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MAXIMAL FUNCTIONS ALONG TWISTED SURFACES ON PRODUCT DOMAINS

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Abstract. In this paper, we introduce a class of maximal functions along twisted surfaces in $\mathbb{R}^n \times \mathbb{R}^m$ of the form

$$\{(\phi(|v|)u, \varphi(|u|)v) : (u, v) \in \mathbb{R}^n \times \mathbb{R}^m\}.$$

We prove L^p bounds when the kernels lie in the space $L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. As a consequence, we establish the L^p boundedness for such class of operators provided that the kernels are in $L \log L(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ or in the Block spaces $B_q^{0,0}\left(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}\right)$ $(q > 1)$.

1. Introduction and statement of results

Let $\mathbb{R}^d$, $(d \geq 2)$ be the d-dimensional Euclidean space and $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ equipped with the normalized Lebesgue measure $d\sigma_d$. Let $\mathbb{R}_+ = [0, \infty)$ and let U be the class of all measurable functions $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ that satisfy

$$\|h\|_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, r^{-1}s^{-1}drds)} = \left(\int_0^\infty \int_0^\infty |h(r, s)|^2 r^{-1} s^{-1} drds\right)^{\frac{1}{2}} \leq 1.$$

Let $\Gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ be a mapping given by

$$\Gamma(u, v) = (\phi(|v|)u, \varphi(|u|)v), \tag{1}$$

where ϕ and φ are real valued functions defined on $[0, \infty)$. Let Ω be an integrable function on $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ that satisfies

$$\Omega(tx, sy) = \Omega(x, y) \text{ for all } t, s > 0 \tag{2}$$

and

$$\int_{\mathbb{S}^{n-1}} \Omega\left(u', \cdot\right) d\sigma_n\left(u'\right) = \int_{\mathbb{S}^{m-1}} \Omega\left(\cdot, v'\right) d\sigma_m\left(v'\right) = 0. \tag{3}$$

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Consider the maximal function $M_{\Omega,\phi,\varphi}$ given by

$$M_{\Omega,\phi,\varphi}(f)(x,y) = \sup_{h\in U} \left|S_{\Gamma,\Omega,h}(f)(x,y)\right|, \tag{4}$$

where

$$S_{\Gamma,\Omega,h}(f)(x,y) = \iint\limits_{\mathbb{R}^n\times\mathbb{R}^m} f((x,y)-\Gamma(u,v))\frac{h(|u|,|v|)\Omega(u',v')}{|u|^n|v|^m}dudv.$$

When $\phi(t) = \varphi(t) = c$-constant, the operator $M_{\Omega,\phi,\varphi}$ reduces to the classical operator M_Ω introduced by Ding in 1999 in [12]. Ding proved that the operator M_Ω is bounded on $L^2(\mathbb{R}^n\times\mathbb{R}^m)$ provided that the function $|\Omega|(\log^+|\Omega|)^2$ is integrable on $\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$. Subsequently, Al-Salman proved the L^p boundedness for all $2 \le p < \infty$ under the weaker condition that $\Omega \in L\log L(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$, i.e., the function $|\Omega|(\log^+|\Omega|)$ is integrable on $\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$ [3]. In the same paper, Al-Salman showed that the condition $\Omega \in L\log L(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ can not be replaced by any condition of the form $\Omega \in L(\log L)^{1-\varepsilon}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$, $\varepsilon > 0$. For further results concerning the operator M_Ω, we advise readers to consult [2, 3, 10], among others.

For non constant functions ϕ and φ, the L^p boundedness of the corresponding operator $M_{\Omega,\phi,\varphi}$ is not known even for power functions. In fact, the operator $M_{\Omega,\phi,\varphi}$ is considered to be hard and its treatment is very involved due to the twisted nature of the surface Γ. It is our aim in this paper to consider the L^p boundedness of the operator $M_{\Omega,\phi,\varphi}$ for non classical surfaces Γ.

By duality, it follows that the maximal function $M_{\Omega,\phi,\varphi}$ is given by

$$M_{\Omega,\phi,\varphi}(f)(x,y) = \left(\int_0^\infty\int_0^\infty \left|N_{\phi,\varphi,\Omega}(f)(r,s)\right|^2 \frac{drds}{rs}\right)^{\frac{1}{2}}, \tag{5}$$

where

$$N_{\phi,\varphi,\Omega}(f)(r,s) = \int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{m-1}} f(x-\phi(s)ru', y-\varphi(r)sv')\Omega(u',v')\,d\sigma_n d\sigma_m.$$

We are interested in surfaces Γ where the functions ϕ and φ satisfy certain growth conditions. Let $\mathcal{F}$ be the class of smooth functions $\Phi : (0,\infty) \to \mathbb{R}$ which satisfy the following growth conditions:

$$|\Phi(t)| \le C_1 t^{d_\Phi},\ C_2 t^{d_\Phi-2} \le \left|\Phi''(t)\right| \le C_3 t^{d_\Phi-2} \tag{6}$$

for some $d_\Phi \neq 0$. We notice here that if $\varphi(t) = \phi(t) = t$, then there exists a smooth f such that $M_{\Omega,\phi,\varphi}(f) = \infty$. On the other hand if $\varphi(t) = t$ and $\phi(t) = t^d, d \neq 1$, then the corresponding operator $M_{\Omega,\phi,\varphi}$ is bounded on L^p for all $2 \le p < \infty$ under the weak condition $\Omega \in L\log L(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$. In fact, it can be shown that $M_{\Omega,\phi,\varphi}(f)(x,y) = (1/|1-d|)M_\Omega(f)(x,y)$. Thus, we are interested in surfaces Γ where the functions $\varphi, \phi \in \mathcal{F}$ with $d_\varphi \neq 1$ and $d_\phi \neq 1$. For convenience, we shall let $\mathcal{F}_1$ be the class of all $\varphi \in \mathcal{F}$ with $d_\varphi \neq 1$. Examples of functions in the class $\mathcal{F}_1$ are widely available such as the power functions $\varphi(t)=t^{\beta}(\beta\neq 1)$ and the function $\varphi(t)=t^{2}\left(1+e^{-\frac{1}{t^{2}}}\right)$. Our main result is the following:

Theorem 1.1. *Suppose that* $\Omega\in L^{q}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$, $q>1$ *and satisfies* (2)-(3) *with* $\|\Omega\|_{1}\leq 1$ *and* $\|\Omega\|_{q}\leq 2^{a}$ *for some* $a>1$. *If* $\varphi,\phi\in\mathcal{F}_{1}$ *with* $d_{\varphi}d_{\phi}\neq 1$, *then*

$$\left\|M_{\Omega,\phi,\varphi}(f)\right\|_{p}\leq aC_{p}\left\|f\right\|_{p} \tag{7}$$

for $p\geq 2$ *with constant* C_{p} *independent of* a.

As a consequence of the above result and suitable decomposition of the function Ω, we have the following result:

Theorem 1.2. *Suppose that* $\Omega\in L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ *and satisfies* (2)-(3). *If* $\varphi,\phi\in\mathcal{F}_{1}$ *with* $d_{\varphi}d_{\phi}\neq 1$, *then* $M_{\Omega,\phi,\varphi}$ *is bounded on* $L^{p}(\mathbb{R}^{n}\times\mathbb{R}^{m})$ *for all* $2\leq p<\infty$.

An immediate consequence of Theorem 1.2 is the following result concerning singular integral operators:

Corollary 1.3. *Let* $\varphi,\phi\in\mathcal{F}_{1}$ *with* $d_{\varphi}d_{\phi}\neq 1$. *Suppose that* $h\in L^{2}(\mathbb{R}_{+}\times\mathbb{R}_{+},$ $r^{-1}s^{-1}drds)$. *If* $\Omega\in L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ *and satisfies* (2)-(3), *then the singular integral operator*

$$T_{\phi,\varphi}(f)(x,y)=\iint\limits_{\mathbb{R}^{n}\times\mathbb{R}^{m}}f(x-\phi(|v|)u,y-\varphi(|u|)v)\frac{h(|u|,|v|)\Omega\left(u',v'\right)}{|u|^{n}\,|v|^{m}}dudv$$

is bounded on $L^{p}(\mathbb{R}^{n}\times\mathbb{R}^{m})$ *for all* $1<p<\infty$.

Singular integrals on product domains have been extensively studied by many authors, we cite [6], [7], [13], [14], [17], [18], among others.

By Theorem 1.2 and change of variables, we immediately obtain the following result concerning Marcinkiewicz integral operators considered in [9, 11]:

Corollary 1.4. *Let* $\varphi,\phi\in\mathcal{F}_{1}$ *with* $d_{\varphi}d_{\phi}\neq 1$. *Suppose that* $h\in L^{2}(\mathbb{R}_{+}\times\mathbb{R}_{+},$ $r^{-1}s^{-1}drds)$. *If* $\Omega\in L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ *and satisfies* (2)-(3), *then the Marcinkiewicz integral operator*

$$\mu_{\phi,\varphi}f(x,y)$$
$$=\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_{|u|\leq 2^{t}}\int_{|v|\leq 2^{s}}f(x-\phi(|v|)u,y-\varphi(|u|)v)\frac{\Omega\left(u',v'\right)}{|u|^{n-1}\,|v|^{m-1}}dudv\right|^{2}\frac{dtds}{2^{2(t+s)}}\right)^{\frac{1}{2}}$$

is bounded on $L^{p}(\mathbb{R}^{n}\times\mathbb{R}^{m})$ *for all* $2\leq p<\infty$.

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Throughout this paper the letter C will stand for a constant that may vary at each occurrence, but it is independent of the essential variables.

2. Preparation

The twisted nature of the surface Γ involves lacunary sequences of multi-indexes. This requires a fundamental extension of existing theory. To this end, we prove the following generalization of Lemma 2.1 in [3]:

Lemma 2.1. *Let $L:\mathbb{R}^n\to\mathbb{R}^n$ and $Q:\mathbb{R}^m\to\mathbb{R}^m$ be nonzero linear transformations. Suppose that $\gamma_1,\gamma_2>0$ with $\gamma_1\gamma_2\neq 1$, $a>1$, and $\alpha,\beta,C>0$. Suppose that $\sigma_a=\{\sigma_{a,r,s}:r,s\in\mathbb{R}\}$ is a family of measures satisfying*

(i) $\sup\limits_{r,s\in\mathbb{R}}\|\sigma_{a,r,s}\|\leq C$;

(ii) $$\int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}}|\hat{\sigma}_{a,r,s}(\xi,\eta)|^2\,\frac{dsdr}{sr}$$
$$\leq a^2C\min\left\{\left|2^{aj}2^{a\gamma_1k}L(\xi)\right|^{-\frac{\alpha}{a}},\left|2^{ak}2^{a\gamma_2j}Q(\eta)\right|^{-\frac{\beta}{a}}\right\}$$
for $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^m$;

(iii) $$\int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}}|\hat{\sigma}_{a,r,s}(\xi,\eta)|^2\,\frac{dsdr}{sr}$$
$$\leq a^2C\min\left\{\left|2^{a(j+1)}2^{a\gamma_1(k+1)}\,|L(\xi)|\right|^{\frac{\alpha}{a}},\left|2^{a(k+1)}2^{a\gamma_2(j+1)}\,|Q(\eta)|\right|^{\frac{\beta}{a}}\right\}$$
for $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^m$;

(iv) *For $1<p<\infty$, there exists a constant $C_p>0$ that is independent of a such that the maximal function*

$$\sigma_a^*(f)(x,y)=\sup_{j,k\in\mathbb{Z}}\int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}}||\sigma_{a,r,s}|*f(x)|\,\frac{dsdr}{sr}\tag{8}$$

satisfies

$$\|\sigma_a^*(f)\|_p\leq a^2C_p\,\|f\|_p\,.$$

Then the square function

$$S_\sigma(f)(x,y)=\left(\int_0^\infty\int_0^\infty|\sigma_{a,r,s}*f(x,y)|^2\,\frac{dsdr}{rs}\right)^{\frac{1}{2}}$$

satisfies

$$\|S_\sigma(f)\|_p\leq aC\,\|f\|_p$$

for all $2\leq p<\infty$ with L^p bounds independent of the parameter a and the linear transformations L and Q.

Proof. By similar argument as in [15] (see also [8]), we may assume that $L(\xi)=\pi^n_{n_L}(\xi)$ and $Q_s(\eta)=\pi^m_{m_Q}(\eta)$ where $n_L=\mathrm{rank}(L)$, $m_Q=\mathrm{rank}(Q)$,

$$\pi^n_{n_L}(\xi_1,\ldots,\xi_n)=(\xi_1,\ldots,\xi_{n_L})\text{ and }\pi^m_{m_Q}(\eta_1,\ldots,\eta_m)=(\eta_1,\ldots,\eta_{m_Q}).$$

Let

$$D(\mathbb{Z}\times\mathbb{Z})=\{(u,v):u=j+\gamma_1k\text{ and }v=k+\gamma_2j\text{ for some }j,k\in\mathbb{Z}\}.$$

It is clear that $D(\mathbb{Z}\times\mathbb{Z})$ is infinite countable. In fact, since $\gamma_1\gamma_2 \neq 1$, the mapping $(j,k) \to (j+\gamma_1 k, k+\gamma_2 j) \in D(\mathbb{Z}\times\mathbb{Z})$ is a bijection. Hence, $|D(\mathbb{Z}\times\mathbb{Z})| = |\mathbb{Z}\times\mathbb{Z}|$. Moreover, it can be shown that $\left|D^{(1)}(\mathbb{Z}\times\mathbb{Z})\right| = \left|D^{(2)}(\mathbb{Z}\times\mathbb{Z})\right| = |D(\mathbb{Z}\times\mathbb{Z})|$ where

$$D^{(1)}(\mathbb{Z}\times\mathbb{Z}) = \{u : (u,v) \in D(\mathbb{Z}\times\mathbb{Z}) \text{ for some } v\}$$

and

$$D^{(2)}(\mathbb{Z}\times\mathbb{Z}) = \{v : (u,v) \in D(\mathbb{Z}\times\mathbb{Z}) \text{ for some } u\}.$$

We also remark that the set $D(\mathbb{Z}\times\mathbb{Z})$ is closed under the usual addition of vectors in the plane, i.e., if $(u,v),(u',v') \in D(\mathbb{Z}\times\mathbb{Z})$, then $(u,v)+(u',v') = (u+u', v+v') \in D(\mathbb{Z}\times\mathbb{Z})$.

Now, we construct a sequence $\{\psi_u^{(1)}(t)\psi_v^{(2)}(s) : (u,v) \in D(\mathbb{Z}\times\mathbb{Z})\}$ where $\psi_u^{(1)}(t)$ and $\psi_v^{(2)}(s)$ are real valued functions on $\mathbb{R}$ such that $\psi_u^{(1)}(t), \psi_v^{(2)}(s) \in C^\infty$,

$$0 \le \psi_u^{(1)}(t), \psi_v^{(2)}(s) \le 1, \ \sum_{(u,v)} \left(\psi_u^{(1)}(t)\psi_v^{(2)}(s)\right)^2 = 1, \tag{9}$$

$$\psi_u^{(1)}(t) \subseteq (2^{-a(u+1+\gamma_1)}, 2^{-a(u-1-\gamma_1)}), \tag{10}$$

$$\psi_v^{(2)}(s) \subseteq (2^{-a(v+1+\gamma_2)}, 2^{-a(v-1-\gamma_2)}), \tag{11}$$

$$\left|\frac{d^l\psi_u^{(1)}(t)}{dt^l}\right| \le \frac{C_l}{t^l}, \quad \text{and} \quad \left|\frac{d^l\psi_v^{(2)}(s)}{ds^l}\right| \le \frac{C_l}{s^l}, \tag{12}$$

where C_l is independent of a, u, and v. For $j,k \in \mathbb{Z}$, let $\Psi_{j,k,a}$ be defined by

$$\hat{\Psi}_{j,k,a}(\xi,\eta) = \left(\psi_{j+\gamma_1 k}^{(1)}(|\pi_n^n\xi|^2)\psi_{k+\gamma_2 j}^{(2)}(|\pi_m^m\eta|^2)\right)^2. \tag{13}$$

Then by making use of the identity in (9), we have the following

$$S_\sigma(f)(x,y) \le \sum_{j,k\in\mathbb{Z}} S_{j,k,a}(f)(x,y), \tag{14}$$

where

$$S_{j,k,a}(f)(x,y)$$
$$= \left(\sum_{l,o\in\mathbb{Z}} \int_{2^{al}}^{2^{a(l+1)}} \int_{2^{ao}}^{2^{a(o+1)}} |\Psi_{j+l,k+o,a} * \sigma_{a,r,s} * f(x,y)|^2 \frac{dsdr}{rs}\right)^{\frac{1}{2}}. \tag{15}$$

By Littlewood-Paley theory [20], it can be shown that

$$\left\|\left(\sum_{l,o\in\mathbb{Z}} |\Psi_{j+l,k+o,a} * f|^2\right)^{\frac{1}{2}}\right\|_p \le C_p \|f\|_p \tag{16}$$

for all $p \geq 2$ with constant C_p independent of the parameter a. The independence of the constant C_p on the parameter a is a consequence of the property (12). Now, let

$$A_{a,j,k} = \left\{ \xi \in \mathbb{R}^n : 2^{-a(j+1)} 2^{-a(k+1)\gamma_1} < \left| \pi^n_{n_L} \xi \right| < 2^{-a(j-1)} 2^{-a(k-1)\gamma_1} \right\}$$

and

$$B_{a,j,k} = \left\{ \eta \in \mathbb{R}^m : 2^{-a(k+1)} 2^{-a(j+1)\gamma_2} < \left| \pi^m_{m_Q} \eta \right| < 2^{-a(k-1)} 2^{-a(j-1)\gamma_2} \right\}.$$

Let

$$F_{a,j,k,l,o}(\xi,\eta) = \iint\limits_{A_{a,j+l} \times B_{a,k+o}} \left| \hat{f}(\xi,\eta) \right|^2 \int_{2^{al}}^{2^{a(l+1)}} \int_{2^{a0}}^{2^{a(o+1)}} \left| \hat{\sigma}_{a,r,s}(\xi,\eta) \right|^2 \frac{dsdr}{sr} d\xi d\eta.$$

Now, by the assumptions (i)-(iii), we have

$$F_{a,j,k,l,o}(\xi,\eta) \leq \frac{2^{\alpha(2+\gamma_1+\gamma_2)} a^2}{2^{\alpha|j+k\gamma_1|} 2^{\alpha|k+j\gamma_2|}} \iint\limits_{A_{a,j+l} \times B_{a,k+o}} \left| \hat{f}(\xi,\eta) \right|^2 d\xi d\eta. \tag{17}$$

Thus, by Placherel's theorem and Fubini's theorem, we have

$$\left\| S_{j,k,a}(f) \right\|_2 \leq aC 2^{-\alpha|j+k\gamma_1|} 2^{-\alpha|k+j\gamma_2|} \left\| f \right\|_2. \tag{18}$$

Next, by the assumption (iv), (16), and duality argument, it can be shown that

$$\left\| S_{j,k,a}(f) \right\|_p \leq aC_p \left\| f \right\|_p \tag{19}$$

for all $p > 2$ with constant C_p independent of the essential variables. By interpolation between (18) and (19), we get

$$\left\| S_{j,k,a}(f) \right\|_p \leq aC 2^{-\varepsilon|j+k\gamma_1|} 2^{-\epsilon|k+j\gamma_2|} \left\| f \right\|_p \tag{20}$$

for $2 \leq p < \infty$ where ε, ϵ, and C_p are positive constants independent of a, j, and k (see [3] for details). Thus

$$\left\| S_{j,k,a}(f) \right\|_p \leq aC \left(\sum_{j,k \in \mathbb{Z}} 2^{-\varepsilon|j+k\gamma_1|} 2^{-\epsilon|k+j\gamma_2|} \right) \left\| f \right\|_p \leq aC \left\| f \right\|_p$$

for all $p \geq 2$. This completes the proof. □

In order to obtain estimates of maximal functions in the form (8), we recall the two parameter maximal functions introduced recently in [4]. For fixed points $z_1 \in \mathbb{S}^{n-1}$, $z_2 \in \mathbb{S}^{m-1}$ and $\phi, \varphi \in \mathcal{F}$, let $\mu^{(z_1,z_2)}_{\phi,\varphi}$ be the maximal function given by

$$\mu^{(z_1,z_2)}_{\phi,\varphi}(f)(x,y) = \sup_{j,k} \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} \left| f(x - \phi(s) r z_1, y - \varphi(r) s z_2) \right| \frac{drds}{rs}. \tag{21}$$

The following result can be found in [4]:

Theorem 2.2 ([4]). *The maximal function* $\mu_{\phi,\varphi}^{(z_1,z_2)}$ *is bounded on* $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ *for* $p \in (1,\infty)$ *with* L^p *bounds independent of the points* z_1 *and* z_2.

For convenience and completeness, we shall present in the next section a proof of Theorem 2.2. Our proof here is slightly different from that given in [4]. Our argument here is based on square functions approach.

3. A maximal function

As we pointed out in the previous section, this section is devoted for presenting a proof of Theorem 2.2. We shall start by proving a general lemma which will greatly simplify our argument. Let $\Psi = \{\Psi_{t,s} : t, s \in \mathbb{R}\}$ be a family of real valued $\mathcal{C}^\infty$ functions on $\mathbb{R}^n \times \mathbb{R}^m$. Let $\nu = \{\nu_{t,s} : t, s \in \mathbb{R}\}$ be a family of measures on $\mathbb{R}^n \times \mathbb{R}^m$. For $j, k \in \mathbb{Z}$, let $J_{j,k}^{(\Psi,\nu)}$ and $G_{j,k}^{(\Psi)}$ be given by

$$J_{j,k}^{(\Psi,\nu)}(f)(x,y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nu_{t,s} * \Psi_{t+j,s+k} * f(x,y)|^2 \, dtds \right)^{\frac{1}{2}} \tag{22}$$

and

$$G_{j,k}^{(\Psi)}(f)(x,y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Psi_{t+j,s+k} * f(x)|^2 \, dtds \right)^{\frac{1}{2}}. \tag{23}$$

For the family $\nu = \{\nu_{t,s} : t, s \in \mathbb{R}\}$, we let ν^* be the maximal function

$$\nu^*(f)(x,y) = \sup_{t,s\in\mathbb{R}} |\nu_{t,s} * f(x,y)|. \tag{24}$$

We shall let $\|\nu_{t,s}\|$ to denote the total variation of the measure $\nu_{t,s}$. Our main lemma in this section is the following:

Lemma 3.1. *Let* $\nu = \{\nu_{t,s} : t, s \in \mathbb{R}\}$, $\Psi = \{\Psi_{t,s} : t, s \in \mathbb{R}\}$, $J_{j,k}^{(\Psi,\nu)}$, $G_{j,k}^{(\Psi)}$, *and* ν^* *be as above. Let* γ_1, γ_2, *and* ε *be positive real numbers. Suppose that*

(i) $\sup_{t,s\in\mathbb{R}} \|\nu_{t,s}\| \le 1$;

(ii) $\left\| J_{j,k}^{(\Psi,\nu)}(f) \right\|_2 \le C 2^{-\varepsilon|j+k\gamma_1|} 2^{-\epsilon|k+j\gamma_2|} \|f\|_2$ *for all* $j, k \in \mathbb{Z}$;

(iii) $\|\nu^*(f)\|_q \le A_q \|f\|_q$ *for some* $q > 1$.

(iv) $\left\| G_{j,k}^{(\Psi)}(f) \right\|_p \le B_p \ \|f\|_p$ *for all* $j, k \in \mathbb{Z}$ *and* $1 < p < \infty$.

(v) $\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \Psi_{t+j,s+k} = 1$ *for all* $t, s \in \mathbb{R}$.

Then for $p_0' < p < p_0$ *where* $\left|\frac{1}{2} - \frac{1}{p_0}\right| = \frac{1}{2q}$, *there exists a constant* $C_p > 0$ *such that the operator*

$$S_\nu(f)(x,y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nu_{t,s} * f(x,y)|^2 \, dtds \right)^{\frac{1}{2}} \tag{25}$$

satisfies

$$\|S_\nu f\|_p \le C_p \|f\|_p. \tag{26}$$

Proof. The proof is fairly standard. First, we show that there exists a constant $C_{p_0} > 0$ such that

$$\left\| J_{j,k}^{(\Psi,\nu)}(f) \right\|_{p_0} \leq C_{p_0} \|f\|_{p_0} \tag{27}$$

for all $j, k \in \mathbb{Z}$. To see (27), we argue as follows. By duality, we may assume that $p_0 > 2$. Let $q = (\frac{p_0}{2})'$. Then there exists a non-negative function $h \in L^q(\mathbb{R}^n \times \mathbb{R}^m)$ with $\|h\|_q = 1$ such that

$$\begin{aligned}
\left\| J_{j,k}^{(\Psi,\nu)}(f) \right\|_{p_0}^2 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nu_{t,s} * \Psi_{t+j,s+k} * f(x,y)|^2 h(x,y) dt ds dx dy \\
&\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \left(G_{j,k}^{(\Psi)}(f) \right)^2 (x,y) \boldsymbol{\nu}^*(h)(-x,-y) dx dy \\
&\leq \left\| G_{j,k}^{(\Psi)}(f) \right\|_{p_0}^2 \|\boldsymbol{\nu}^*(h)\|_q \\
&\leq B_{p_0}^2 A_q \|f\|_{p_0}
\end{aligned}$$

which implies (27). Here, the last inequality follows by the assumptions (iii) and (iv).

Next, by the assumption (v) and Minkowski's inequality, we have

$$S_\nu(f)(x,y) \leq \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} J_{j,k}^{(\Psi,\nu)}(f)(x,y). \tag{28}$$

Now, by interpolation between the assumption (ii) and the estimate (27), we get that

$$\left\| J_{j,k}^{(\Psi,\nu)}(f) \right\|_p \leq 2^{-\varepsilon_p |j+k\gamma_1|} 2^{-\epsilon_p |k+j\gamma_2|} C \|f\|_p \tag{29}$$

for all $p_0' < p < p_0$ and some ε_p. Hence, (26) follows by (28) and (29). This completes the proof. □

Proof of Theorem 2.2. First, we shall assume that $d_\phi > 0$ and $d_\varphi > 0$. The case where $d_\phi < 0$ or $d_\varphi < 0$ follows by the same argument with minor modifications.

Now, notice that

$$\begin{aligned}
\mu_{\phi,\varphi}^{(z_1,z_2)}(f)(x,y) &\leq 4 \sup_{t,s\in\mathbb{R}} \frac{1}{2^{t+s}} \int_0^{2^t}\int_0^{2^s} |f(x-\phi(w)rz_1, y-\varphi(r)wz_2)|\, dr dw \\
&= 4 \sup_{t,s\in\mathbb{R}} \left| \nu_{t,s}^{(0)} * f(x,y) \right| = 4\mu_{\nu^{(0)}}(f)(x,y),
\end{aligned} \tag{30}$$

where $\{\nu_{t,s}^{(0)} : t, s \in \mathbb{R}\}$ is the family of measures defined by

$$\int f d\nu_{t,s}^{(0)} = \frac{1}{2^{t+s}} \int_0^{2^t}\int_0^{2^s} f(\phi(w)rz_1, \varphi(r)wz_2) dr dw. \tag{31}$$

Let $\nu^{(1)} = \{\nu^{(1)}_{t,s} : t, s \in \mathbb{R}\}$ and $\nu^{(2)} = \{\nu^{(2)}_{t,s} : t, s \in \mathbb{R}\}$ be the family of measures defined by

$$(\nu^{(1)}_{t,s})\hat{}(\xi, \eta) = \hat{\nu}_{t,s}(\xi, 0) \text{ and}$$
$$(\nu^{(2)}_{t,s})\hat{}(\xi, \eta) = \hat{\nu}_{t,s}(0, \eta).$$

Let

$$(\nu^{(i)})^*(f)(x, y) = \sup_{t,s\in\mathbb{R}} \left|\nu^{(i)}_{t,s} * f(x, y)\right|, \ i = 1, 2. \tag{32}$$

Notice that

$$\begin{aligned}
(\nu^{(1)})^*(f)(x, y) &\leq \sup_{t,s\in\mathbb{R}} \frac{1}{2^{t+s}} \int_0^{2^t}\!\!\int_0^{2^s} |f(x - \phi(w)rz_1, y)|\, drdw \\
&\leq \sup_{t,s\in\mathbb{R}} \frac{1}{2^s} \int_0^{2^s} \left(\frac{1}{2^t\phi(w)} \int_0^{2^t\phi(w)} |f(x - uz_1, y)|\, du \right) dw \\
&\leq C \sup_{t,s\in\mathbb{R}} \frac{1}{2^s} \int_0^{2^s} (\overline{M}_{z_1} + \overline{M}_{-z_1}) f(x, y) dw \\
&= C(\overline{M}_{z_1} + \overline{M}_{-z_1}) f(x, y), \qquad (33)
\end{aligned}$$

where $\overline{M}_z$ is the directional Hardy-Littlewood maximal function in the direction of z acting on the x-variable. Similarly, we can show that

$$(\nu^{(2)})^*(f)(x, y) \leq C(\overline{M}_{z_2} + \overline{M}_{-z_2}) f(x, y). \tag{34}$$

Thus by the boundedness of the directional Hardy-Littlewood maximal function, we get

$$\left\|(\nu^{(i)})^*(f)\right\|_p \leq C_p \|f\|_p \tag{35}$$

for $1 < p < \infty$ with C_p independent of z_i, $i = 1, 2$.

Next, it is straightforward to see that total variation $\|\cdot\|$ of the measure $\sigma_{a,r,s}$ satisfies

$$\sup_{t,s\in\mathbb{R}} \left\|\nu^{(i)}_{t,s}\right\| \leq 1, \ i = 0, 1, 2. \tag{36}$$

Now, we estimate the Fourier transform of the measures $\nu^{(i)}_{t,s}$. Notice that for $w \geq 1$ and $r > 0$, we have

$$\begin{aligned}
&\left|\frac{d^2}{dw^2}\left((\xi \cdot z_1)\phi(2^{s-j}w)r + (\eta \cdot z_2)2^{s-j}w\varphi(r)\right)\right| \\
&= 2^{2(s-j)}\left|(\xi \cdot z_1)\phi''(2^{s-j}w)r\right| \geq C_2 2^{d_\phi(s-j)} |\xi \cdot z_1|\, r.
\end{aligned}$$

Thus by Van der Corput Lemma [20] along with interpolation with the estimate

$$\left|\frac{1}{2^s}\int_0^{2^s} e^{i(\xi,\eta)\cdot(\phi(w)rz_1,\varphi(r)wz_2)} dw\right| \leq 1,$$

we get

$$\begin{aligned}&\left|\frac{1}{2^s}\int_0^{2^s} e^{i(\xi,\eta)\cdot(\phi(w)rz_1,\varphi(r)wz_2)}dw\right|\\ \leq\ & \sum_{j=1}^{\infty}2^{-j}\left|\int_1^2 e^{i(\xi,\eta)\cdot(\phi(2^{s-j}w)rz_1,\varphi(r)2^{s-j}wz_2)}dw\right|\\ \leq\ & \left|C_2 2^{d_\phi s}\left|\xi\cdot z_1\right| r\right|^{-\frac{1}{2(d_\phi+1)}}\sum_{j=1}^{\infty}2^{-\frac{j}{2}}\\ \leq\ & C\left|C_2 2^{d_\phi s}\left|\xi\cdot z_1\right| r\right|^{-\frac{1}{2(d_\phi+1)}}.\end{aligned}$$

Thus, by noticing that

$$\frac{1}{2^t}\int_0^{2^t} r^{-\frac{1}{2(d_\phi+1)}}dr=\frac{2d_\phi+2}{2d_\phi+1}2^{-\frac{1}{2(d_\phi+1)}t},$$

we get

$$\left|(\nu_{t,s}^{(0)}\hat{)}(\xi,\eta)\right|\leq C\left|2^{d_\phi s}2^t\left|\xi\cdot z_1\right|\right|^{-\frac{1}{2(d_\phi+1)}}. \tag{37}$$

Similarly, we can show that the following hold:

$$\left|(\nu_{t,s}^{(0)}\hat{)}(\xi,\eta)\right|\leq C\left|2^{d_\varphi t}2^s\left|\eta\cdot z_2\right|\right|^{-\frac{1}{2(d_\varphi+1)}}, \tag{38}$$

$$\left|(\nu_{t,s}^{(1)}\hat{)}(\xi,\eta)\right|\leq C\left|2^t2^{d_\phi s}\xi\cdot z_1\right|^{-\frac{1}{2(d_\phi+1)}}, \tag{39}$$

$$\left|(\nu_{t,s}^{(2)}\hat{)}(\xi,\eta)\right|\leq C\left|2^{d_\varphi t}2^s\eta\cdot z_2\right|^{-\frac{1}{2(d_\varphi+1)}}. \tag{40}$$

By the definitions of the involved measures, (36), and (37)-(40), we obtain the following estimates

$$\left|(\nu_{t,s}^{(0)}\hat{)}(\xi,\eta)\right|\leq C\left|2^{d_\phi s}2^t\left|\xi\cdot z_1\right|\right|^{-\frac{1}{4(d_\phi+1)}}\left|2^{d_\varphi t}2^s\left|\eta\cdot z_2\right|\right|^{-\frac{1}{4(d_\varphi+1)}}, \tag{41}$$

$$\left|(\nu_{t,s}^{(0)}\hat{)}(\xi,\eta)-(\nu_{t,s}^{(1)}\hat{)}(\xi,\eta)\right|\leq C\left|2^t2^{d_\phi s}\xi\cdot z_1\right|^{-\frac{1}{4(d_\phi+1)}}\left|2^s2^{d_\varphi t}\eta\cdot z_2\right|^{\frac{1}{4(d_\varphi+1)}}, \tag{42}$$

$$\left|(\nu_{t,s}^{(0)}\hat{)}(\xi,\eta)-(\nu_{t,s}^{(2)}\hat{)}(\xi,\eta)\right|\leq C\left|2^t2^{d_\phi s}\xi\cdot z_1\right|^{\frac{1}{4(d_\phi+1)}}\left|2^s2^{d_\varphi t}\eta\cdot z_2\right|^{-\frac{1}{4(d_\varphi+1)}}, \tag{43}$$

$$\begin{aligned}&\left|(\nu_{t,s}^{(0)}\hat{)}(\xi,\eta)-(\nu_{t,s}^{(1)}\hat{)}(\xi,\eta)-(\nu_{t,s}^{(2)}\hat{)}(\xi,\eta)+(\ln 2)^2\right|\\ &\leq C\left|2^t2^{d_\phi s}\xi\cdot z_1\right|^{\frac{1}{4(d_\phi+1)}}\left|2^s2^{d_\varphi t}\eta\cdot z_2\right|^{\frac{1}{4(d_\varphi+1)}},\end{aligned} \tag{44}$$

$$\left|(\nu_{t,s}^{(1)}\hat{)}(\xi,\eta)-(\ln 2)^2\right|\leq C\left|2^t2^{d_\phi s}\xi\cdot z_1\right|^{\frac{1}{4(d_\phi+1)}}, \tag{45}$$

$$\left|(\nu_{t,s}^{(2)}\hat{)}(\xi,\eta)-(\ln 2)^2\right|\leq C\left|2^s2^{d_\varphi t}\eta\cdot z_2\right|^{\frac{1}{4(d_\varphi+1)}}. \tag{46}$$

Now, let $\theta \in \mathcal{S}(\mathbb{R})$ be such that $\theta(t) = 1$ if $|t| < \frac{1}{2}$ and $\theta(t) = 0$ if $|t| > 1$. Define the family of measures $\{\sigma_{t,s} : t, s \in \mathbb{R}\}$ by

$$\begin{aligned}
&\hat{\sigma}_{t,s}(\xi,\eta) \\
&= (\nu^{(0)}_{t,s})\hat{}(\xi,\eta) - \theta(\left|2^t 2^{d_\phi s}\xi\cdot z_1\right|)(\nu^{(1)}_{t,s})\hat{}(\xi,\eta) - \theta(2^s 2^{d_\varphi t}\eta\cdot z_2)(\nu^{(2)}_{t,s})\hat{}(\xi,\eta) \\
&\quad + \theta(\left|2^t 2^{d_\phi s}\xi\cdot z_1\right|)\theta(\left|2^s 2^{d_\varphi t}\eta\cdot z_2\right|)(\ln 2)^2.
\end{aligned} \tag{47}$$

By (36) and (41)-(47), we get

$$\sup_{t,s\in\mathbb{R}} \|\sigma_{t,s}\| \le C, \tag{48}$$

$$|\hat{\sigma}_{t,s}(\xi,\eta)| \le \left|2^t 2^{d_\phi s}\xi\cdot z_1\right|^{\pm\frac{1}{4(d_\phi+1)}} \left|2^s 2^{d_\varphi t}\eta\cdot z_2\right|^{\pm\frac{1}{4(d_\varphi+1)}}. \tag{49}$$

It can be easily shown that the following inequality holds:

$$\mu_{\nu^{(0)}}(f)(x,y) \le 2S_\nu(f)(x,y), \tag{50}$$

where S_ν is given by (25). Let $\left\{\psi^{(1)}_{t,s}\right\}_{-\infty}^{\infty}$ and $\left\{\psi^{(2)}_{t,s}\right\}_{-\infty}^{\infty}$ be two families of $\mathcal{C}^\infty$ functions satisfying the properties (9)-(12) with j and k are replaced by t and s respectively and $a = 2$. Let $\Psi_{t,s}$ be given by (13) with j and k are replaced by t and s respectively and $a = 2$. Then by (47), (50), the property (9) and Minkowski's inequality, we have

$$\mu_{\nu^{(0)}}(f)(x,y) \le \mathcal{G}^{(\Psi,\sigma)}(f)(x,y) + C\sum_{j=1}^{3}\mathcal{M}^{(j)}(f)(x,y), \tag{51}$$

$$\sigma^*(f)(x,y) \le 2\mathcal{G}^{(\Psi,\sigma)}(f)(x,y) + 2C\sum_{j=1}^{3}\mathcal{M}^{(j)}(f)(x,y), \tag{52}$$

where

$$\begin{aligned}
\mathcal{G}^{(\Psi,\sigma)}(f)(x,y) &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} J^{(\Psi,\sigma)}_{j,k}(f)(x,y), \\
\mathcal{M}^{(1)}(f)(x,y) &= ((M_{\mathbb{R}}\otimes I_{\mathbb{R}^{n-1}})\otimes I_{\mathbb{R}^m})\,((\nu^{(1)})^*(f))(x,y), \\
\mathcal{M}^{(2)}(f)(x,y) &= ((M_{\mathbb{R}}\otimes I_{\mathbb{R}^{m-1}})\otimes I_{\mathbb{R}^n})\,((\nu^{(2)})^*(f))(x,y), \\
\mathcal{M}^{(3)}(f)(x,y) &= ((M_{\mathbb{R}}\otimes I_{\mathbb{R}^{n-1}})\otimes (M_{\mathbb{R}}\otimes I_{\mathbb{R}^{m-1}}))\,(f)(x,y),
\end{aligned}$$

$M_{\mathbb{R}}$ is the classical Hardy-Littlewood maximal function on $\mathbb{R}$, and $I_{\mathbb{R}^d}$ denote the identity operator on $\mathbb{R}^d$ $(d \ge 1)$.

By a well known argument (see [17, 20]), it can be shown that

$$\left\|G^{(\Psi)}_{j,k}(f)\right\|_p \le C_p\,\|f\|_p \tag{53}$$

for all $1 < p < \infty$ with constant C_p independent of j and k.

By (48)-(49) and Plancherel's theorem, we get that $J_{j,k}^{(\Psi,\sigma)}$ satisfies condition (ii) in Lemma 3.1. Thus, by (35), (48)-(52), L^p boundedness of Hardy-Littlewood maximal function, Lemma 3.1, and the well known bootstrapping argument as in [5] (see also [15]), the proof is complete. □

4. Proof of main results

Proof of Theorem 1.1. Suppose that $\Omega \in L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$, $q > 1$ and satisfies (2)-(3) with $\|\Omega\|_1 \leq 1$ and $\|\Omega\|_q \leq 2^a$ for some $a > 1$. Let $\sigma_a = \{\sigma_{a,r,s} : r, s \in \mathbb{R}\}$ be the family of measures defined by

$$\hat{\sigma}_{a,r,s}(\xi,\eta) = \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} e^{-i\{\xi\cdot u' r\phi(s)+\eta\cdot v' s\varphi(r)\}}\Omega(u',v')d\sigma(u')d\sigma(v'). \tag{54}$$

Then

$$M_{\Omega,\phi,\varphi}(f)(x,y) = \left(\int_0^\infty\int_0^\infty \left|\sigma_{a,r,s} * f(x,y)\right|^2 \frac{drds}{rs}\right)^{\frac{1}{2}}. \tag{55}$$

Now, we show that the family σ_a satisfies the following estimates

$$\sup_{r,s\in\mathbb{R}} \|\sigma_{a,r,s}\| \leq C; \tag{56}$$

$$\begin{aligned}&\int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} \left|\hat{\sigma}_{a,r,s}(\xi,\eta)\right|^2 \frac{dsdr}{sr}\\ &\leq a^2C\min\left\{\left|2^{aj}2^{ad_\phi k}\xi\right|^{-\frac{1}{2aq'}}, \left|2^{ak}2^{ad_\varphi j}\eta\right|^{-\frac{1}{2aq'}}\right\},\end{aligned} \tag{57}$$

$$\begin{aligned}&\int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} \left|\hat{\sigma}_{a,r,s}(\xi,\eta)\right|^2 \frac{dsdr}{sr}\\ &\leq a^2C\min\left\{\left|2^{a(j+1)}2^{d_\phi(k+1)}\xi\right|^{\frac{1}{2aq'}}, \left|2^{a(k+1)}2^{ad_\varphi(j+1)}\eta\right|^{\frac{1}{2aq'}}\right\}\end{aligned} \tag{58}$$

for $(\xi,\eta) \in \mathbb{R}^n\times\mathbb{R}^m$. Here, $\|\sigma_{a,r,s}\|$ denote the total variation of the measure $\sigma_{a,r,s}$.

The estimate (56) is clear. To verify the estimate in (57), we notice that

$$\begin{aligned}&\int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} \left|\hat{\sigma}_{a,r,s}(\xi,\eta)\right|^2 \frac{dsdr}{sr}\\ &\leq \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \left|\Omega(u',v')\right|\left|\Omega(z',w')\right| I_{a,j,k}(u',v',z',w')d\sigma(u')d\sigma(v')d\sigma(z')d\sigma(w').\end{aligned}$$

The last inequality and Hölder's inequality imply that

$$\int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} \left|\hat{\sigma}_{a,r,s}(\xi,\eta)\right|^2 \frac{dsdr}{sr} \tag{59}$$

$$\le \|\Omega\|_q^2 \left(\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} I_{a,j,k}(z',w')d\sigma(z')d\sigma(w') \right)^{\frac{1}{q'}},$$

where

(60) $I_{a,j,k}(z',w')$

$$= \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \left| \int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} e^{-i\{\xi\cdot(u'-z')r\phi(s)+\eta\cdot(v'-w')s\varphi(r)\}}\frac{drds}{rs} \right|^{q'} d\sigma(u')d\sigma(v')$$

and $q' = 1-(1/q)$. By Van der Corput Lemma [20], we have

$$|I_{a,j,k}(u',v',z',w')|$$

$$\le a\min\left\{ \left|2^{aj}2^{akd_\phi}\xi\cdot(u'-z')\right|^{-\frac{1}{2}}, \left|2^{ak}2^{ajd_\varphi}\eta\cdot(v'-w')\right|^{-\frac{1}{2}} \right\}. \tag{61}$$

By (61) and the observation that

$$|I_{a,j,k}(u',v',z',w')| \le a^2, \tag{62}$$

we get

$$|I_{a,j,k}(u',v',z',w')|$$

$$\le a^2\min\left\{ \left|2^{aj}2^{akd_\phi}\xi\cdot(u'-z')\right|^{-\frac{1}{2q'}}, \left|2^{ak}2^{ajd_\varphi}\eta\cdot(v'-w')\right|^{-\frac{1}{2q'}} \right\}. \tag{63}$$

By (59), (63), and the fact that

$$\sup_{\zeta'\in\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} \left|\zeta'\cdot(x'-y')\right|^{-\varepsilon} d\sigma_n d\sigma_n \le C < \infty \tag{64}$$

for all small $\varepsilon > 0$, $d \ge 2$, we obtain

$$\int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} |\hat{\sigma}_{a,r,s}(\xi,\eta)|^2 \frac{dsdr}{sr}$$

$$\le a^2\|\Omega\|_q^2 \min\left\{ \left|2^{aj}2^{akd_\phi}\xi\right|^{-\frac{1}{2q'}}, \left|2^{ak}2^{ajd_\varphi}\eta\right|^{-\frac{1}{2q'}} \right\}. \tag{65}$$

By interpolation between (65) and the trivial estimate

$$\int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} |\hat{\sigma}_{a,r,s}(\xi,\eta)|^2 \frac{dsdr}{sr} \le a^2, \tag{66}$$

we get (57). Here, we used the observation that $(\|\Omega\|_q^2)^{\frac{1}{a}} \le (2^{2a})^{\frac{1}{a}} = 4$. To verify (58), we first observe that

$$\hat{\sigma}_{a,r,s}(0,\eta) = \hat{\sigma}_{a,r,s}(\xi,0) = 0. \tag{67}$$

Thus

$$\int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} |\hat{\sigma}_{a,r,s}(\xi,\eta)|^2 \frac{dsdr}{sr}$$

$$\leq \int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} \min\left\{\left|\hat{\sigma}_{a,r,s}(\xi,\eta)-\hat{\sigma}_{a,r,s}(0,\eta)\right|,\left|\hat{\sigma}_{a,r,s}(\xi,\eta)-\hat{\sigma}_{a,r,s}(\xi,0)\right|\right\}^2 \frac{dsdr}{sr}$$
$$\leq a^2\left\|\Omega\right\|_1^2 \min\left\{\left|2^{a(j+1)}2^{a(k+1)d_\phi}\xi\right|,\left|2^{a(k+1)}2^{a(j+1)d_\varphi}\eta\right|\right\}^2.$$

By combining the last inequality, the estimate (66), and the fact that $\left\|\Omega\right\|_1 \leq 1$, we get (58).

Next, we estimate the L^p norms of the maximal function σ_a^*. Notice that

$$\sigma_a^*(f)(x,y) \tag{68}$$
$$= \sup_{j,k\in\mathbb{Z}} \int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} \left||\sigma_{a,r,s}|*f(x,y)\right| \frac{dsdr}{sr}$$
$$\leq \sup_{j,k\in\mathbb{Z}} \int_{2^{aj}}^{2^{a(j+1)}}\int_{2^{ak}}^{2^{a(k+1)}} \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |f(x-\phi(s)ru',y-\varphi(r)sv')|\,|\Omega(u',v')|\,d\sigma(u')d\sigma(v')\frac{dsdr}{sr}$$
$$\leq a^2 \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |\Omega(u',v')|\,\mu_{\phi,\varphi}^{(u',v')}(f)(x,y)d\sigma(u')d\sigma(v'),$$

where $\mu_{\phi,\varphi}^{(u',v')}$ is given by (21) with z_1 and z_2 replaced by u' and v', respectively. Therefore, by Minkowski's inequality, Theorem 2.2, and (68), we have

$$\left\|\sigma_a^*(f)\right\|_p \leq a^2 \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |\Omega(u',v')| \left\|\mu_{\phi,\varphi}^{(u',v')}(f)\right\|_p d\sigma(u')d\sigma(v')$$
$$\leq a^2 C_p \left\|\Omega\right\|_1 \left\|f\right\|_p \leq a^2 C_p \left\|f\right\|_p \tag{69}$$

for all $1 < p < \infty$ with constants C_p independent of the essential variables. Hence, (7) follows by (56)-(58), (69), and Lemma 2.1. This completes the proof. □

Now, we move to the proof of Theorem 1.2.

Proof of Theorem 1.2. Assume that $\Omega \in L(\log^+ L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ and satisfies (2)-(3). We shall use the estimates obtained in Theorem 1.2. To this end, we decompose the function Ω into a sum of L^2 functions with suitable sizes. By similar argument as in [3], there exist a sequence of numbers $\{\lambda_l : l \in \mathbb{N}\cup\{0\}$ and a sequence $\{\Omega_l : l \in \mathbb{N}\cup\{0\}\}$ of functions on $\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$ such that

$$\int_{\mathbb{S}^{n-1}} \Omega_l(u,\cdot)\,d\sigma(u) = \int_{\mathbb{S}^{m-1}} \Omega_l(\cdot,v)\,d\sigma(v) = 0, \tag{70}$$

$$\Omega_l(tx,sy) = \Omega_l(x,y) \text{ for any } t,s>0; \tag{71}$$

$$\left\|\Omega_l\right\|_1 \leq C,\ \left\|\Omega_l\right\|_2 \leq C2^{4(l+1)}, \tag{72}$$

$$\Omega(x,y) = \sum_{l=0}^{\infty} \lambda_l \Omega_l(x,y), \tag{73}$$

(74) $$\sum_{l=0}^{\infty}(l+1)\lambda_l \leq C\|\Omega\|_{L(\log L)(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}.$$

Let $M_{\Omega_l,\phi,\varphi}$ be the maximal function given by (4) with Ω replaced by Ω_l. Then

(75) $$M_{\Omega,\phi,\varphi}(f)(x,y) \leq \sum_{l=0}^{\infty}\lambda_l M_{\Omega_l,\phi,\varphi}(f)(x,y).$$

Now, we can apply Theorem 1.1 with $a = l+1$ to get

$$\begin{aligned}\|M_{\Omega,\phi,\varphi}(f)\|_p &\leq \sum_{l=0}^{\infty}\lambda_l \|M_{\Omega_l,\phi,\varphi}(f)\|_p \\ &\leq \left(\sum_{l=0}^{\infty}(l+1)\lambda_l\right) C_p\|f\|_p \leq C_p\|f\|_p\end{aligned}$$

for all $p \geq 2$. This completes the proof. □

We end this section by pointing out that Corollary 1.3 and Corollary 1.4 follow by Theorem 1.1, the observation (5), and simple change of variables. We omit the details.

5. Block spaces

In [18], Jiang and Lu introduced a special class of block spaces

$$B_q^{0,\upsilon}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}) \text{ (for } \upsilon > -1 \text{ and } q > 1).$$

A cap I on $\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$ is a subset defined by

$$I = \left\{x' \in \mathbb{S}^{n-1} : |x' - x_0'| < \alpha\right\} \times \left\{y' \in \mathbb{S}^{m-1} : |y' - y_0'| < \beta\right\}$$

for some α, $\beta > 0$, $x_0' \in \mathbb{S}^{n-1}$ and $y_0' \in \mathbb{S}^{m-1}$.

Definition 5.1. For $1 < q \leq \infty$, a measurable function $b(x,y)$ on $\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$ is called a q-block if $\operatorname{supp}(b) \subseteq I$ and $\|b\|_{L^q} \leq |I|^{-\frac{1}{q'}}$ where $1/q + 1/q' = 1$ and I is a cap on $\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$.

Block functions can be defined using the Block decomposition which is given by the following definition:

Definition 5.2. A function $\Omega \in L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ is in $B_q^{0,\upsilon}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$, $1 < q \leq \infty$, if

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu,$$

where each c_μ is a complex number; each b_μ is a q-block supported on a cap I_μ on $S^{n-1}\times S^{m-1}$; and

(76) $$M_q^{0,\upsilon}\left(\{c_\mu\}\right) = \sum_{\mu=1}^{\infty}|c_\mu|\left(1 + (\log\frac{1}{|I_\mu|})^{1+\upsilon}\right) < \infty.$$

It is known that

$$L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)\subset B^{0,v}\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)\subset L^1\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right); \tag{77}$$

$$B_q^{0,v}\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)\subseteq L^p\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right) \text{ for any } v>-1; \tag{78}$$

$$B_q^{0,v}\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)=L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right); \tag{79}$$

$$\bigcup_{q>1}L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)\subset\bigcup_{q>1}B_q^{0,v}\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right). \tag{80}$$

For more information about Block spaces we refer the reader to consult [1], [18], and [19], among others. By making use of Theorem 1.1 and the Block decomposition above, we immediately obtain the following result:

Theorem 5.3. *Suppose that* $\Omega\in B_q^{0,0}\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)(q>1)$ *and satisfies* (2)-(3). *If* $\varphi,\phi\in\mathcal{F}_1$ *with* $d_\varphi d_\phi\neq1$, *then* $M_{\Omega,\phi,\varphi}$ *is bounded on* $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ *for all* $2\leq p<\infty$.

By following similar argument as that led to Corollaries 1.3 and 1.4, we obtain the following:

Corollary 5.4. *Let* $\varphi,\phi\in\mathcal{F}_1$ *with* $d_\varphi d_\phi\neq1$. *Suppose that* $h\in L^2(\mathbb{R}_+\times\mathbb{R}_+, r^{-1}s^{-1}drds)$. *If* $\Omega\in B_q^{0,0}\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)(q>1)$ *and satisfies* (2)-(3), *then the singular integral operator* $T_{\phi,\phi}$ *given in Corollary 1.3 is bounded on* $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ *for all* $1<p<\infty$.

Corollary 5.5. *Let* $\varphi,\phi\in\mathcal{F}_1$ *with* $d_\varphi d_\phi\neq1$. *Suppose that* $h\in L^2(\mathbb{R}_+\times\mathbb{R}_+, r^{-1}s^{-1}drds)$. *If* $\Omega\in B_q^{0,0}\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)(q>1)$ *and satisfies* (2)-(3), *then the Marcinkiewicz integral operator* $\mu_{\phi,\phi}$ *given in Corollary 1.4 is bounded on* $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ *for all* $2\leq p<\infty$.

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