j \) or \( \mathcal{P} = L' \) for some prime ideal \( L' \) of \( g(A) + j' \). Without loss of generality we may assume that \( \mathcal{P} = L \). Note that \( \mathcal{P}B = \{ f(a) + j \mid f(a) + j, g(a) + j' \in L \} \) is an \( A \)-submodule of \( B \). As \( B \) is an \( S \)-finite \( A \)-module and \( A \) is an \( S \)-Noetherian ring, there exist \( s_1 \in S \) and \( f(a_1) + j_1, f(a_2) + j_2, \ldots, f(a_n) + j_n \in \mathcal{P}B(L) \) such that

\[
    f(s_1)pB(L) \subseteq f(A)(f(a_1) + j_1) + \cdots + f(A)(f(a_n) + j_n) \subseteq pB(L).
\]

Let \( (f(a) + j, g(a) + j') \in L \). Then

\[
    (f(s_1), g(s_1))(f(a) + j, g(a) + j') = (\sum_{i=1}^n f(a_i)(f(a_i) + j_i), g(s_1)g(a) + g(s_1)j')
\]

for some \( \alpha_1, \alpha_2, \ldots, \alpha_n \in A \). Note that \( g(s_1)g(a) - \sum_{i=1}^n g(\alpha_i)g(a_1) \in J' \), and hence \( s_1a - \sum_{i=1}^n \alpha_ia_1 \in g^{-1}(J') \). As \( g^{-1}(J') \) is an ideal of \( A \) and \( A \) is an \( S \)-Noetherian ring, there exist an element \( s_2 \in S \) and \( b_1, b_2, \ldots, b_m \in A \) such that

\[
    s_2g^{-1}(J') \subseteq \{ [b_1, b_2, \ldots, b_m] \} \subseteq g^{-1}(J').
\]

Hence \( s_2(s_1a - \sum_{i=1}^n \alpha_ia_1) = \sum_{k=1}^m \beta_kb_k \) for some \( \beta_1, \beta_2, \ldots, \beta_m \in A \). So \( s_2s_1a = s_2^2 \sum_{i=1}^n \alpha_ia_i + m \sum_{k=1}^m \beta_kb_k \). Moreover, as \( C \) is an \( S \)-finite \( A \)-module, there exists \( s_3 \in S \) such that \( g(s_3)J' \subseteq g(A)j_1 + g(A)j_2 + \cdots + g(A)j_p \subseteq J' \). Putting \( s = s_1s_2s_3 \), we have:

\[
    (f(s), g(s))(f(a) + j, g(a) + j')
\]

\[
= (f(s_2s_3)(\sum_{i=1}^n f(\alpha_i)(f(a_i) + j_i)), g(s_1s_2s_3)g(a) + g(s_1s_2s_3)j')
\]

\[
= (\sum_{i=1}^n f(s_2s_3)f(\alpha_i)f(a_i) + \sum_{i=1}^n f(s_2s_3)f(\alpha_i)j_i,
\]

\[
= (g(s_2s_3)(\sum_{i=1}^n g(\alpha_i)g(a_i)) + g(s_3)(\sum_{k=1}^m g(\beta_k)g(b_k) + g(s_1s_2)(\sum_{l=1}^p g(\eta_l)j_l))
\]

\[
= \sum_{i=1}^n (f(s_2s_3)f(\alpha_i), g(s_2s_3)g(\alpha_i))(f(a_i), g(a_i))
\]

\[
+ \sum_{i=1}^n (f(s_2s_3)f(\alpha_i), g(s_2s_3)g(\alpha_i))(j_i, 0)
\]

\[
+ \sum_{k=1}^m (f(s_3)f(\beta_k), g(s_3)g(\beta_k))(0, g(b_k))
\]

\[
+ \sum_{l=1}^p (f(s_1s_2)f(\eta_l), g(s_1s_2)g(\eta_l))(0, j_l)
\]

\[
\in \{ \{(f(a_i), g(a_i)), (j_i, 0), (0, g(b_k)), (0, j_l) \mid 1 \leq i \leq n, 1 \leq k \leq m, \text{ and } 1 \leq l \leq p \} \}.
\]
Then it follows from all cases that $\mathcal{P}$ is $S'$-finite, which implies that $A \triangleright f,g (J, J')$ is an $S'$-Noetherian ring. \hfill \Box

We close this note by the following examples.

**Example 2.8.** Let $A$ be an $S$-Noetherian ring and $I, K$ two proper ideals of $A$ such that $I \subseteq K$. Set $B := A/I$ and $C := A \times A$. Let $f : A \rightarrow B$ be the canonical ring homomorphism and $g : A \rightarrow C$ be the canonical embedding on the first component. Consider $J := K/I$ and $J' := K \times 0$ the ideals of $B$ and $C$ respectively. Note that $B$ and $C$ are $S$-finite as $A$-modules. Then by Theorem 2.7, $A \triangleright f,g (J, J')$ is an $S'$-Noetherian ring.

The following example uses the $A \bowtie E$ construction introduced and studied in [13]. Let $A$ be a ring and $E$ an $A$-module. Then $A \bowtie E = A \times E$ is a ring with identity $(1, 0)$ under addition defined by $(a, e) + (b, f) = (a + b, e + f)$ and multiplication defined by $(a, e)(b, f) = (ab, af + be)$. See for instance [2, 10].

**Example 2.9.** Let $A$ be an $S$-Noetherian ring and $E$ and $E'$ two $A$-modules such that $E \neq E'$ which are $S$-finite. Consider the natural injective ring homomorphisms $f : A \rightarrow A \bowtie E$ and $g : A \rightarrow A \bowtie E'$ and set $J := I \bowtie E$ and $J' := I \bowtie E'$ to be two ideals of $A \bowtie E$ and $A \bowtie E'$ respectively for some ideal $I$ of $A$. We claim that the bi-amalgamation $A \bowtie f,g (J, J')$ is an $S'$-Noetherian ring. Indeed, notice first that $f^{-1}(J) = g^{-1}(J') = I$, $f(A) + J = A \bowtie E$, and $g(A) + J = A \bowtie E'$. Furthermore, $f(A) + J$ is an $f(S)$-Noetherian ring and $g(A) + J'$ is a $g(S)$-Noetherian ring by [16, Theorem 3.8.]. Now the assertion follows immediately from Theorem 2.2(3).

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