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# RESTRICTED POLYNOMIAL EXTENSIONS

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Abstract. Let $\mathbb{F}$ be a commutative ring. A restricted skew polynomial extension over $\mathbb{F}$ is a class of iterated skew polynomial $\mathbb{F}$-algebras which include well-known quantized algebras such as the quantum algebra $U_q(\mathfrak{sl}_2)$, Weyl algebra, etc. Here we obtain a necessary and sufficient condition in order to be restricted skew polynomial extensions over $\mathbb{F}$. We also introduce a restricted Poisson polynomial extension which is a class of iterated Poisson polynomial algebras and observe that a restricted Poisson polynomial extension appears as semiclassical limits of restricted skew polynomial extensions. Moreover, we obtain usual as well as unusual quantized algebras of the same Poisson algebra as applications.

## 1. Introduction

Let $\mathbb{F}$ be a commutative ring. Given an $\mathbb{F}$-endomorphism $\beta$ on an $\mathbb{F}$-algebra $R$, an $\mathbb{F}$-linear map $\nu$ is said to be a *left $\beta$-derivation* on $R$ if $\nu(ab) = \beta(a)\nu(b) + \nu(a)b$ for all $a, b \in R$. For such a pair $(\beta, \nu)$, a free left $R$-module with basis $\{z^i\}_{i=0}^{\infty}$ becomes an $\mathbb{F}$-algebra with multiplication

$$za = \beta(a)z + \nu(a), \quad a \in R,$$

which is a *skew polynomial* $\mathbb{F}$*-algebra* or an *Ore extension* of $R$ and denoted by $R[z; \beta, \nu]$. Refer to [7, Chapter 2] for details of a skew polynomial algebra.

**Definition.** An iterated skew polynomial $\mathbb{F}$-algebra

$$A_k = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \cdots [x_k; \beta_k, \nu_k]$$

is called a *$k$-restricted skew polynomial extension* over $\mathbb{F}$ if the pairs $(\beta_j, \nu_j)$ satisfy that, for all $1 \le i < j \le k$,

$$\beta_j(1) = 1, \qquad \beta_j(x_i) = a_{ji}x_i, \ a_{ji} \in \mathbb{F}, \tag{1.1}$$

$$\nu_j(1) = 0, \qquad \nu_j(x_i) = u_{ji} \in A_{j-1}. \tag{1.2}$$

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(Note that the conditions $\beta_j(1) = 1$ and $\nu_j(1) = 0$ in (1.1) and (1.2) are natural since $\beta_j$ is an algebra homomorphism and $\nu_j$ is a derivation.)

Let $A_{k-1} = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \cdots [x_{k-1}; \beta_{k-1}, \nu_{k-1}]$ be a $(k-1)$-restricted skew polynomial extension over $\mathbb{F}$ and let $\beta_k, \nu_k$ be $\mathbb{F}$-linear maps from $A_{k-1}$ into itself. In this article, we find necessary and sufficient conditions for $\beta_k$ and $\nu_k$ such that there exists a restricted skew polynomial extension $A_k = A_{k-1}[x_k; \beta_k, \nu_k]$ over $\mathbb{F}$. See Lemma 2.2 and Theorem 2.4. Hence, using induction on $k$, we get a restricted skew polynomial extension $A_k$ over $\mathbb{F}$ from the result.

Suppose that $A$ is an $\mathbb{F}$-algebra and let $\hbar \in A$ be a nonzero, nonunit, central and non-zero-divisor such that $A/\hbar A$ is commutative. Then $\overline{A} := A/\hbar A$ is a nontrivial commutative $\mathbf{k}$-algebra as well as a Poisson algebra with the Poisson bracket

$$\{\overline{a}, \overline{b}\} = \overline{\hbar^{-1}(ab - ba)} \tag{1.3}$$

for $\overline{a}, \overline{b} \in A/\hbar A$ by [1, III.5.4]. The algebra $A$ is called a *quantization* of the Poisson algebra $\overline{A}$ and $\overline{A}$ is called a *semiclassical limit* of $A$ in [6, §2] and [10, Definition 3]. Moreover, if $\hbar - q$, $q \in \mathbb{F}$, is a nonzero and nonunit, then the nontrivial algebra $A/(\hbar - q)A$ is called a *deformation* of $A$ or $\overline{A}$ in [6, §2] and [10, Definition 3]. An interested reader is referred to [5], [12] and [2] for a deformation quantization.

By analogy with the restricted skew polynomial extension, we define a restricted Poisson polynomial extension for iterated Poisson polynomial algebras. See Definition 3. Here we obtain a condition for $\beta_k$ and $\nu_k$ such that a restricted Poisson polynomial extension is a semiclassical limit of $A_k$. See Theorem 3.2 and Corollary 3.3. As applications, we will show that there are well-known restricted skew polynomial extensions as well as unusual restricted skew polynomial extensions such that their semiclassical limits are the same as a Poisson polynomial algebra in §3.

## 2. A construction of restricted skew polynomial extensions

Set $A_1 = \mathbb{F}[x_1]$ and let $A_n$, $n > 1$, be an iterated skew polynomial $\mathbb{F}$-algebra

$$A_n = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \cdots [x_n; \beta_n, \nu_n].$$

By monomials in $A_n$ we mean finite products of $x_i$'s together with the unity 1. A monomial $X$ is said to be *standard* if $X$ is of the form

$$X = 1 \text{ or } X = x_{i_1} x_{i_2} \cdots x_{i_k} \qquad (1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n).$$

Note that the set of all standard monomials of $A_n$ forms an $\mathbb{F}$-basis.

Let $\beta$ and $\nu$ be $\mathbb{F}$-linear maps from an $\mathbb{F}$-algebra $R$ into itself. The following lemma is well known, e.g. see [9, p. 177].

**Lemma 2.1.** *The following conditions are equivalent*:

(1) *The $\mathbb{F}$-linear map $\phi : R \to M_2(R)$ given by*

$$\phi(r) = \begin{pmatrix} \beta(r) & \nu(r) \\ 0 & r \end{pmatrix}$$

*for all $r \in R$, is an $\mathbb{F}$-algebra homomorphism.*

(2) *$\beta$ and $\nu$ are an endomorphism and a left $\beta$-derivation on $R$, respectively.*

For a $(k-1)$-restricted skew polynomial extension

$$A_{k-1} = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \cdots [x_{k-1}; \beta_{k-1}, \nu_{k-1}]$$

over $\mathbb{F}$, we are going to construct a $k$-restricted skew polynomial extension

$$A_k = A_{k-1}[x_k; \beta_k, \nu_k] = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \cdots [x_k; \beta_k, \nu_k]$$

over $\mathbb{F}$. Let $k = 2$. Then there exists an $\mathbb{F}$-algebra homomorphism $\phi : \mathbb{F}[x_1] \to M_2(\mathbb{F}[x_1])$ defined by

$$\phi(x_1) = \begin{pmatrix} \beta_2(x_1) & \nu_2(x_1) \\ 0 & x_1 \end{pmatrix}.$$

Hence $\beta_2$ is an $\mathbb{F}$-algebra endomorphism and $\nu_2$ is a left $\beta_2$-derivation on $A_1 = \mathbb{F}[x_1]$ by Lemma 2.1 and thus there exists the skew polynomial $\mathbb{F}$-algebra $A_2 = A_1[x_2; \beta_2, \nu_2]$. Henceforth we assume $k \geq 3$. The following statement gives us necessary conditions for the existence of a $k$-restricted skew polynomial extension $A_k = A_{k-1}[x_k; \beta_k, \nu_k]$ over $\mathbb{F}$.

**Lemma 2.2.** *Let $A_{k-1} = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \cdots [x_{k-1}; \beta_{k-1}, \nu_{k-1}]$ be a $(k-1)$-restricted skew polynomial extension over $\mathbb{F}$. If there exists a $k$-restricted skew polynomial extension*

$$A_k = A_{k-1}[x_k; \beta_k, \nu_k] = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \cdots [x_k; \beta_k, \nu_k]$$

*over $\mathbb{F}$, then $\beta_k$, $\nu_k$ satisfy the following conditions*

$$\beta_k(u_{ji}) = a_{kj} a_{ki} u_{ji} \qquad (1 \leq i < j < k), \tag{2.1}$$

$$a_{kj} x_j u_{ki} + u_{kj} x_i = a_{ji} a_{ki} x_i u_{kj} + a_{ji} u_{ki} x_j + \nu_k(u_{ji}) \qquad (1 \leq i < j < k). \tag{2.2}$$

*Proof.* Let $1 \leq i < j \leq k-1$. Since $\beta_k$ is an $\mathbb{F}$-algebra endomorphism, we have that

$$\beta_k(x_j x_i) = \beta_k(\beta_j(x_i) x_j + \nu_j(x_i)) = a_{kj} a_{ki} a_{ji} x_i x_j + \beta_k(u_{ji})$$

and

$$\begin{aligned} \beta_k(x_j x_i) &= \beta_k(x_j) \beta_k(x_i) = a_{kj} a_{ki} x_j x_i \\ &= a_{kj} a_{ki} (\beta_j(x_i) x_j + \nu_j(x_i)) = a_{kj} a_{ki} a_{ji} x_i x_j + a_{kj} a_{ki} u_{ji} \end{aligned}$$

by (1.1), (1.2). Hence we get (2.1).

Similarly, since $\nu_k$ is a left $\beta_k$-derivation, we have that

$$\nu_k(x_j x_i) = \beta_k(x_j) \nu_k(x_i) + \nu_k(x_j) x_i = a_{kj} x_j u_{ki} + u_{kj} x_i$$

and

$$\begin{aligned}\nu_k(x_jx_i) &= \nu_k(\beta_j(x_i)x_j + \nu_j(x_i)) = \nu_k(a_{ji}x_ix_j + u_{ji})\\ &= a_{ji}(\beta_k(x_i)\nu_k(x_j) + \nu_k(x_i)x_j) + \nu_k(u_{ji})\\ &= a_{ji}a_{ki}x_iu_{kj} + a_{ji}u_{ki}x_j + \nu_k(u_{ji})\end{aligned}$$

by (1.1), (1.2). Hence we get (2.2). □

**Lemma 2.3.** *Let $A_{k-1} = \mathbb{F}[x_1][x_2;\beta_2,\nu_2]\cdots[x_{k-1};\beta_{k-1},\nu_{k-1}]$ be a $(k-1)$-restricted skew polynomial extension over $\mathbb{F}$ and let $\beta_k,\nu_k$ be $\mathbb{F}$-linear maps from $A_{k-1}$ into itself subject to the conditions* (1.1), (1.2). *If $\beta_k$ and $\nu_k$ satisfy* (2.1) *and* (2.2), *then the following conditions hold.*

$$\beta_k(x_j)\beta_k(x_i) = \beta_k\beta_j(x_i)\beta_k(x_j) + \beta_k\nu_j(x_i), \tag{2.3}$$

$$\beta_k(x_j)\nu_k(x_i) + \nu_k(x_j)x_i = \beta_k\beta_j(x_i)\nu_k(x_j) + \nu_k\beta_j(x_i)x_j + \nu_k\nu_j(x_i). \tag{2.4}$$

*Proof.* Since $A_{k-1}$ is a $(k-1)$-restricted skew polynomial extension over $\mathbb{F}$, the equations (2.3) and (2.4) follow from (2.1) and (2.2), respectively, by (1.1), (1.2). □

In the following theorem, we see that (2.1) and (2.2) are sufficient conditions for the existence of the restricted skew polynomial extension $A_k = A_{k-1}[x_k;\beta_k,\nu_k]$ over $\mathbb{F}$.

**Theorem 2.4.** *Let $A_{k-1} = \mathbb{F}[x_1][x_2;\beta_2,\nu_2]\cdots[x_{k-1};\beta_{k-1},\nu_{k-1}]$ be a $(k-1)$-restricted skew polynomial extension over $\mathbb{F}$. Given $\mathbb{F}$-linear maps $\beta_k,\nu_k$ from $A_{k-1}$ into itself subject to the conditions* (1.1), (1.2), *if $\beta_k$ and $\nu_k$ satisfy the conditions* (2.1), (2.2), *then there exists a $k$-restricted skew polynomial extension*

$$A_k = A_{k-1}[x_k;\beta_k,\nu_k] = \mathbb{F}[x_1][x_2;\beta_2,\nu_2]\cdots[x_k;\beta_k,\nu_k]$$

*over $\mathbb{F}$.*

*Proof.* It is enough to show that there exist an $\mathbb{F}$-algebra endomorphism $\beta_k$ on $A_{k-1}$ and a left $\beta_k$-derivation $\nu_k$ subject to the conditions (1.1) and (1.2). Note that the set of all standard monomials forms an $\mathbb{F}$-basis of $A_{k-1}$. For any standard monomials $x_{i_1}\cdots x_{i_r} \in A_{k-1}$, define $\mathbb{F}$-linear maps $\beta_k$ and $\nu_k$ from $A_{k-1}$ into itself by

$$\beta_k(1) = 1, \quad \beta_k(x_{i_1}\cdots x_{i_r}) = (a_{ki_1}x_{i_1})\cdots(a_{ki_r}x_{i_r}), \tag{2.5}$$

$$\nu_k(1) = 0, \quad \nu_k(x_{i_1}\cdots x_{i_r}) = \sum_{\ell=1}^{r}(a_{ki_1}x_{i_1})\cdots(a_{ki_{\ell-1}}x_{i_{\ell-1}})u_{ki_\ell}(x_{i_{\ell+1}}\cdots x_{i_r}), \tag{2.6}$$

where $a_{ki_\ell} \in \mathbb{F}$ and $u_{ki_\ell} \in A_{k-1}$ for $\ell = 1,\ldots,r$. Observe that these $\mathbb{F}$-linear maps $\beta_k$ and $\nu_k$ satisfy (1.1) and (1.2). We claim that the map $\beta_k$ defined by (2.5) is an $\mathbb{F}$-algebra endomorphism and the map $\nu_k$ defined by (2.6) is a left $\beta_k$-derivation by using Lemma 2.1.

Let $\mathbb{F}\langle S_{k-1}\rangle$ be the free $\mathbb{F}$-algebra on the set $S_{k-1} = \{x_1, \ldots, x_{k-1}\}$. Define an $\mathbb{F}$-algebra homomorphism $f : \mathbb{F}\langle S_{k-1}\rangle \to M_2(A_{k-1})$ by

$$f(x_i) = \begin{pmatrix} \beta_k(x_i) & \nu_k(x_i) \\ 0 & x_i \end{pmatrix} \qquad (1 \le i < k).$$

Let us show that

$$f(\nu_j(x_i)) = \begin{pmatrix} \beta_k \nu_j(x_i) & \nu_k \nu_j(x_i) \\ 0 & \nu_j(x_i) \end{pmatrix} \tag{2.7}$$

for $1 \le i < j < k$. For any standard monomial $X = x_{i_1} \cdots x_{i_r}$ in $A_{k-1}$, by (2.5) and (2.6),

$$\begin{aligned} \nu_k(X) &= \sum_{\ell=1}^{r} \beta_k(x_{i_1} \cdots x_{i_{\ell-1}})\nu_k(x_{i_\ell})(x_{i_{\ell+1}} \cdots x_{i_r}) \\ &= \sum_{\ell=1}^{r-1} \beta_k(x_{i_1} \cdots x_{i_{\ell-1}})\nu_k(x_{i_\ell})(x_{i_{\ell+1}} \cdots x_{i_r}) + \beta_k(x_{i_1} \cdots x_{i_{r-1}})\nu_k(x_{i_r}) \\ &= \nu_k(x_{i_1} \cdots x_{i_{r-1}})x_{i_r} + \beta_k(x_{i_1} \cdots x_{i_{r-1}})\nu_k(x_{i_r}). \end{aligned}$$

In particular, if $Xx_j$ is standard (thus $i_r \le j$), then

$$\nu_k(Xx_j) = \beta_k(X)\nu_k(x_j) + \nu_k(X)x_j. \tag{2.8}$$

Let us verify first that

$$f(X) = \begin{pmatrix} \beta_k(X) & \nu_k(X) \\ 0 & X \end{pmatrix} \tag{2.9}$$

for any standard monomial $X = x_{i_1} \cdots x_{i_r}$ in $A_{k-1}$ of length $r$. We proceed by induction on $r$. If $r = 1$, then (2.9) is true trivially. Assume that $r > 1$ and that (2.9) holds for any standard monomial of length $< r$. Set $Y = x_{i_1} \cdots x_{i_{r-1}}$. Then $Y$ is a standard monomial of length $r - 1$ and $X = Yx_{i_r}$. Thus (2.9) holds as follows:

$$\begin{aligned} f(X) &= f(Yx_{i_r}) = f(Y)f(x_{i_r}) \\ &= \begin{pmatrix} \beta_k(Y) & \nu_k(Y) \\ 0 & Y \end{pmatrix} \begin{pmatrix} \beta_k(x_{i_r}) & \nu_k(x_{i_r}) \\ 0 & x_{i_r} \end{pmatrix} \quad \text{(by induction hypothesis)} \\ &= \begin{pmatrix} \beta_k(Y)\beta_k(x_{i_r}) & \beta_k(Y)\nu_k(x_{i_r}) + \nu_k(Y)x_{i_r} \\ 0 & Yx_{i_r} \end{pmatrix} \\ &= \begin{pmatrix} \beta_k(X) & \nu_k(X) \\ 0 & X \end{pmatrix}. \qquad \text{(by (2.5), (2.8))} \end{aligned}$$

Let $\nu_j(x_i) = \sum_\ell b_\ell X_\ell$, where all $b_\ell \in \mathbb{F}$ and $X_\ell$ are standard monomials of $A_{j-1}$. Since $f$ is an $\mathbb{F}$-algebra homomorphism, we have

$$\begin{aligned} f(\nu_j(x_i)) &= \sum_\ell b_\ell f(X_\ell) \\ &= \sum_\ell b_\ell \begin{pmatrix} \beta_k(X_\ell) & \nu_k(X_\ell) \\ 0 & X_\ell \end{pmatrix} \qquad \text{(by (2.9))} \\ &= \begin{pmatrix} \beta_k(\sum_\ell b_\ell X_\ell) & \nu_k(\sum_\ell b_\ell X_\ell) \\ 0 & \sum_\ell b_\ell X_\ell \end{pmatrix} \\ &= \begin{pmatrix} \beta_k \nu_j(x_i) & \nu_k \nu_j(x_i) \\ 0 & \nu_j(x_i) \end{pmatrix}. \end{aligned}$$

Thus (2.7) holds.

Note that $A_{k-1}$ is an $\mathbb{F}$-algebra generated by $x_1, \ldots, x_{k-1}$ with relations

$$x_j x_i - \beta_j(x_i) x_j - \nu_j(x_i) \qquad (1 \le i < j < k).$$

Namely, $A_{k-1}$ is isomorphic to the $\mathbb{F}$-algebra $\mathbb{F}\langle S_{k-1}\rangle / I$, where $I$ is the ideal generated by

$$x_j x_i - \beta_j(x_i) x_j - \nu_j(x_i) \quad (1 \le i < j < k).$$

Since $f$ is an $\mathbb{F}$-algebra homomorphism, it is easy to check that $I \subseteq \ker f$ by (2.3), (2.4) and (2.7). Hence there exists an $\mathbb{F}$-algebra homomorphism $\phi : A_{k-1} \to M_2(A_{k-1})$ such that

$$\phi(x_i) = \begin{pmatrix} \beta_k(x_i) & \nu_k(x_i) \\ 0 & x_i \end{pmatrix}$$

for $1 \le i < k$. By Lemma 2.1, $\beta_k$ is an $\mathbb{F}$-algebra endomorphism on $A_{k-1}$ and $\nu_k$ is a left $\beta_k$-derivation on $A_{k-1}$ as claimed. □

*Remark* 2.5. Retain the notations of Theorem 2.4. If $a_{ki} \neq 0$ for all $1 \le i < k$, then $\beta_k$ is a monomorphism.

*Proof.* Note that $\beta_i$, $\nu_i$ are $\mathbb{F}$-linear for all $i = 1, \ldots, k$. Let $g = \sum_i a_i X_i \in A_{k-1}$, where $a_i \in \mathbb{F}$ and $X_i$ are standard monomials for all $i$, and suppose that $\beta_k(g) = 0$. Then $\beta_k(X_i) = b_i X_i$ for some $0 \neq b_i \in \mathbb{F}$ by (2.5) and thus

$$0 = \beta_k(g) = \sum_i a_i b_i X_i.$$

It follows that all $a_i = 0$ since the standard monomials of $A_k$ form an $\mathbb{F}$-basis. Thus $g = 0$. □

## 3. Application

Let $\hbar$ be an indeterminate throughout the section.

**Example 3.1.** Set $\mathbb{F} = \mathbb{C}[\hbar, \hbar^{-1}]$ and

$$\begin{aligned} &a_{21} = \hbar^2, && a_{31} = \hbar^{-2}, && a_{32} = \hbar^2, \\ &u_{21} = -(\hbar^2 - 1), && u_{31} = -(\hbar^{-2} - 1), && u_{32} = -(\hbar^2 - 1). \end{aligned}$$

By Theorem 2.4, there exists a restricted skew polynomial extension

$$A_3 = \mathbb{F}[x_1][x_2; \beta_2, \nu_2][x_3; \beta_3, \nu_3]$$

over $\mathbb{F}$, where

$$\begin{aligned} &\beta_2(x_1) = a_{21}x_1 = \hbar^2 x_1, && \beta_3(x_1) = a_{31}x_1 = \hbar^{-2}x_1, \\ &\beta_3(x_2) = a_{32}x_2 = \hbar^2 x_2, && \nu_2(x_1) = u_{21} = -(\hbar^2 - 1), \\ &\nu_3(x_1) = u_{31} = -(\hbar^{-2} - 1), && \nu_3(x_2) = u_{32} = -(\hbar^2 - 1) \end{aligned}$$

since $\beta_2$, $\nu_2$, $\beta_3$ and $\nu_3$ satisfy (1.1), (1.2), (2.1), (2.2). Note that $A_3$ is the $\mathbb{F}$-algebra generated by $x_1, x_2, x_3$ subject to the relations

$$\hbar^2 x_1x_2 - x_2x_1 = \hbar^2 - 1,\ \hbar^2 x_3x_1 - x_1x_3 = \hbar^2 - 1,\ \hbar^2 x_2x_3 - x_3x_2 = \hbar^2 - 1, \tag{3.1}$$

which is the relation appearing in [8].

Observe that $\hbar - 1$ is a nonzero, nonunit, non-zero-divisor and central element of $A_3$ such that the factor $\overline{A_3} := A_3/(\hbar - 1)A_3$ is commutative. Hence $\overline{A_3}$ is a Poisson $\mathbb{C}$-algebra with Poisson bracket

$$\{\overline{a}, \overline{b}\} = \overline{(\hbar - 1)^{-1}(ab - ba)}$$

for all $\overline{a}, \overline{b} \in \overline{A_3}$ by (1.3), which is a semiclassical limit of $A_3$. More precisely, $\overline{A_3}$ is Poisson isomorphic to the Poisson algebra $\mathbb{C}[x_1, x_2, x_3]$ with Poisson bracket

$$\{x_2, x_1\} = 2x_1x_2 - 2,\ \{x_1, x_3\} = 2x_1x_3 - 2,\ \{x_3, x_2\} = 2x_2x_3 - 2. \tag{3.2}$$

Refer to [11, Example 4.2] for the Poisson bracket (3.2). □

A derivation $\alpha$ on a Poisson algebra $R$ is said to be a Poisson derivation if

$$\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$$

for all $a, b \in R$. Let $\alpha$ be a Poisson derivation on $R$ and let $\delta$ be a derivation on $R$ such that

$$\delta(\{a, b\}) - \{\delta(a), b\} - \{a, \delta(b)\} = \alpha(a)\delta(b) - \delta(a)\alpha(b)$$

for all $a, b \in R$. By [14, 1.1], the commutative polynomial $\mathbb{C}$-algebra $R[z]$ is a Poisson algebra with Poisson bracket $\{z, a\} = \alpha(a)z + \delta(a)$ for all $a \in R$. Such a Poisson polynomial algebra $R[z]$ is denoted by $R[z; \alpha, \delta]_p$ in order to distinguish it from skew polynomial algebras. If $\alpha = 0$, then we write $R[z; \delta]_p$ for $R[z; 0, \delta]_p$ and if $\delta = 0$, then we write $R[z; \alpha]_p$ for $R[z; \alpha, 0]_p$.

**Definition.** An iterated Poisson polynomial $\mathbb{C}$-algebra

$$B_k = \mathbb{C}[x_1][x_2; \alpha_2, \delta_2]_p \cdots [x_k; \alpha_k, \delta_k]_p$$

is called a *k-restricted Poisson polynomial extension* over $\mathbb{C}$ if the pairs $(\alpha_j, \delta_j)$ satisfy that, for $i = 1, \ldots, j-1$ and $i < j \le k$,

$$\alpha_j(x_i) = c_{ji}x_i, \text{ where } c_{ji} \in \mathbb{C}, \tag{3.3}$$

$$\delta_j(x_i) = d_{ji} \in B_{j-1} = \mathbb{C}[x_1, \ldots, x_{j-1}]. \tag{3.4}$$

Let $\mathbb{F}$ be the ring $\mathbb{C}[[\hbar]]$ of the formal power series over $\mathbb{C}$. Here, by using Theorem 2.4, we obtain a Poisson algebra $\mathbb{C}[x_1, \ldots, x_k]$, which is a semiclassical limit of $A_k$, in the following.

**Theorem 3.2.** *Let $\mathbb{F} = \mathbb{C}[[\hbar]]$ and let $A_k = \mathbb{F}[x_1][x_2; \beta_2, \nu_2] \cdots [x_k; \beta_k, \nu_k]$ be a restricted skew polynomial extension over $\mathbb{F}$ where*

$$\beta_j(x_i) = a_{ji}x_i, (a_{ji} \in \mathbb{F}), \;\; \nu_j(x_i) \in A_{j-1},$$

*for $1 \le i < j \le k$. Suppose that*

$$a_{ji} - 1 \in \hbar\mathbb{F}, \quad \nu_j(x_i) \in \hbar A_k \tag{3.5}$$

*for all $1 \le i < j \le k$. Then $\overline{A}_k = A_k/\hbar A_k$ is Poisson isomorphic to a restricted Poisson polynomial extension*

$$\mathbb{C}[x_1][x_2; \alpha_2, \delta_2]_p \cdots [x_k; \alpha_k, \delta_k]_p$$

*over $\mathbb{C}$, where*

$$\alpha_j(x_i) = c_{ji}x_i = \left(\frac{da_{ji}}{d\hbar}|_{\hbar=0}\right)x_i, \quad \delta_j(x_i) = d_{ji} = \frac{d\nu_j(x_i)}{d\hbar}|_{\hbar=0} \tag{3.6}$$

*for all $1 \le i < j \le k$. (Derivatives are formal derivatives of power series in $\hbar$.)*

*Proof.* Note that $A_k$ is generated by $x_1, \ldots, x_k$ and that $\hbar \in \mathbb{F}$ is a nonzero central element of $A_k$. Since

$$\begin{aligned} x_jx_i - x_ix_j &= \beta_j(x_i)x_j + \nu_j(x_i) - x_ix_j \\ &= (a_{ji}-1)x_ix_j + \nu_j(x_i) \in \hbar A_k \quad (i<j) \end{aligned} \tag{3.7}$$

by (3.5), $\overline{A}_k$ is a commutative $\mathbb{C}$-algebra $\mathbb{C}[x_1, \ldots, x_k]$. Moreover we have

$$\begin{aligned} \{\overline{x}_j, \overline{x}_i\} &= \overline{\hbar^{-1}(x_jx_i - x_ix_j)} \\ &= \overline{\left(\frac{a_{ji}-1}{\hbar}\right)x_ix_j + \left(\frac{\nu_j(x_i)}{\hbar}\right)} && \text{(by (3.7))} \\ &= \left(\frac{da_{ji}}{d\hbar}|_{\hbar=0}\right)\overline{x}_i\overline{x}_j + \overline{\left(\frac{d\nu_j(x_i)}{d\hbar}|_{\hbar=0}\right)} && \text{(by (3.5))} \end{aligned}$$

for all $1 \le i < j \le k$. Hence the result follows. □

**Corollary 3.3.** *Let $\mathbb{F} = \mathbb{C}[[\hbar]]$. The restricted skew polynomial extension $A_k$ in Theorem 3.2 is a quantization of the restricted Poisson polynomial extension*

$$B_k = \mathbb{C}[x_1][x_2; \alpha_2, \delta_2]_p \cdots [x_k; \alpha_k, \delta_k]_p$$

*in Theorem 3.2.*

*Proof.* The result follows by the proof of Theorem 3.2. □

**Example 3.4.** Here we obtain a quantization of the Poisson algebra $B_3 = \mathbb{C}[x_1, x_2, x_3]$ with the Poisson bracket (3.2). Note that $B_3$ is a restricted Poisson polynomial extension

$$B_3 = \mathbb{C}[x_1][x_2; \alpha_2, \delta_2]_p[x_3; \alpha_3, \delta_3]_p$$

over $\mathbb{C}$, where

$$\begin{aligned} \alpha_2 &= 2x_1\frac{\partial}{\partial x_1}, & \delta_2 &= -2\frac{\partial}{\partial x_1},\\ \alpha_3 &= -2x_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2}, & \delta_3 &= 2\frac{\partial}{\partial x_1} - 2\frac{\partial}{\partial x_2}. \end{aligned}$$

Set $\mathbb{F} = \mathbb{C}[[\hbar]]$ and

$$\begin{aligned} a_{21} &= e^{2\hbar}, & a_{31} &= e^{-2\hbar}, & a_{32} &= e^{2\hbar},\\ u_{21} &= -\sin(2\hbar), & u_{31} &= \sin(2\hbar), & u_{32} &= -\sin(2\hbar). \end{aligned}$$

By Theorem 2.4, there exists a restricted skew polynomial extension

$$A_3 = \mathbb{F}[x_1][x_2; \beta_2, \nu_2][x_3; \beta_3, \nu_3]$$

over $\mathbb{F}$, where

$$\begin{aligned} \beta_2(x_1) &= a_{21}x_1 = e^{2\hbar}x_1, & \beta_3(x_1) &= a_{31}x_1 = e^{-2\hbar}x_1,\\ \beta_3(x_2) &= a_{32}x_2 = e^{2\hbar}x_2, & \nu_2(x_1) &= u_{21} = -\sin(2\hbar),\\ \nu_3(x_1) &= u_{31} = \sin(2\hbar), & \nu_3(x_2) &= u_{32} = -\sin(2\hbar). \end{aligned}$$

since $\beta_2$, $\nu_2$, $\beta_3$ and $\nu_3$ satisfy (1.1), (1.2), (2.1), (2.2).

Moreover $A_3$ is a quantization of $B_3$ by Corollary 3.3 since $a_{ji}, u_{ji}$ satisfy (3.5) and (3.6). Note that $A_3$ is the $\mathbb{F}$-algebra generated by $x_1, x_2, x_3$ subject to the relations

$$e^{2\hbar}x_1x_2 - x_2x_1 = \sin(2\hbar),\ e^{2\hbar}x_3x_1 - x_1x_3 = e^{2\hbar}\sin(2\hbar),\ e^{2\hbar}x_2x_3 - x_3x_2 = \sin(2\hbar),$$

which is different from (3.1). □

*Remark* 3.5. Note that the quantized algebras in Example 3.1 and Example 3.4 are distinct quantizations of the Poisson algebra $\mathbb{C}[x_1, x_2, x_3]$ with the Poisson bracket (3.2).

Let us find quantizations of the Poisson Weyl algebra. The Poisson Weyl algebra is the Poisson polynomial algebra $B_{2k} = \mathbb{C}[x_1, x_2, \dots, x_{2k-1}, x_{2k}]$ with Poisson bracket

$$\{f, g\} = \sum_{i=1}^{k}\left(-\frac{\partial f}{\partial x_{2i-1}}\frac{\partial g}{\partial x_{2i}} + \frac{\partial g}{\partial x_{2i-1}}\frac{\partial f}{\partial x_{2i}}\right),$$

cf., [4, 1.1.A] and [13, 1.3]. Namely, the Poisson bracket is

$$\{x_j, x_i\} = \begin{cases} 1, & \text{if } j = 2\ell, i = 2\ell - 1, \\ 0, & \text{otherwise} \end{cases}$$

for $j > i$. Hence $B_{2k}$ is a restricted Poisson polynomial extension

$$B_{2k} = \mathbb{C}[x_1][x_2; \delta_2]_p \cdots [x_{2k-1}]_p[x_{2k}; \delta_{2k}]_p$$

over $\mathbb{C}$, where

$$\delta_{2\ell}(x_i) = \begin{cases} 1, & \text{if } i = 2\ell - 1, \\ 0, & \text{if } i \neq 2\ell - 1. \end{cases}$$

**Example 3.6** (Moyal-Weyl quantization)**.** Set $\mathbb{F} = \mathbb{C}[[\hbar]]$ and

$$a_{ji} = 1,\ u_{ji} = \begin{cases} \hbar, & \text{if } j = 2\ell, i = 2\ell - 1, \\ 0, & \text{otherwise} \end{cases} \tag{3.8}$$

for all $1 \leq i < j \leq 2k$. By Theorem 2.4, there exists the restricted skew polynomial extension

$$A_{2k} = \mathbb{F}[x_1][x_2; \nu_2] \cdots [x_{2k-1}][x_{2k}; \nu_{2k}]$$

over $\mathbb{F}$, where

$$\beta_j(x_i) = a_{ji}x_i = x_i,\ (1 \leq i < j \leq 2k),\ \nu_{2\ell}(x_i) = u_{2\ell i} = \begin{cases} \hbar, & \text{if } i = 2\ell - 1, \\ 0, & \text{if } i \neq 2\ell - 1, \end{cases}$$

since all $\beta_j$ and $\nu_j$ satisfy (1.1), (1.2), (2.1), (2.2). By Corollary 3.3, $A_{2k}$ is a quantization of the Poisson Weyl algebra $B_{2k}$, since all $a_{ji}$ and $u_{ji}$ satisfy (3.5) and

$$\frac{da_{ji}}{d\hbar}|_{\hbar=0} = 0,\ \ \frac{du_{ji}}{d\hbar}|_{\hbar=0} = \begin{cases} 1, & \text{if } j = 2\ell, i = 2\ell - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $A_{2k}$ is an $\mathbb{F}$-algebra generated by $x_1, x_2, \ldots, x_{2k}$ subject to the relations

$$x_jx_i - x_ix_j = \begin{cases} \hbar, & \text{if } j = 2\ell, i = 2\ell - 1, \\ 0, & \text{otherwise,} \end{cases} \tag{3.9}$$

which is the so-called *Moyal-Weyl quantization* [3, §20.1]. □

**Example 3.7.** Here we obtain a quantization of the Poisson Weyl algebra $B_{2k}$ appearing in quantum physics. Set $\mathbb{F} = \mathbb{C}[[\hbar]]$ and

$$a_{ji} = \begin{cases} \cos\hbar, & \text{if } i + j \text{ is odd,} \\ \sec\hbar, & \text{if } i + j \text{ is even,} \end{cases} \qquad u_{ji} = \begin{cases} \sin\hbar, & \text{if } j = 2\ell, i = 2\ell - 1, \\ 0, & \text{otherwise} \end{cases} \tag{3.10}$$

for all $1 \leq i < j \leq 2k$. Note that $a_{ji}, u_{ji} \in \mathbb{F}$ satisfy (3.5) and

$$\frac{da_{ji}}{d\hbar}|_{\hbar=0} = 0,\ \frac{du_{ji}}{d\hbar}|_{\hbar=0} = \begin{cases} 1, & \text{if } j = 2\ell, i = 2\ell - 1, \\ 0, & \text{otherwise} \end{cases}$$

by elementary calculus.

We will show that there exists a restricted skew polynomial extension

$$A_{2k} = \mathbb{F}[x_1][x_2;\beta_2,\nu_2]\cdots[x_{2k-1};\beta_{2k-1}][x_{2k};\beta_{2k},\nu_{2k}]$$

over $\mathbb{F}$, where

$$\beta_j(x_i) = a_{ji}x_i,\ \nu_j(x_i) = u_{ji},\ (1 \le i < j \le 2k).$$

For all $1 \le i < j \le 2k$, since all $\beta_j$ and $\nu_j$ satisfy (1.1) and (1.2), it is enough to show that all $\beta_j$ and $\nu_j$ satisfy (2.1) and (2.2). Assume that there exists a restricted skew polynomial extension $A_{2k-2}$ over $\mathbb{F}$. Note that, for any positive integers $i, j, \ell$,

$$\begin{aligned}&i+j \text{ is odd if and only if}\\ &(\ell+j \text{ is odd and } \ell+i \text{ is even}) \text{ or } (\ell+j \text{ is even and } \ell+i \text{ is odd}).\end{aligned} \tag{3.11}$$

Observe that $\mathbb{F}$-linear maps $\beta_{2k-1}$ and $\nu_{2k-1}$ satisfy (2.2) trivially since $\nu_{2k-1}(u_{ji}) = 0$ and $u_{2k-1,i} = 0$ for all $1 \le i < 2k-1$ and that they also satisfy (2.1) by (3.11) since $\beta_{2k-1}(u_{ji}) = u_{ji}$. Hence there exists a restricted skew polynomial extension $A_{2k-2}[x_{2k-1};\beta_{2k-1}]$ over $\mathbb{F}$ by Theorem 2.4. For $\mathbb{F}$-linear maps $\beta_{2k}$ and $\nu_{2k}$, they satisfy (2.1) and (2.2) by (3.11) since $\beta_{2k}(u_{ji}) = u_{ji}$ and $\nu_{2k}(u_{ji}) = 0$ and thus there exists $A_{2k} = A_{2k-2}[x_{2k-1};\beta_{2k-1}][x_{2k};\beta_{2k},\nu_{2k}]$ by Theorem 2.4. Moreover $A_{2k}$ is a quantization of the Poisson Weyl algebra $B_{2k}$ by Corollary 3.3 since $a_{ji}, u_{ji}$ satisfy (3.5) and (3.6).

Note that $A_{2k}$ is an $\mathbb{F}$-algebra generated by $x_1, x_2, \ldots, x_{2k-1}, x_{2k}$ subject to the relations

$$\begin{aligned} x_{2\ell}x_{2\ell-1} - (\cos\hbar)x_{2\ell-1}x_{2\ell} &= \sin\hbar, && (\ell = 1,\ldots,k),\\ x_jx_i - (\sec\hbar)x_ix_j &= 0, && (i<j,\ \ i+j \text{ is even}),\\ x_jx_i - (\cos\hbar)x_ix_j &= 0, && \begin{pmatrix} i<j,\ i+j \text{ is odd,}\\ \text{if } j = 2\ell, \text{ then } i \ne 2\ell-1\end{pmatrix}. \end{aligned} \tag{3.12}$$

In particular, we obtain a deformation, a $\mathbb{C}$-algebra generated by $x_1, x_2, \ldots,$ $x_{2k-1}, x_{2k}$ subject to the relations

$$x_jx_i + x_ix_j = 0\ (j > i),$$

by setting $\hbar = \pi + 2n\pi$ $(n \in \mathbb{Z})$ in the relation (3.12) in which $x_i^2$ is a central element for $i = 1, \ldots, 2k$. In (3.9), we obtain a deformation, a $\mathbb{C}$-algebra $W$ generated by $x_1, x_2, \ldots, x_{2k-1}, x_{2k}$ subject to the relations

$$x_jx_i - x_ix_j = \begin{cases} 1, & \text{if } j = 2\ell, i = 2\ell-1,\\ 0, & \text{otherwise,}\end{cases}$$

by setting $\hbar = 1$. This is the $k$-th Weyl algebra. This implies that (3.12) is a quantization different from (3.9). □

*Remark* 3.8. Note that the quantized algebras in Example 3.6 and Example 3.7 are distinct quantizations of the Poisson Weyl algebra.

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