Bull. Korean Math. Soc. **58** (2021), No. 4, pp. 1031–1038
https://doi.org/10.4134/BKMS.b200809
pISSN: 1015-8634 / eISSN: 2234-3016

ON FINITE GROUPS WITH THE SAME ORDER TYPE AS SIMPLE GROUPS $F_4(q)$ WITH q EVEN

Ashraf Daneshkhah, Fatemeh Moameri, and Hosein Parvizi Mosaed

Abstract. The main aim of this article is to study quantitative structure of finite simple exceptional groups $F_4(2^n)$ with $n > 1$. Here, we prove that the finite simple exceptional groups $F_4(2^n)$, where $2^{4n}+1$ is a prime number with $n > 1$ a power of 2, can be uniquely determined by their orders and the set of the number of elements with the same order. In conclusion, we give a positive answer to J. G. Thompson's problem for finite simple exceptional groups $F_4(2^n)$.

1. Introduction

For a finite group G, the set $\mathsf{nse}(G)$ of the number of elements in G with the same order links to a well-known problem posed by J. G. Thompson (1987) which is related to algebraic number fields [8, Problem 12.37]:

For a finite group G and a natural number n, set $G(n) = \{g \in G \mid g^n = 1\}$ and define the type of G to be the function whose value at n is the size of $G(n)$. Is it true that a group is solvable if its type is the same as that of a solvable one?

It immediately turns out that if two groups G and H are of the same type, then $|G| = |H|$ and $\mathsf{nse}(G) = \mathsf{nse}(H)$. Therefore, if a group G has been uniquely determined by its order and $\mathsf{nse}(G)$, then Thompson's problem is true for G. One may ask this problem for non-solvable groups, in particular, finite simple groups. In this direction, Shao and et al. [9] studied finite simple groups with at most four prime divisors of their orders and nse. Following this investigation, such problem has been studied for some families of simple groups [1,2] including Suzuki groups $\mathrm{Sz}(q)$ and Small Ree groups $\mathrm{R}(q)$. In this paper, we prove that:

Theorem 1.1. *Let G be a group with $\mathsf{nse}(G) = \mathsf{nse}(F_4(2^n))$ and $|G| = |F_4(2^n)|$, where $2^{4n} + 1$ is a prime number and $n > 1$ is a power of 2. Then G is isomorphic to* $F_4(2^n)$.

Received September 21, 2020; Accepted January 14, 2021.
2010 *Mathematics Subject Classification.* Primary 20D60; Secondary 20D06.
Key words and phrases. Exceptional groups of Lie type, prime graph, the set of the number of elements with the same order.

In order to prove Theorem 1.1, we determine the number of elements in $\mathrm{F}_4(2^n)$ with the same order in Proposition 3.1. Then we prove that the prime graph of the group G satisfying hypothesis of Theorem 1.1 has at least two components, see Proposition 3.2, and then we show that a section of G is isomorphic to $\mathrm{F}_4(2^n)$. Finally, we prove that G is isomorphic to $\mathrm{F}_4(2^n)$.

1.1. Definitions and notation

All sets and groups in this paper are finite. The symmetric and alternating groups on n letters are denoted by S_n and A_n, respectively. A Frobenius group G with kernel K and complement H is a semidirect product $G = K \rtimes H$ such that K is a normal subgroup in G, and $\mathbf{C}_K(x) = 1$ for every non-identity element x of H. A group G is a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively.

For finite simple groups of Lie type, we adopt the standard notation as in [5], and in particular, we use the notation recorded in Table 1 to denote the finite simple classical groups.

Table 1. Finite simple classical groups

X	d	$\lvert X\rvert$	$\lvert\mathbf{Out}(X)\rvert$
$\mathrm{PSL}_n(q)$, $n \geqslant 3$	$\gcd(n, q-1)$	$d^{-1}q^{\frac{n(n-1)}{2}}\mathrm{p}_2^n(q)$	$2ad$
$\mathrm{PSL}_2(q)$, $q \neq 2, 3$	$\gcd(2, q-1)$	$d^{-1}q(q^2-1)$	ad
$\mathrm{PSU}_n(q)$, $n \geqslant 3$, $(n, q) \neq (3, 2)$	$\gcd(n, q+1)$	$d^{-1}q^{\frac{n(n-1)}{2}}\mathrm{u}_2^n(q)$	$2ad$
$\mathrm{PSp}_{2m}(q)$, $m \geqslant 3$	$\gcd(2, q-1)$	$d^{-1}q^{m^2}\mathrm{p}_1^m(q^2)$	ad
$\mathrm{PSp}_4(q)$, $q \neq 2$	$\gcd(2, q-1)$	$d^{-1}q^4(q^2-1)(q^4-1)$	$2a$
$\mathrm{P\Omega}_{2m+1}(q)$, q odd and $m \geqslant 3$	2	$2^{-1}q^{m^2}\mathrm{p}_1^m(q^2)$	$2a$
$\mathrm{P\Omega}_{2m}^+(q)$, $m \geqslant 5$	$\gcd(4, q^m-1)$	$d^{-1}q^{m(m-1)}(q^m-1)\mathrm{p}_1^{m-1}(q^2)$	$2ad$
$\mathrm{P\Omega}_8^+(q)$	$\gcd(4, q^4-1)$	$d^{-1}q^{12}(q^4-1)\prod_{i=1}^3(q^{2i}-1)$	$6ad$
$\mathrm{P\Omega}_{2m}^-(q)$, $m \geqslant 4$	$\gcd(4, q^m+1)$	$d^{-1}q^{m(m-1)}(q^m+1)\mathrm{p}_1^{m-1}(q^2)$	$2ad$

Note: $\mathrm{p}_t^n(q) = \prod_{i=t}^n(q^i-1)$ and $\mathrm{u}_t^n(q) = \prod_{i=t}^n(q^i-(-1)^i)$, where $q = p^a$ with p prime.

In this manner, the only repetitions are

$$\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong \mathrm{A}_5, \quad \mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2), \quad \mathrm{PSL}_2(9) \cong \mathrm{A}_6,$$
$$\mathrm{PSL}_4(2) \cong \mathrm{A}_8, \quad \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2).$$

For a positive integer n, the set of prime divisors of n is denoted by $\pi(n)$, and if G is a finite group, $\pi(G) := \pi(|G|)$, where $|G|$ is the order of G. We denote the set of elements' orders of G by $\omega(G)$ (known as spectrum of G). Recall that $\mathsf{nse}(G)$ is the set of the numbers elements in G with the same order. In other word, $\mathsf{nse}(G)$ consists of the number $\mathsf{m}_i(G)$ of elements of order i in G for $i \in \omega(G)$. Also, we denote a Sylow p-subgroup of G by G_p and the number of Sylow p-subgroups of G by $\mathsf{n}_p(G)$. The prime graph $\Gamma(G)$ of a finite group G is a graph whose vertex set is $\pi(G)$, and two distinct vertices u and v are adjacent if and only if $uv \in \omega(G)$. Assume further that $\Gamma(G)$ has $t(G)$ connected components $\pi_i(G)$ for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1(G)$, and $\pi_1(G)$ is said to be the even component of G. Also we denote by $\omega_i(G)$ the subset of $\omega(G)$ consisting of all the numbers such that their prime divisors are in $\pi_i(G)$. Further, the largest element in each $\omega_i(G)$ is called the order component of G.

2. Preliminaries

In this section, we give some useful results which will be used in the proof of Theorem 1.1.

Lemma 2.1 ([3, Theorem 2]). *Let G be a Frobenius group of even order with kernel K and complement H. Then the following statements hold*:

(a) *K is a nilpotent group*;
(b) *$|H|$ divides $|K|-1$*;
(c) *$t(G)=2$, $\pi(H)$ and $\pi(K)$ are the connected components of $\Gamma(G)$.*

Lemma 2.2 ([3, Theorem 2]). *Let G be a 2-Frobenius group of even order. Then the following statements hold*:

(a) *$t(G)=2$, $\pi_1(G)=\pi(H)\cup\pi(G/K)$, and $\pi_2(G)=\pi(K/H)$*;
(b) *G/K and K/H are cyclic groups, $|G/K|$ divides $|\mathsf{Aut}(K/H)|$, $\gcd(|G/K|,|K/H|)=1$ and $|G/K|<|K/H|$*;
(c) *H is a nilpotent group and G is a solvable group.*

Lemma 2.3 ([10, Lemma 3 and Theorem A]). *Let G be a finite group with $t(G)\geqslant 2$. Then one of the following statements holds*:

(a) *G is a Frobenius group*;
(b) *G is a 2-Frobenius group*;
(c) *G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group, $|G/K|$ divides $|\mathsf{Out}(K/H)|$, $t(K/H)\geqslant t(G)$, and for any $i\in\{2,\dots,t(G)\}$, there exists $j\in\{2,\dots,t(K/H)\}$ such that $\pi_i(G)=\pi_j(K/H)$.*

Lemma 2.4 ([6, Page 4]). *Let G be a finite group, and let n be a positive integer dividing $|G|$. If $G(n)=\{g\in G \mid g^n=1\}$, then n divides $|G(n)|$.*

In what follows, φ is the *Euler totient* function. The proof of the following result is straightforward by Lemma 2.4.

Lemma 2.5. *Let G be a finite group, and let $i\in\omega(G)$. Then $\mathsf{m}_i(G)=k\varphi(i)$, where k is the number of cyclic subgroups of order i in G, and i divides $\sum_{j|i}\mathsf{m}_j(G)$. Moreover, if $i>2$, then $\mathsf{m}_i(G)$ is even.*

Lemma 2.6 ([11, Lemma 6]). *Let a,m,n be natural numbers. Then*

(a) $\gcd(a^m-1,a^n-1)=a^{\gcd(m,n)}-1$;
(b) $\gcd(a^m+1,a^n+1)=\begin{cases} a^{\gcd(m,n)}+1, & \text{if both } \frac{m}{\gcd(m,n)} \text{ and } \frac{n}{\gcd(m,n)} \text{ are odd};\\ \gcd(2,a+1), & \text{otherwise.}\end{cases}$

(c) $\gcd(a^m-1, a^n+1) = \begin{cases} a^{\gcd(m,n)}+1, & \text{if } \frac{m}{\gcd(m,n)} \text{ is even and } \frac{n}{\gcd(m,n)} \text{ is odd;} \\ \gcd(2, a+1), & \text{otherwise.} \end{cases}$

In particular, for every $a \geq 2$ and $m \geq 1$, the inequality $\gcd(a^m-1, a^m+1) \leq 2$ holds.

A group G is called a C_{pp}-group if the centralizers of its elements of order p in G are p-groups.

Lemma 2.7 ([4]). *Let $p = 2^\alpha 3^\beta + 1$ be a prime number. Then the finite simple C_{pp}-groups are as in* Table 2.

TABLE 2. Finite simple C_{pp}-groups

p	*Group*	*Conditions*
2	A_5, A_6	
2	$PSL_2(q)$, $PSL_3(2^2)$	q *Fermat or Mersenne prime*, $q = 2^n \geqslant 8$
2	$Sz(2^{2n+1})$	$n \geqslant 1$
3	A_5, A_6	
3	$PSL_2(q)$, $PSL_2(2^3)$, $PSL_3(2^2)$	$q = 3^{n+1}$, $q = 2 \cdot 3^n \pm 1$ *prime*, $n \geqslant 1$
5	A_5, A_6, A_7	
5	$PSL_2(q)$, $PSL_2(7^2)$, $PSL_3(2^2)$, $PSU_4(3)$, $PSp_4(3)$, $PSp_4(7)$	$q = 5^n$, $q = 2 \cdot 5^n \pm 1$ *prime*, $n \geqslant 1$
5	$Sz(2^3)$, $Sz(2^5)$	
5	M_{11}, M_{22}	
7	A_7, A_8, A_9	
7	$PSL_2(q)$, $PSL_2(2^3)$, $PSL_3(2^2)$, $PSU_3(3)$, $PSU_3(5)$, $PSU_3(19)$, $PSU_4(3)$, $PSU_6(2)$, $PSp_6(2)$, $P\Omega_8^+(2)$	$q = 7^n$, $q = 2 \cdot 7^n - 1$ *prime*, $n \geqslant 1$
7	$G_2(3)$, $G_2(19)$, $Sz(2^3)$	
7	M_{22}, J_1, J_2, HS	
13	A_{13}, A_{14}, A_{15}	
13	$PSL_2(q)$, $PSL_2(3^3)$, $PSL_2(5^2)$, $PSL_3(3)$, $PSL_4(3)$, $PSU_3(2^2)$, $PSU_3(23)$, $PSp_4(5)$, $PSp_6(3)$, $P\Omega_7(3)$, $P\Omega_8^+(3)$	$q = 13^n$, $q = 2 \cdot 13^n - 1$ *prime*, $n \geqslant 1$
13	$F_4(2)$, $G_2(2^2)$, $G_2(3)$, $Sz(2^3)$, ${}^3D_4(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$	
13	Suz, Fi_{22}	
17	A_{17}, A_{18}, A_{19}	
17	$PSL_2(q)$, $PSL_2(2^4)$, $PSp_4(4)$, $PSp_8(2)$, $P\Omega_8^-(2)$, $P\Omega_{10}^-(2)$	$q = 17^n$, $q = 2 \cdot 17^n \pm 1$ *prime*, $n \geqslant 1$
17	$F_4(2)$, ${}^2E_6(2)$	
17	J_3, He, Fi_{23}, Fi_{24}	
19	A_{19}, A_{20}, A_{21}	
19	$PSL_2(q)$, $PSL_3(7)$, $PSU_3(2^3)$	$q = 19^n$, $2 \cdot 19^n - 1$ *prime*, $n \geqslant 1$
19	$R(3^3)$, ${}^2E_6(2)$	
19	J_1, J_3, O′N, Th, HN	
37	A_{37}, A_{38}, A_{39}	
37	$PSL_2(q)$, $PSU_3(11)$	$q = 37^n$, $2 \cdot 37^n - 1$ *prime*, $n \geqslant 1$
37	$R(3^3)$, ${}^2F_4(2^3)$	
37	J_4, Ly	
73	A_{73}, A_{74}, A_{75}	
73	$PSL_2(q)$, $PSL_3(2^3)$, $PSU_3(3^2)$, $PSp_6(2^3)$	$q = 73^n$, $2 \cdot 73^n - 1$ *prime*, $n \geqslant 1$
73	$G_2(2^3)$, $G_2(3^2)$, $F_4(3)$, $E_6(2)$, $E_7(2)$, ${}^3D_4(3)$	
109	A_{109}, A_{110}, A_{111}	
109	$PSL_2(q)$	$q = 109^n, 2 \cdot 109^n - 1$ *prime*, $n \geqslant 1$
109	${}^2F_4(2^3)$	
2^m+1	A_p, A_{p+1}, A_{p+2}	$m = 2^s$
2^m+1	$PSL_2(q)$	$m = 2^s$, $q = 2^m$, $q = p^n$, $q = 2 \cdot p^n \pm 1$ *prime*, $s \geqslant n \geqslant 1$
2^m+1	$PSp_a(2^b)$	$m = 2^s$, $a = 2^{c+1}$, $b = 2^d$, $c \geqslant 1$, $c+d = s$
2^m+1	$P\Omega_{2(m+1)}^-(2)$	$m = 2^s$, $s \geqslant 1$
2^m+1	$P\Omega_a^-(2^b)$	$m = 2^s$, $a = 2^{c+1}$, $b = 2^d$, $c \geqslant 2$, $c+d = s$
2^m+1	$F_4(2^e)$	$4e = m = 2^s$, $e \geqslant 1$
other	A_p, A_{p+1}, A_{p+2}	
other	$PSL_2(q)$	$q = p^n$, $2 \cdot p^n - 1$ *prime* , $n \geqslant 1$

3. Proof of the main result

In this section, we prove Theorem 1.1. Here we set $F := F_4(q)$, where $q = 2^n$, and $p = q^4 + 1$. Let also G be a finite group with $\mathsf{nse}(G) = \mathsf{nse}(F)$ and $|G| = |F| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$. In the following proposition, we determine the set of the number of elements in F with the same order.

Proposition 3.1. *Let* F *be the finite simple exceptional group* $F_4(2^n)$*, where* $p = 2^{4n} + 1$ *is a prime number and* $n > 1$ *is a power of* 2. *Then the following properties hold*:

(a) $\mathsf{m}_p(F) = (p - 1)|F|/(8p)$;

(b) p *divides* $\mathsf{m}_i(F)$ *for all* $i \in \omega(F) \setminus \{1, p\}$.

Proof. (a) Suppose that F_p is a Sylow p-subgroup of F. Since F_p is a cyclic group of order p, Lemma 2.5 implies that $\mathsf{m}_p(F) = \varphi(p)\mathsf{n}_p(F) = (p - 1)\mathsf{n}_p(F)$. We now obtain $\mathsf{n}_p(F)$. By [7], p is an isolated vertex of the prime graph $\Gamma(F)$ of F. Then $|\mathbf{C}_F(F_p)| = p$ and $|\mathbf{N}_F(F_p)| = kp$ for some positive integer k, and so k divides $p - 1$ because $\mathbf{N}_F(F_p)/\mathbf{C}_F(F_p) \lesssim \mathsf{Aut}(F_p)$. On the other hand, by the Sylow's theorem, we have that $p \mid (8 - k)$. This follows that $k = 8$ and $\mathsf{n}_p(F) = |F|/(8p)$, as claimed.

(b) Let $i \in \omega(F) \setminus \{1, p\}$. As p is an isolated vertex of $\Gamma(F)$, it follows that p does not divide i and $pi \notin \omega(F)$. Therefore, F_p acts fixed point freely by conjugation on the set of elements of order i, and this implies that $|F_p| \mid \mathsf{m}_i(F)$, as desired. □

Proposition 3.2. *Let* F *be the finite simple exceptional group* $F_4(2^n)$*, where* $p = 2^{4n} + 1$ *is a prime number and* $n > 1$ *is a power of* 2. *If* G *is a finite group with* $\mathsf{nse}(G) = \mathsf{nse}(F)$ *and* $|G| = |F|$*, then the following properties of group* G *hold*:

(a) $\mathsf{m}_2(G) = \mathsf{m}_2(F)$;

(b) $\mathsf{m}_p(G) = \mathsf{m}_p(F)$;

(c) $\mathsf{n}_p(G) = \mathsf{n}_p(F)$;

(d) p *is an isolated vertex of* $\Gamma(G)$;

(e) $p \mid \mathsf{m}_i(G)$ *for all* $i \in \omega(G) \setminus \{1, p\}$.

Proof. (a) By Lemma 2.5, $\mathsf{m}_i(G)$ is odd if and only if $i = 1$ or 2, and so $\mathsf{m}_2(G) = \mathsf{m}_2(F)$.

(b) As $p \nmid \mathsf{m}_p(G)$ and $\mathsf{nse}(G) = \mathsf{nse}(F)$, Proposition 3.1 implies that $\mathsf{m}_p(G) = \mathsf{m}_p(F)$.

(c) We know that both G_p and F_p are cyclic groups of order p. So by part (b), we have that $\mathsf{m}_p(G) = \varphi(p)\mathsf{n}_p(G) = \varphi(p)\mathsf{n}_p(F) = \mathsf{m}_p(F)$, which gives $\mathsf{n}_p(G) = \mathsf{n}_p(F)$.

(d) Assuming to the contrary that p is not an isolated vertex of $\Gamma(G)$. Then there exists $i \in \pi(G) - \{p\}$ such that $ip \in \omega(G)$. We now obtain $\mathsf{m}_{ip}(G)$. We know that $\mathsf{m}_{ip}(G) = \phi(ip)\mathsf{n}_p(G)k$, where k is the number of cyclic subgroups of order i in $\mathbf{C}_G(G_p)$, and since $\mathsf{n}_p(G) = \mathsf{n}_p(F)$, this implies that $\mathsf{m}_{ip}(G) =$

$(i-1)(p-1)|\mathrm{F}|k/(8p)$. Assume that $\mathsf{m}_{ip}(G)$ is coprime to p, that is to say, $p \nmid \mathsf{m}_{ip}(G)$. Then, by Proposition 3.1, $\mathsf{m}_{ip}(G) = \mathsf{m}_p(G)$, and so $i = 2$ and $k = 1$. Lemma 2.5 implies that p divides $\mathsf{m}_2(G) + \mathsf{m}_{2p}(G)$, and since $\mathsf{m}_2(G) = \mathsf{m}_2(\mathrm{F})$ and $p \mid \mathsf{m}_2(\mathrm{F})$, we deduce that p divides $\mathsf{m}_{2p}(G)$, which is a contradiction. Therefore, $p \mid \mathsf{m}_{ip}(G)$, and hence p divides $(i-1)k$. Thus the fact that $\mathsf{m}_{ip}(G) < |G|$ yields $p - 1 \leq 8$, but this is impossible as $p = 2^{4n} + 1$ and $n > 1$ is a power of 2. Therefore, p is an isolated vertex of $\Gamma(G)$.

(e) It follows from part (d) that p is an isolated vertex of $\Gamma(G)$. Then $p \nmid i$ and $pi \notin \omega(G)$, and so G_p acts fixed point freely by conjugation on the set of elements of order i. Thus $|G_p|$ divides $\mathsf{m}_i(G)$, and hence p divides $\mathsf{m}_i(G)$ as claimed. □

Proposition 3.3. *Let* F *be the finite simple exceptional group* $\mathrm{F}_4(2^n)$, *where* $p = 2^{4n} + 1$ *is a prime number and* $n > 1$ *is a power of* 2. *If* G *is a finite group with* $\mathsf{nse}(G) = \mathsf{nse}(\mathrm{F})$ *and* $|G| = |\mathrm{F}|$, *then* G *is neither a Frobenius group, nor a* 2*-Frobenius group.*

Proof. Suppose to the contrary that G is a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now by Proposition 3.2, p is an isolated vertex of $\Gamma(G)$, and hence either $|H| = p$ and $|K| = |G|/p$, or $|H| = |G|/p$ and $|K| = p$, with $p = 2^{4n} + 1$ prime. The latter case can be ruled out as $|H|$ must divide $|K| - 1$. Therefore, $|H| = p$ and $|K| = |G|/p$, and hence $p = 2^{4n} + 1$ divides 7, which is impossible.

Suppose to the contrary that G is a 2-Frobenius group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively. Since p is an isolated vertex of $\Gamma(G)$, we conclude that $|K/H| = p$. Thus Lemma 2.2 implies that $|G/K|$ divides $|\mathsf{Aut}(K/H)| = p - 1$. Therefore, $2^n + 1$ divides $|H|$. As H is a nilpotent group, $H_t \rtimes L$ is a Frobenius group with kernel H_t and complement L, where L is the complement of Frobenius group K and $t \in \pi(2^n + 1)$. Therefore $p = 2^{4n} + 1$ divides $t - 1$, which is impossible. □

Proof of Theorem 1.1. Let F be the finite simple exceptional group $\mathrm{F}_4(2^n)$, where $p = 2^{4n} + 1$ is a prime number and $n > 1$ is a power of 2. Let also G be a finite group with $\mathsf{nse}(G) = \mathsf{nse}(\mathrm{F})$ and $|G| = |\mathrm{F}|$. By Propositions 3.2 and 3.3, the prime graph of G has at least two connected components and G is neither a Frobenius group nor a 2-Frobenius group. Thus Lemma 2.3 implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$. Moreover, any odd order component of G is also an odd order component of K/H.

We first prove that K/H is isomorphic to F. Since p is an odd order component of G, Lemma 2.3 follows immediately that p is an odd order component of K/H. Thus K/H is a simple C_{pp}-group, and hence K/H is isomorphic to one of the groups recorded in Table 2. In what follows, we discuss the alternating and the classical cases and other cases can be treated in a similar manner.

(1) K/H is not isomorphic to alternating groups.

If $K/H \cong \mathrm{A}_n$, then according to Table 2, $n \in \{p, p+1, p+2\}$. We know that $|K/H| \mid |G|$, so q^4-2 divides $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$, but since $q = 2^n$ with $n > 1$ power of 2, it is impossible by Lemma 2.6.

(2) K/H is not isomorphic to projective special linear groups.

If K/H is isomorphic to $\mathrm{PSL}_2(q')$, then by Table 2, we have the following three cases to consider:

(2.1) $q' = q^4$. Since $|G/K|$ divides $|\mathsf{Out}(K/H)|$, and by Table 1, $|\mathsf{Out}(K/H)| = 4n$ and n is a power of 2, we deduce that $|G/K|$ is a divisor of $4n$ and $2(q^{12}-1)(q^6-1)(q^2-1)$ divides $|H|$. Thus for every $i \in \pi(q^4-q^2+1)$, we have that $H_i = G_i$. This implies that $\mathsf{m}_i(H) = \mathsf{m}_i(G)$. On the other hand, H has only one Sylow i-subgroup since H is nilpotent. Thus $\mathsf{m}_i(H) \leqslant q^4-q^2+1$, which is impossible because $\mathsf{m}_i(H) = \mathsf{m}_i(G)$ and $p \mid \mathsf{m}_i(G)$.

(2.2) $q' = p^k$. Then $p+1$ divides $|K/H|$, but $p+1$ does not divide $|G|$, which is a contradiction.

(2.3) $q' = 2 \cdot p^k \pm 1$. Then $k = 1$ because p^2 divides $|K/H|$ and $p^2 \nmid |G|$. So $|K/H| = 2p(2p \pm 1)(2p \pm 2)$, which is a contradiction as $2p \pm 1 \nmid |G|$.

(3) K/H is not isomorphic to projective symplectic groups.

By Table 2, $K/H \cong \mathrm{PSp}_a(2^b)$, where $a = 2^{c+1}$ and $b = 2^d$ with $c \geqslant 1$, $c + d = s$. If $c > 2$, then $2^{10b} - 1 \mid |K/H|$ but $2^{10b} - 1$ does not divide $|G|$, which is a contradiction. Thus $c = 1$ or 2. Since $|G/K|$ divides $|\mathsf{Out}(K/H)|$, $|\mathsf{Out}(K/H)| \mid 2b$ and b is a power of 2, we deduce that $|G/K|$ divides $2b$ and $q^4 - q^2 + 1 \mid |H|$. Thus for every $i \in \pi(q^4-q^2+1)$, we have that $H_i = G_i$. This implies that $\mathsf{m}_i(H) = \mathsf{m}_i(G)$. Note that H is nilpotent. Thus H has only one Sylow i-subgroup, and so $\mathsf{m}_i(H) \leqslant q^4-q^2+1$. This is impossible as $\mathsf{m}_i(H) = \mathsf{m}_i(G)$ and $p \mid \mathsf{m}_i(G)$.

(4) K/H is not isomorphic to simple groups of orthogonal type.

If $K/H \cong \mathrm{P}\Omega^-_{2(m+1)}(2)$, then $2q^4 + 1 \mid |K/H|$, which is a contradiction as $2q^4+1$ is not a divisor of $|G|$. If $K/H \cong \mathrm{P}\Omega^-_a(2^b)$, where $a = 2^{c+1} \geqslant 8$ and $b = 2^d$ with $c+d = s$, then since $(q^4)^{2^c-1} \mid |K/H|$, it follows that $2^c - 1 \leqslant 6$, and hence $c = 2$. Since also $|G/K|$ divides $|\mathsf{Out}(K/H)| = 2b$ and b is a power of 2, we deduce that $|G/K| \mid 2b$ and $q^4-q^2+1 \mid |H|$. Thus for every $i \in \pi(q^4-q^2+1)$, we have $H_i = G_i$. This implies that $\mathsf{m}_i(H) = \mathsf{m}_i(G)$, but H has only one Sylow i-subgroup. Thus $\mathsf{m}_i(H) \leqslant q^4-q^2+1$, which is impossible as $\mathsf{m}_i(H) = \mathsf{m}_i(G)$ and $p \mid \mathsf{m}_i(G)$.

As noted above, by a similar argument, we conclude that K/H is isomorphic to $\mathrm{F}_4(2^e)$ with $4e = 2^s$. Thus $q = 2^e$, and hence $|K/H| = |G|$. Therefore, $H = 1$, and consequently, $G = K = \mathrm{F}_4(q)$. $\square$

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Ashraf Daneshkhah
Department of Mathematics
Faculty of Science
Bu-Ali Sina University
Hamedan, Iran
Email address: adanesh@basu.ac.ir

Fatemeh Moameri
Department of Mathematics
Faculty of Science
Bu-Ali Sina University
Hamedan, Iran
Email address: f.moameri@basu.ac.ir

Hosein Parvizi Mosaed
Alvand Institute of Higher Education
Hamedan, Iran
Email address: h.parvizi.mosaed@gmail.com