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ON ϕ-w-FLAT MODULES AND THEIR HOMOLOGICAL DIMENSIONS

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Abstract. In this paper, we introduce and study the class of ϕ-w-flat modules which are generalizations of both ϕ-flat modules and w-flat modules. The ϕ-w-weak global dimension ϕ-w-w.gl.dim(R) of a commutative ring R is also introduced and studied. We show that, for a ϕ-ring R, ϕ-w-w.gl.dim$(R) = 0$ if and only if w-dim$(R) = 0$ if and only if R is a ϕ-von Neumann ring. It is also proved that, for a strongly ϕ-ring R, ϕ-w-w.gl.dim$(R) \leq 1$ if and only if each nonnil ideal of R is ϕ-w-flat, if and only if R is a ϕ-PvMR, if and only if R is a PvMR.

Throughout this paper, R denotes a commutative ring with $1 \neq 0$ and all modules are unitary. We denote by $\mathrm{Nil}(R)$ the nilpotent radical of R, $\mathrm{Z}(R)$ the set of all zero-divisors of R and $\mathrm{T}(R)$ the localization of R at the set of all regular elements. The R-submodules I of $\mathrm{T}(R)$ such that $sI \subseteq R$ for some regular element s are said to be *fractional ideals*. Recall from [3] that a ring R is an NP-*ring* if $\mathrm{Nil}(R)$ is a prime ideal, and a ZN-*ring* if $\mathrm{Z}(R) = \mathrm{Nil}(R)$. A prime ideal P is said to be *divided prime* if $P \subsetneq (x)$ for every $x \in R - P$. Set $\mathcal{H} = \{R \mid R \text{ is a commutative ring and Nil(R) is a divided prime ideal of } R\}$. A ring R is a ϕ-*ring* if $R \in \mathcal{H}$. Moreover, a ZN ϕ-ring is said to be a *strongly ϕ-ring*. For a ϕ-ring R, there is a ring homomorphism $\phi : \mathrm{T}(R) \to R_{\mathrm{Nil}(R)}$ such that $\phi(a/b) = a/b$ where $a \in R$ and b is a regular element. Denote by the ring $\phi(R)$ the image of ϕ restricted to R. In 2001, Badawi [4] investigated ϕ-chain rings (ϕ-CRs for short) and ϕ-pseudo-valuation rings as a ϕ-version of chain rings and pseudo-valuation rings. In 2004, Anderson and Badawi [1] introduced the concept of ϕ-Prüfer rings and showed that a ϕ-ring R is ϕ-Prüfer if and only if $R_{\mathfrak{m}}$ is a ϕ-chain ring for any maximal ideal $\mathfrak{m}$ of R if and only if $R/\mathrm{Nil}(R)$ is a Prüfer domain if and only if $\phi(R)$ is Prüfer. Later, the authors in [2, 5] generalized the concepts of Dedekind domains, Krull domains and Mori domains to the context of rings that are in the class $\mathcal{H}$. In 2013, Zhao et al. [19] introduced and studied the conceptions of ϕ-flat modules and ϕ-von Neumann rings and obtained that a ϕ-ring is ϕ-von Neumann if and only if its

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Krull dimension is 0. Recently, Zhao [18] gave a homological characterization of ϕ-Prüfer rings as follows: a strongly ϕ-ring R is ϕ-Prüfer, if and only if each submodule of a ϕ-flat module is ϕ-flat, if and only if each nonnil ideal of R is ϕ-flat.

Some other important generalizations of classical notions are their w-versions. In 1997, Wang and McCasland [15] introduced the w-modules over strong Mori domains (SM domains for short) which can be seen as a w-version of Noetherian domains. In 2011, Yin *et al.* [17] extended w-theories to commutative rings containing zero divisors. The notion of w-flat modules appeared first in [10] for integral domains and was extended to arbitrary commutative rings in [13]. In 2012, Kim and Wang [7] introduced ϕ-SM rings which can be seen as both a ϕ-version and a w-version of Noetherian domains and obtained that a ϕ-ring R is ϕ-SM if and only if $R/\mathrm{Nil}(R)$ is an SM domain if and only if $\phi(R)$ is an SM ring. In 2014, Wang and Kim [12] introduced w-w.gl.dim(R) as a generalization of the classical weak global dimension and obtained that a ring R is a von Neumann ring if and only if each R-module is w-flat, i.e., w-w.gl.dim$(R) = 0$. In 2015, Wang and Qiao [16] studied several properties of the w-weak global dimension, and proved that an integral domain R is a Prüfer v-multiplication domain (PvMD for short) if and only if w-w.gl.dim$(R) \leq 1$ if and only if $R_{\mathfrak{m}}$ is a valuation domain for any maximal w-ideal $\mathfrak{m}$ of R. As ϕ-rings are natural extensions of integral domains, we introduce and study the ϕ-versions of w-flat modules, von Neumann rings and PvMDs in this article. As our work involves w-theories, we give a review as below.

Let R be a commutative ring and J a finitely generated ideal of R. Then J is called a GV-*ideal* if the natural homomorphism $R \to \mathrm{Hom}_R(J, R)$ is an isomorphism. The set of all GV-ideals is denoted by $\mathrm{GV}(R)$. An R-module M is said to be GV-*torsion* if for any $x \in M$ there is a GV-ideal J such that $Jx = 0$; an R-module M is said to be GV-*torsion free* if $Jx = 0$, then $x = 0$ for any $J \in \mathrm{GV}(R)$ and $x \in M$. A GV-torsion free module M is said to be a w-*module* if for any $x \in E(M)$ there is a GV-ideal J such that $Jx \subseteq M$ where $E(M)$ is the injective envelope of M. The w-*envelope* M_w of a GV-torsion free module M is defined by the minimal w-module that contains M. Therefore, a GV-torsion free module M is a w-module if and only if $M_w = M$. A *maximal* w-*ideal* for which is maximal among the w-submodules of R is proved to be prime (see [17, Proposition 3.8]). The set of all maximal w-ideals is denoted by w-Max(R). The w-*dimension* w-dim(R) of a ring R is defined to be the supremum of the heights of all maximal w-ideals.

An R-homomorphism $f : M \to N$ is said to be a w-*monomorphism* (resp., w-*epimorphism*, w-*isomorphism*) if for any $p \in w$-Max(R), $f_p : M_p \to N_p$ is a monomorphism (resp., an epimorphism, an isomorphism). Note that f is a w-monomorphism (resp., w-epimorphism) if and only if $\mathrm{Ker}(f)$ (resp., $\mathrm{Coker}(f)$) is GV-torsion. A sequence $A \to B \to C$ is said to be w-*exact* if for any $p \in w$-Max(R), $A_p \to B_p \to C_p$ is exact. A class $\mathcal{C}$ of R-modules is said to be *closed under* w-*isomorphisms* provided that for any w-isomorphism $f : M \to N$, if one of the modules M and N is in $\mathcal{C}$, so is the other. An R-module M is said to be of *finite type* if there exist a finitely generated free module F and a w-epimorphism $g : F \to M$, or equivalently, if there exists a finitely generated R-submodule N of M such that $N_w = M_w$. Certainly, the class of finite type modules is closed under w-isomorphisms. Now we proceed to introduce the notion of ϕ-w-flat modules.

1. ϕ-w-flat modules

We say an ideal I of R is nonnil provided that there is a non-nilpotent element in I. Denote by $\mathrm{NN}(R)$ the set of all nonnil ideals of R. Certainly, GV-ideals are nonnil. Let R be an NP-ring. It is easy to verify that $\mathrm{NN}(R)$ is a multiplicative system of ideals. That is $R \in \mathrm{NN}(R)$ and for any $I \in \mathrm{NN}(R)$, $J \in \mathrm{NN}(R)$, we have $IJ \in \mathrm{NN}(R)$. Let M be an R-module. Define

$$\phi\text{-}tor(M) = \{x \in M \mid Ix = 0 \text{ for some } I \in \mathrm{NN}(R)\}.$$

An R-module M is said to be *ϕ-torsion* (resp., *ϕ-torsion free*) provided that ϕ-$tor(M) = M$ (resp., ϕ-$tor(M) = 0$). Clearly, if R is an NP-ring, the class of ϕ-torsion modules is closed under submodules, quotients, direct sums and direct limits. Thus an NP-ring R is ϕ-torsion free if and only if every flat module is ϕ-torsion free if and only if R is a ZN-ring (see [18, Proposition 2.2]). The classes of ϕ-torsion modules and ϕ-torsion free modules constitute a hereditary torsion theory of finite type. For more details, refer to [9].

Lemma 1.1. *Let R be an* NP*-ring, $\mathfrak{m}$ a maximal w-ideal of R and I an ideal of R. Then $I \in \mathrm{NN}(R)$ if and only if $I_{\mathfrak{m}} \in \mathrm{NN}(R_{\mathfrak{m}})$.*

Proof. Let $I \in \mathrm{NN}(R)$ and x a non-nilpotent element in I. We will show the element $x/1$ in $I_{\mathfrak{m}}$ is a non-nilpotent element of $R_{\mathfrak{m}}$. If $(x/1)^n = x^n/1 = 0$ in $R_{\mathfrak{m}}$ for some positive integer n, there is an $s \in R - \mathfrak{m}$ such that $sx^n = 0$ in R. Since R is an NP-ring, $\mathrm{Nil}(R)$ is the minimal prime w-ideal of R. In the integral domain $R/\mathrm{Nil}(R)$, we have $\overline{s}\overline{x^n} = \overline{0}$, thus $\overline{x^n} = \overline{0}$ since $s \notin \mathrm{Nil}(R)$. So $x \in \mathrm{Nil}(R)$, a contradiction.

Let x/s be a non-nilpotent element in $I_{\mathfrak{m}}$ where $x \in I$ and $s \in R - \mathfrak{m}$. Clearly, x is non-nilpotent and thus $I \in \mathrm{NN}(R)$. □

Proposition 1.2. *Let R be an* NP*-ring, $\mathfrak{m}$ a maximal w-ideal of R and M an R-module. Then M is ϕ-torsion over R if and only $M_{\mathfrak{m}}$ is ϕ-torsion over $R_{\mathfrak{m}}$.*

Proof. Let M be an R-module and $x \in M$. If $M_{\mathfrak{m}}$ is ϕ-torsion over $R_{\mathfrak{m}}$, there is an ideal $I_{\mathfrak{m}} \in \mathrm{NN}(R_{\mathfrak{m}})$ such that $I_{\mathfrak{m}}x/1 = 0$ in $R_{\mathfrak{m}}$. Let I be the preimage of $I_{\mathfrak{m}}$ in R. Then I is nonnil by Lemma 1.1. Thus there is a non-nilpotent element $t \in I$ such that $tkx = 0$ for some $k \notin m$. Let $s = tk$. Then we have $(s) \in \mathrm{NN}(R)$ and $(s)x = 0$. Thus M is ϕ-torsion. Suppose M is ϕ-torsion over R. Let x/s be an element in $M_{\mathfrak{m}}$. Then there is an ideal $I \in \mathrm{NN}(R)$ such that $Ix = 0$, and thus $I_{\mathfrak{m}}x/s = 0$ with $I_{\mathfrak{m}} \in \mathrm{Nil}(R_{\mathfrak{m}})$ by Lemma 1.1. It follows that $M_{\mathfrak{m}}$ is ϕ-torsion over $R_{\mathfrak{m}}$. □

Recall from [13] that an R-module M is said to be *w-flat* if for any w-monomorphism $f : A \to B$, the induced sequence $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is also a w-monomorphism. Obviously, GV-torsion modules and flat modules are all w-flat. It was proved that the class of w-flat modules is closed under w-isomorphisms (see [14, Corollary 6.7.4]). Following [19, Definition 3.1], an R-module M is said to be *ϕ-flat* if for every monomorphism $f : A \to B$ with $\mathrm{Coker}(f)$ ϕ-torsion, $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is a monomorphism. Obviously flat modules are both ϕ-flat and w-flat. Now we give a generalization of both ϕ-flat modules and w-flat modules.

Definition 1.3. Let R be a ring. An R-module M is said to be *ϕ-w-flat* if for every monomorphism $f : A \to B$ with $\mathrm{Coker}(f)$ ϕ-torsion, $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is a w-monomorphism; equivalently, if $0 \to A \to B \to C \to 0$ is an exact exact sequence with C ϕ-torsion, then $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is w-exact.

Clearly ϕ-flat modules and w-flat modules are ϕ-w-flat. It is well known that an R-module M is flat if and only if the induced homomorphism $1 \otimes_R f : M \otimes_R I \to M \otimes_R R$ is exact for any (finitely generated) ideal I, if and only if the multiplication homomorphism $i : I \otimes_R M \to IM$ is an isomorphism for any (finitely generated) ideal I, if and only if $\mathrm{Tor}_1^R(R/I, M) = 0$ for any (finitely generated) ideal I of R. Some similar characterizations of w-flat modules and ϕ-flat modules are given in [12, Proposition 1.1] and [19, Theorem 3.2], respectively. We can also obtain some similar characterizations of ϕ-w-flat modules.

Theorem 1.4. *Let R be an* NP*-ring. The following statements are equivalent for an R-module M:*

(1) *M is ϕ-w-flat;*
(2) *$M_{\mathfrak{m}}$ is ϕ-flat over $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in w$-$Max(R)$;*
(3) *$\mathrm{Tor}_1^R(T, M)$ is* GV*-torsion for all* (*finite type*) *ϕ-torsion R-modules T;*
(4) *$\mathrm{Tor}_1^R(R/I, M)$ is* GV*-torsion for all* (*finite type*) *nonnil ideals I of R;*
(5) *$f \otimes_R 1 : I \otimes_R M \to R \otimes_R M$ is w-exact for all* (*finite type*) *nonnil ideals I of R;*
(6) *the multiplication homomorphism $i : I \otimes_R M \to IM$ is a w-isomorphism for all* (*finite type*) *ideals I;*
(7) *let $0 \to K \to F \to M \to 0$ be an exact sequence of R-modules, where F is free. Then $(K \cap FI)_w = (IK)_w$ for all* (*finite type*) *nonnil ideals I of R.*

Proof. (1) $\Rightarrow$ (2): Let $\mathfrak{m}$ be a maximal w-ideal of R, $f : A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ an $R_{\mathfrak{m}}$-homomorphism with $\mathrm{Coker}(f)$ ϕ-torsion over $R_{\mathfrak{m}}$. Then $\mathrm{Coker}(f)$ is ϕ-torsion over R by Proposition 1.2. It follows that $f \otimes_R M : A_{\mathfrak{m}} \otimes_R M \to B_{\mathfrak{m}} \otimes_R M$ is a w-monomorphism over R. Localizing at $\mathfrak{m}$, we have $f \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} : A_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \to B_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ is a monomorphism over $R_{\mathfrak{m}}$ since $N_{\mathfrak{m}} \otimes_R M_{\mathfrak{m}} \cong N_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ for any R-module N. It follows that $M_{\mathfrak{m}}$ is ϕ-flat over $R_{\mathfrak{m}}$.

(2) $\Rightarrow$ (1): Let $f : A \to B$ be a monomorphism with Coker(f) ϕ-torsion. For any $\mathfrak{m} \in w$-Max(R), we have $f_{\mathfrak{m}} : A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is a monomorphism with coker($f_{\mathfrak{m}}$) ϕ-torsion over $R_{\mathfrak{m}}$ by Proposition 1.2. Since $M_{\mathfrak{m}}$ is ϕ-flat over $R_{\mathfrak{m}}$, $f_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} : A_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \to B_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ is a monomorphism. Thus $f \otimes_R M : A \otimes_R M \to B \otimes_R M$ is a w-monomorphism. Consequently, M is ϕ-w-flat.

The equivalences of (2)-(7) hold from [19, Theorem 3.2] by localizing at all maximal w-ideals. □

Corollary 1.5. *Let R be an* NP*-ring. The class of ϕ-w-flat modules is closed under w-isomorphisms.*

Proof. Let $f : M \to N$ be a w-isomorphism and T a ϕ-torsion module. There exist two exact sequences $0 \to T_1 \to M \to L \to 0$ and $0 \to L \to N \to T_2 \to 0$ with T_1 and T_2 GV-torsion. Considering the induced two long exact sequences $\mathrm{Tor}_1^R(T, T_1) \to \mathrm{Tor}_1^R(T, M) \to \mathrm{Tor}_1^R(T, L) \to T \otimes T_1$ and $\mathrm{Tor}_2^R(T, T_2) \to \mathrm{Tor}_1^R(T, L) \to \mathrm{Tor}_1^R(T, N) \to \mathrm{Tor}_1^R(T, T_2)$, we have M is ϕ-w-flat if and only if N is ϕ-w-flat by Theorem 1.4. □

Lemma 1.6. *Let R be a ϕ-ring and I a nonnil ideal of R. Then* $\mathrm{Nil}(R) = I\mathrm{Nil}(R)$.

Proof. Let I be a nonnil ideal of R with a non-nilpotent element $s \in I$. Then $\mathrm{Nil}(R) \subseteq (s)$. Thus for any $a \in \mathrm{Nil}(R)$, there exists $b \in R$ such that $a = sb$. Thus $\overline{a} = \overline{s}\overline{b}$ in the integral domain $R/\mathrm{Nil}(R)$. Since $\overline{a} = 0$ and $\overline{s} \neq 0$, we have $\overline{b} = 0$. So $b \in \mathrm{Nil}(R)$ and then $\mathrm{Nil}(R) \subseteq s\mathrm{Nil}(R) \subseteq I\mathrm{Nil}(R) \subseteq \mathrm{Nil}(R)$. It follows that $\mathrm{Nil}(R) = I\mathrm{Nil}(R)$. □

Proposition 1.7. *Let R be a ϕ-ring and M an R-module. Then $M/\mathrm{Nil}(R)M$ is ϕ-flat over R if and only if $M/\mathrm{Nil}(R)M$ is flat over $R/\mathrm{Nil}(R)$. Consequently, $R/\mathrm{Nil}(R)$ is always ϕ-flat over R.*

Proof. For the "only if" part, let $\overline{I} = I/\mathrm{Nil}(R)$ be an ideal of $\overline{R} = R/\mathrm{Nil}(R)$. If $\overline{I}$ is zero, certainly $\mathrm{Tor}_1^{\overline{R}}(\overline{R}/\overline{I}, M/\mathrm{Nil}(R)M) = 0$. Let $\overline{I}$ be a non-zero ideal of $\overline{R}$ with $I \in \mathrm{NN}(R)$. Since $M/\mathrm{Nil}(R)M$ is ϕ-flat over R,

$$\mathrm{Tor}_1^R(R/I, M/\mathrm{Nil}(R)M) = 0.$$

By Lemma 1.6,

$$\mathrm{Tor}_1^R(R/\mathrm{Nil}(R), R/I) \cong I \cap \mathrm{Nil}(R)/I\mathrm{Nil}(R) = \mathrm{Nil}(R)/I\mathrm{Nil}(R) = 0.$$

We have $\mathrm{Tor}_1^{\overline{R}}(\overline{R}/\overline{I}, M/\mathrm{Nil}(R)M) \cong \mathrm{Tor}_1^R(R/I, M/\mathrm{Nil}(R)M) = 0$ by change of rings.

For the "if" part, let I be a nonnil ideal of R. Similarly to the proof of "only if" part, since $\mathrm{Tor}_1^R(R/\mathrm{Nil}(R), R/I) = 0$, we have $\mathrm{Tor}_1^R(R/I, M/\mathrm{Nil}(R)M) \cong \mathrm{Tor}_1^{\overline{R}}(\overline{R}/\overline{I}, M/\mathrm{Nil}(R)M) = 0$. It follows that $M/\mathrm{Nil}(R)M$ is ϕ-flat over R. □

By localizing at all maximal w-ideals, we obtain the following corollary.

Corollary 1.8. *Let R be a ϕ-ring and M an R-module. Then $M/\mathrm{Nil}(R)M$ is ϕ-w-flat over R if and only if $M/\mathrm{Nil}(R)M$ is w-flat over $R/\mathrm{Nil}(R)$.*

Proof. See Proposition 1.7, Theorem 1.4 and [8, Theorem 3.3]. □

Certainly if R is an integral domain, every ϕ-w-flat module is w-flat. Conversely, this property characterizes integral domains.

Theorem 1.9. *The following statements are equivalent for a ϕ-ring R:*

(1) *R is an integral domain;*
(2) *every ϕ-w-flat module is w-flat;*
(3) *every ϕ-flat module is w-flat.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let s be a nilpotent element of R. Then

$$\mathrm{Tor}_1^R(R/(s), R/\mathrm{Nil}(R)) \cong (s) \cap \mathrm{Nil}(R)/s\mathrm{Nil}(R) = (s)/s\mathrm{Nil}(R)$$

is GV-torsion since $R/\mathrm{Nil}(R)$ is w-flat by (3) and Proposition 1.7. Thus there is a GV-ideal J such that $sJ \subseteq s\mathrm{Nil}(R)$. Since J is a nonnil ideal, $\mathrm{Nil}(R) = J\mathrm{Nil}(R)$ by Lemma 1.6. Thus $sJ \subseteq s\mathrm{Nil}(R) = sJ\mathrm{Nil}(R) \subseteq sJ$. That is, $sJ = sJ\mathrm{Nil}(R)$. Since sJ is finitely generated, $sJ = 0$ by Nakayama's lemma. Since $J \in \mathrm{GV}(R)$, $s \in R$ is GV-torsion free, then $s = 0$. Consequently, $\mathrm{Nil}(R) = 0$ and R is an integral domain. □

Recall from [11] that a ring R is said to be a DW *ring* if every ideal of R is a w-ideal. Then a ring R is a DW ring if and only if every R-module is a w-module, if and only if $\mathrm{GV}(R) = \{R\}$ (see [11, Theorem 3.8]). Certainly if R is a DW ring, every ϕ-w-flat module is ϕ-flat. Conversely, this property characterizes DW rings.

Theorem 1.10. *The following statements are equivalent for an* NP*-ring R:*

(1) *R is a* DW *ring;*
(2) *every ϕ-w-flat module is ϕ-flat;*
(3) *every w-flat module is ϕ-flat.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): For any $J \in \mathrm{GV}(R)$, R/J is GV-torsion, and thus w-flat. By (3), R/J is ϕ-flat. Since every GV-ideal J is a nonnil ideal of R, we have $\mathrm{Tor}_1^R(R/J, R/J) \cong J/J^2 = 0$. It follows that J is a finitely generated idempotent ideal, and thus J is projective. So $J = J_w = R$ by [14, Exercise 6.10(1)] and thus R is a DW ring by [14, Theorem 6.3.12]. □

Some non-integral domain examples are provided by the idealization construction $R(+)M$ where M is an R-module (see [6]). We recall this construction. Let $R(+)M = R \oplus M$ as an R-module, and define

(1) $(r, m)+(s, n)=(r + s, m + n)$.
(2) $(r, m)(s, n)=(rs, sm + rn)$.

Under these definitions, $R(+)M$ becomes a commutative ring with identity. Denote by $(0:_R M)$ the set $\{r \in R \mid rM = 0\}$. Now we compute some examples of GV-ideals of $R(+)M$.

Proposition 1.11. *Let T be a commutative ring and E a w-module over T such that $(0:_T E) = 0$. Set $R = T(+)E$. Then $J(+)E$ is a* GV*-ideal of R for any $J \in \mathrm{GV}(T)$.*

Proof. Let J be a GV-ideal of T. Then we claim that $J(+)E \in \mathrm{GV}(R)$. Indeed, since $T(+)E/J(+)E \cong T/J$, for any $i = 0, 1$, we have

$$\mathrm{Ext}^i_R(T(+)E/J(+)E, R) \cong \mathrm{Ext}^i_T(T/J, \mathrm{Hom}_R(T, R)).$$

Note that

$$\mathrm{Hom}_R(T, R) = \mathrm{Hom}_R(R/0(+)E, R) \cong 0(+)E \cong E$$

since $(0:_T E) = 0$. Thus $\mathrm{Ext}^i_R(T(+)E/J(+)E, R) \cong \mathrm{Ext}^i_T(T/J, E)$ for any $i = 0, 1$. If $J \in \mathrm{GV}(T)$ then $J(+)E \in \mathrm{GV}(R)$ since E is a w-module over T. □

Now we give an example to show the notion of ϕ-w-flat modules is a strict generalization of ϕ-flat modules and w-flat modules.

Example 1.12. Let D be a non-DW integral domain and K its quotient field. Then $R = D(+)K$ is a ϕ-ring (see [2, Remark 1]). However, by Proposition 1.11, R is neither an integral domain nor a DW ring. Consequently, there is a ϕ-w-flat module over R which is neither ϕ-flat nor w-flat by Theorem 1.9 and Theorem 1.10.

2. Homological properties of ϕ-w-flat modules

Let R be a ring. It is well known that the flat dimension of an R-module M is defined as the shortest flat resolution of M and the weak global dimension of R is the supremum of the flat dimensions of all R-modules. The w-flat dimension $w\text{-fd}_R(M)$ of an R-module M and w-weak global dimension $w\text{-w.gl.dim}(R)$ of a ring R were introduced and studied in [16]. We now introduce the notion of ϕ-w-flat dimension of an R-module as follows.

Definition 2.1. Let R be a ring and M an R-module. We write $\phi\text{-}w\text{-fd}_R(M) \leq n$ (ϕ-w-fd abbreviates *ϕ-w-flat dimension*) if there is a w-exact sequence of R-modules

$$(\diamondsuit) \qquad 0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where each F_i is w-flat for $i = 0, \ldots, n-1$ and F_n is ϕ-w-flat. The w-exact sequence $(\diamondsuit)$ is said to be a ϕ-w-flat w-resolution of length n of M. If such finite w-resolution does not exist, then we say $\phi\text{-}w\text{-fd}_R(M) = \infty$; otherwise, define $\phi\text{-}w\text{-fd}_R(M) = n$ if n is the length of the shortest ϕ-w-flat w-resolution of M.

It is obvious that an R-module M is ϕ-w-flat if and only if ϕ-w-$\mathrm{fd}_R(M)=0$. Certainly, ϕ-w-$\mathrm{fd}_R(M)\leq w$-$\mathrm{fd}_R(M)$. If R is an integral domain, then ϕ-w-$\mathrm{fd}_R(M)=w$-$\mathrm{fd}_R(M)$.

Lemma 2.2 ([16, Lemma 2.2]). *Let N be an R-module and $0\to A\to F\to C\to 0$ a w-exact sequence of R-modules with F a w-flat module. Then for any $n>0$, the induced map $\mathrm{Tor}^R_{n+1}(C,N)\to\mathrm{Tor}^R_n(A,N)$ is a w-isomorphism. Hence, $\mathrm{Tor}^R_{n+1}(C,N)$ is* GV*-torsion if and only if so is $\mathrm{Tor}^R_n(A,N)$.*

Proposition 2.3. *Let R be an* NP*-ring. The following statements are equivalent for an R-module M:*

(1) *ϕ-w-$fd_R(M)\leq n$;*
(2) *$\mathrm{Tor}^R_{n+k}(M,N)$ is* GV*-torsion for all ϕ-torsion R-modules N and all $k>0$;*
(3) *$\mathrm{Tor}^R_{n+1}(M,N)$ is* GV*-torsion for all ϕ-torsion R-modules N;*
(4) *$\mathrm{Tor}^R_{n+1}(M,R/I)$ is* GV*-torsion for all nonnil ideals I;*
(5) *$\mathrm{Tor}^R_{n+1}(M,R/I)$ is* GV*-torsion for all finite type nonnil ideals I;*
(6) *if $0\to F_n\to\cdots\to F_1\to F_0\to M\to 0$ is an exact sequence, where $F_0,F_1,\ldots,F_{n-1}$ are flat R-modules, then F_n is ϕ-w-flat;*
(7) *if $0\to F_n\to\cdots\to F_1\to F_0\to M\to 0$ is an w-exact sequence, where $F_0,F_1,\ldots,F_{n-1}$ are w-flat R-modules, then F_n is ϕ-w-flat;*
(8) *if $0\to F_n\to\cdots\to F_1\to F_0\to M\to 0$ is an exact sequence, where $F_0,F_1,\ldots,F_{n-1}$ are w-flat R-modules, then F_n is ϕ-w-flat;*
(9) *if $0\to F_n\to\cdots\to F_1\to F_0\to M\to 0$ is an w-exact sequence, where $F_0,F_1,\ldots,F_{n-1}$ are flat R-modules, then F_n is ϕ-w-flat.*

Proof. (1) $\Rightarrow$ (2): We prove (2) by induction on n. For the case $n=0$, (2) holds by Theorem 1.4 as M is ϕ-w-flat. If $n>0$, then there is a w-exact sequence $0\to F_n\to\cdots\to F_1\to F_0\to M\to 0$, where each F_i is w-flat for $i=0,\ldots,n-1$ and F_n is ϕ-w-flat. Set $K_0=\ker(F_0\to M)$. Then both $0\to K_0\to F_0\to M\to 0$ and $0\to F_n\to F_{n-1}\to\cdots\to F_1\to K_0\to 0$ are w-exact, and ϕ-w-$\mathrm{fd}_R(K_0)\leq n-1$. By induction, $\mathrm{Tor}^R_{n-1+k}(K_0,N)$ is GV-torsion for all ϕ-torsion R-modules N and all $k>0$. Thus, it follows from Lemma 2.2 that $\mathrm{Tor}^R_{n+k}(M,N)$ is GV-torsion.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (6): Let $K_0=\ker(F_0\to M)$ and $K_i=\ker(F_i\to F_{i-1})$, where $i=1,\ldots,n-1$. Then $K_{n-1}=F_n$. Since all $F_0,F_1,\ldots,F_{n-1}$ are flat, $\mathrm{Tor}^R_1(F_n,R/I)\cong\mathrm{Tor}^R_{n+1}(M,R/I)$ is GV-torsion for all finite type nonnil ideal I. Hence F_n is a ϕ-w-flat module by Theorem 1.4.

(6) $\Rightarrow$ (1): Obvious.

(3) $\Rightarrow$ (7): Set $L_n=F_n$ and $L_i=\mathrm{Im}(F_i\to F_{i-1})$, where $i=1,\ldots,n-1$. Then both $0\to L_{i+1}\to F_i\to L_i\to 0$ and $0\to L_1\to F_0\to M\to 0$ are w-exact sequences. By using Lemma 2.2 repeatedly, we can obtain that $\mathrm{Tor}^R_1(F_n,N)$ is GV-torsion for all ϕ-torsion R-modules N. Thus F_n is ϕ-w-flat.

(7) $\Rightarrow$ (8) $\Rightarrow$ (6), (7) $\Rightarrow$ (9) and (9) $\Rightarrow$ (6): Trivial. □

Definition 2.4. The *ϕ-w-weak global dimension* of a ring R is defined by

$$\phi\text{-}w\text{-w}.gl.\dim(R) = \sup\{w_\phi\text{-}fd_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Obviously, by definition, ϕ-w-w.gl.dim$(R) \leq w$-w.gl.dim(R). Notice that if R is an integral domain, then ϕ-w-w.gl.dim$(R) = w$-w.gl.dim(R).

Proposition 2.5. *Let R be an* NP*-ring. The following statements are equivalent for R.*

(1) *ϕ-w-$fd_R(M) \leq n$ for all R-modules M.*
(2) $\mathrm{Tor}^R_{n+k}(M,N)$ *is* GV*-torsion for all R-modules M and ϕ-torsion N and all $k > 0$.*
(3) $\mathrm{Tor}^R_{n+1}(M,N)$ *is* GV*-torsion for all R-modules M and ϕ-torsion N.*
(4) $\mathrm{Tor}^R_{n+1}(M,R/I)$ *is* GV*-torsion for all R-modules M and nonnil ideals I of R.*
(5) $\mathrm{Tor}^R_{n+1}(M,R/I)$ *is* GV*-torsion for all R-modules M and finite type nonnil ideals I of R.*
(6) *ϕ-w-$fd_R(R/I) \leq n$ for all nonnil ideals I of R.*
(7) *ϕ-w-$fd_R(R/I) \leq n$ for all finite type nonnil ideals I of R.*
(8) *ϕ-w-$w.gl.$*$\dim(R) \leq n$.

Consequently, the ϕ-w-weak global dimension of R is determined by the formulas:

$$\begin{aligned}\phi\text{-}w\text{-}w.gl.\dim(R) &= \sup\{\phi\text{-}w\text{-}fd_R(R/I) \mid I \textit{ is a nonnil ideal of } R\}\\ &= \sup\{\phi\text{-}w\text{-}fd_R(R/I) \mid I \textit{ is a finite type nonnil ideal of } R\}.\end{aligned}$$

Proof. (1) $\Leftrightarrow$ (8) and (1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8): Trivial.

(1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1): Follows from Proposition 2.3.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Trivial.

(8) $\Rightarrow$ (1): Let M be an R-module and $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat R-modules. To complete the proof, it suffices, by Proposition 2.3, to prove that F_n is ϕ-w-flat. Let I be a finite type nonnil ideal of R. Thus ϕ-w-fd$_R(R/I) \leq n$ by (8). It follows from Lemma 2.2 that $\mathrm{Tor}^R_1(R/I, F_n) \cong \mathrm{Tor}^R_{n+1}(R/I, M)$ is GV-torsion. □

3. Rings with ϕ-w-weak global dimension at most one

It is well known that a commutative ring R with weak global dimension 0 is exactly a *von Neumann regular ring*, equivalently $a \in (a^2)$ for any $a \in R$. It was proved in [12, Theorem 4.4] that a commutative ring R has w-weak global dimension 0, if and only if $a \in (a^2)_w$ for any $a \in R$, if and only if $R_\mathfrak{m}$ is a field for any maximal w-ideal $\mathfrak{m}$ of R, if and only if R is a von Neumann regular ring. Recall from [19] that a ϕ-ring R is said to be *ϕ-von Neumann regular* provided that every R-module is ϕ-flat. A ϕ-ring R is ϕ-von Neumann regular, if and only if there is an element $x \in R$ such that $a = xa^2$ for any non-nilpotent element $a \in R$, if and only if $R/\mathrm{Nil}(R)$ is a von Neumann regular ring, if and only if R is zero-dimensional (see [19, Theorem 4.1]). Now, we give some more characterizations of ϕ-von Neumann regular rings.

Theorem 3.1. *Let R be a ϕ-ring. The following statements are equivalent for R:*

(1) ϕ-w-$w.gl.\dim(R) = 0$;
(2) *every R-module is ϕ-w-flat*;
(3) $a \in (a^2)_w$ *for any non-nilpotent element* $a \in R$;
(4) w-$\dim(R) = 0$;
(5) $\dim(R) = 0$;
(6) *R is ϕ-von Neumann regular.*

Proof. (1) $\Leftrightarrow$ (2) By definition.

(2) $\Rightarrow$ (3): Let a be a non-nilpotent element in R. Then Ra is a nonnil ideal of R. It follows that $\mathrm{Tor}_1^R(R/Ra, R/Ra)$ is GV-torsion since R/Ra is ϕ-torsion and ϕ-w-flat. That is, Ra/Ra^2 is GV-torsion, and thus $a \in Ra \subseteq (Ra)_w = (Ra^2)_w$.

(3) $\Rightarrow$ (4): Since R is a ϕ-ring, $\mathrm{Nil}(R)$ is the minimal prime w-ideal R. We claim that the ring $\overline{R}_{\overline{\mathfrak{m}}} := (R/\mathrm{Nil}(R))_{\mathfrak{m}/\mathrm{Nil}(R)}$ is a field for any $\mathfrak{m} \in w$-$\mathrm{Max}(R)$. Indeed, let a be a non-nilpotent element in R. By (3), $(a)_w = (a^2)_w$. Thus $(a)_\mathfrak{m} = (a^2)_\mathfrak{m}$. We have $(\overline{a})_{\overline{\mathfrak{m}}} = (\overline{a^2})_{\overline{\mathfrak{m}}}$ as an ideal of $\overline{R}_{\overline{\mathfrak{m}}}$. So $\overline{R}_{\overline{\mathfrak{m}}}$ is a local von Neumann regular ring, and thus a field. Note that $\overline{R}_{\overline{\mathfrak{m}}} = R_\mathfrak{m}/\mathrm{Nil}(R_\mathfrak{m})$. It follows that $R_\mathfrak{m}$ is 0-dimensional (see [6, Theorem 3.1]). Thus w-$\dim(R) = 0$.

(4) $\Rightarrow$ (1): By Theorem 1.4, we just need to show $\mathrm{Tor}_1^R(R/I, R/J)$ is GV-torsion for all nonnil ideals I and all ideals J of R. Since R is a ϕ-ring with w-$\dim(R) = 0$, $\mathrm{Nil}(R)$ is the unique maximal w-ideal of R. We just need to show $\mathrm{Tor}_1^R(R/I, R/J)_{Nil(R)} = 0$. That is, $(I \cap J/IJ)_{Nil(R)} = 0$.

If J is a nonnil ideal of R, there are non-nilpotent elements $s \in I$ and $t \in J$ such that $st \in IJ$. Since $st \notin \mathrm{Nil}(R)$, $(I \cap J/IJ)_{Nil(R)} = 0$. If J is a nilpotent ideal of R, $I \cap J = J$. Thus $\mathrm{Tor}_1^R(R/I, R/J)_{\mathrm{Nil}(R)} = (I \cap J/IJ)_{\mathrm{Nil}(R)} = (J/IJ)_{\mathrm{Nil}(R)}$. Let s be a non-nilpotent element in I. We have $s(j + IJ) = 0 + (IJ)$ in J/IJ for any $j \in J$. Thus $(I \cap J/IJ)_{Nil(R)} = 0$.

(4) $\Rightarrow$ (5): By (4), $\mathrm{Nil}(R)$ is the unique w-maximal ideal of R. If $\mathrm{Nil}(R)$ is a maximal ideal of R, (6) holds obviously. Otherwise, there is a non-unit element a which is not nilpotent. Since (a) is not a GV-ideal, there is maximal w-ideal $\mathfrak{m}$ such that $\mathrm{Nil}(R) \subsetneq (a) \subseteq (a)_w \subseteq \mathfrak{m}$, Thus w-$\dim(R) \geq 1$, which is a contradiction.

(5) $\Rightarrow$ (4): Trivial.

(5) $\Leftrightarrow$ (6): See [19, Theorem 4.1]. □

Recall from [6] that a ring R is said to be a *Prüfer ring* provided that every finitely generated regular ideal I is invertible, i.e., $II^{-1} = R$ where $I^{-1} = \{x \in T(R) \mid Ix \subseteq R\}$, or equivalently, there is a fractional ideal J of R such that $IJ = R$. It is well known that an integral domain is a Prüfer domain if and only if the weak global dimension of $R \leq 1$. Recall that a ring R is said to be a PvMR if every finitely generated regular ideal I is w-invertible, i.e., $(II^{-1})_w = R$, or equivalently, there is a fractional ideal J of R such that $(IJ)_w = R$. PvMDs are exactly integral domains which are PvMRs. It is known that an integral domain R is a PvMD if and only if $R_\mathfrak{m}$ is a valuation domain for each $\mathfrak{m} \in w$-Max(R) if and only if w-w.gl.dim$(R) \leq 1$ (see [12, 16]).

Following [4], a ϕ-ring R is said to be a *ϕ-chain ring* (ϕ-CR for short) if for any $a, b \in R - \text{Nil}(R)$, either $a \mid b$ or $b \mid a$ in R. A ϕ-ring R is said to be a *ϕ-Prüfer ring* if every finitely generated nonnil ideal I is ϕ-invertible, i.e., $\phi(I)\phi(I^{-1}) = \phi(R)$. It follows from [1, Corollary 2.10] that a ϕ-ring R is ϕ-Prüfer, if and only if $R_\mathfrak{m}$ is a ϕ-CR for any maximal ideal $\mathfrak{m}$ of R, if and only if $R/\text{Nil}(R)$ is a Prüfer domain, if and only if $\phi(R)$ is Prüfer. For a strongly ϕ-ring R, Zhao [18, Theorem 4.3] showed that R is a ϕ-Prüfer ring if and only if all ϕ-torsion free R-modules are ϕ-flat, if and only if each submodule of a ϕ-flat R-module is ϕ-flat, if and only if each nonnil ideal of R is ϕ-flat.

Let R be a ϕ-ring. Recall from [7] that a nonnil ideal J of R is said to be a *ϕ-GV-ideal* (resp., *ϕ-w-ideal*) of R if $\phi(J)$ is a GV-ideal (resp., w-ideal) of $\phi(R)$. A ϕ-ring R is called a *ϕ-SM ring* if it satisfies the ACC on ϕ-w-ideals. An ideal I of R is *ϕ-w-invertible* if $(\phi(I)\phi(I)^{-1})_W = \phi(R)$ where W is the w-operation of $\phi(R)$. A ϕ-ring is ϕ-Krull provided that any nonnil ideal is ϕ-w-invertible (see [7, Theorem 2.23]). By extending ϕ-Krull rings and PvMDs, we give the definition of ϕ-Prüfer v-multiplication rings.

Definition 3.2. Let R be a ϕ-ring. R is said to be a *ϕ-Prüfer v-multiplication ring* (ϕ-PvMR for short) provided that any finitely generated nonnil ideal is ϕ-w-invertible.

Now we characterize ϕ-Prüfer multiplication rings in terms of ϕ-w-flat modules.

Theorem 3.3. *Let R be a ϕ-ring. The following statements are equivalent for R:*

(1) *R is a* ϕ-PvMR;
(2) *$R_\mathfrak{m}$ is a ϕ-CR for any* $\mathfrak{m} \in w$-Max(R);
(3) $R/\text{Nil}(R)$ *is a* PvMD;
(4) $\phi(R)$ *is a* PvMR.

Moreover, if R is a strongly ϕ-ring, all above are equivalent to

(5) *R is a ϕ-w-$w.gl.$*dim$(R) \leq 1$;
(6) *every submodule of a w-flat module is ϕ-w-flat*;
(7) *every submodule of a flat module is ϕ-w-flat*;
(8) *every ideal of R is ϕ-w-flat*;
(9) *every nonnil ideal of R is ϕ-w-flat*;
(10) *every finite type nonnil ideal of R is ϕ-w-flat.*

Proof. Let R be a ϕ-ring. Denote by W, w and $\overline{w}$ the w-operations of $\phi(R)$, R and $R/\text{Nil}(R)$ respectively. We will prove the equivalences of (1)-(4) and (5)-(10).

(1) $\Rightarrow$ (4): Let K be a finitely generated regular ideal of $\phi(R)$. Then $K = \phi(I)$ for some finitely generated nonnil ideal I of R by [1, Lemma 2.1]. Since R is a ϕ-PvMR, $(KK^{-1})_W = (\phi(I)\phi(I)^{-1})_W = \phi(R)$. Thus $\phi(R)$ is a PvMR.

(4) $\Rightarrow$ (1): Let I be a finitely generated nonnil ideal of R. We will show I is ϕ-w-invertible. By [1, Lemma 2.1], $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$. Thus $(\phi(I)\phi(I)^{-1})_W = \phi(R)$ since $\phi(R)$ is a PvMR.

(2) $\Leftrightarrow$ (3): By [1, Theorem 3.7, Corollary 2.10], $R_{\mathfrak{m}}$ is a ϕ-CR for any $\mathfrak{m} \in w$-Max(R) if and only if $R_{\mathfrak{m}}/Nil(R_{\mathfrak{m}}) = (R/Nil(R))_{\mathfrak{m}}$ is a valuation domain for any $\mathfrak{m} \in w$-Max(R) if and only if $R/Nil(R)$ is a PvMD (see [12, Theorem 4.9]).

(3) $\Rightarrow$ (4): Note that $\phi(R)/\mathrm{Nil}(\phi(R)) \cong R/\mathrm{Nil}(R)$ is a PvMD (see [1, Lemma 2.4]). Let $\phi(I)$ be a finitely generated regular ideal of $\phi(R)$. Then, by [1, Lemma 2.1], I is a nonnil ideal of R. Then $\overline{I} = I/Nil(R)$ is w-invertible over $\overline{R} = R/\mathrm{Nil}(R)$ by (3). That is, $(\overline{II^{-1}})_{\overline{w}} = \overline{R}$. There is a GV ideal $\overline{J}$ of $\overline{R}$ such $\overline{J} \subseteq \overline{II^{-1}}$ (see [14, Exercise 6.10(2)]). So $J \subseteq II^{-1}$ where J is a ϕ-GV ideal of R by [7, Lemma 2.3]. Thus $\phi(J) \subseteq \phi(I)\phi(I)^{-1}$. Since $\phi(J) \in \mathrm{GV}(\phi(R))$, $(\phi(I)\phi(I)^{-1})_W = \phi(R)$.

(4) $\Rightarrow$ (3): Suppose $\phi(R)$ is a PvMR. Let $\overline{I}$ is a finitely generated nonzero ideal of $\overline{R}$. Then I is a nonnil ideal of R. Thus $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$ by [1, Lemma 2.1]. So $(\phi(I)\phi(I)^{-1})_W = \phi(R)$ by (4). Hence $J \subseteq II^{-1}$ in R for some ϕ-GV ideal J of R and thus $\overline{J} \subseteq \overline{II^{-1}}$ in $\overline{R}$. By [7, Lemma 2.3], $\overline{J} \in \mathrm{GV}(\overline{R})$, and thus $(\overline{II^{-1}})_{\overline{w}} = \overline{R}$. So $R/\mathrm{Nil}(R)$ is a PvMD.

(5) $\Rightarrow$ (6): Let K be a submodule of a w-flat module F. Then ϕ-w-$\mathrm{fd}_R(F/K) \leq 1$ by (5). Thus K is ϕ-w-flat by Proposition 2.3.

(6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10): Trivial.

(10) $\Rightarrow$ (5): Let I be a finite type nonnil ideal of R. Then ϕ-w-$\mathrm{fd}_R(R/I) \leq 1$ by Proposition 2.3. It follows from Proposition 2.5 that ϕ-w-w.gl.dim$(R) \leq 1$.

Now, let R be a strongly ϕ-ring.

(2) $\Rightarrow$ (9): Let $\mathfrak{m}$ be a maximal w-ideal of R and I a nonnil ideal of R. Then $I_{\mathfrak{m}}$ is a nonnil ideal of $R_{\mathfrak{m}}$ by Lemma 1.1 and thus is ϕ-flat by [18, Theorem 4.3]. So I is ϕ-w-flat by Theorem 1.4.

(9) $\Rightarrow$ (2): Let $\mathfrak{m}$ be a maximal w-ideal of R , $I_{\mathfrak{m}}$ a nonnil ideal of $R_{\mathfrak{m}}$. Then I is a nonnil ideal of R by Lemma 1.1. By (9), I is ϕ-w-flat and so $I_{\mathfrak{m}}$ is ϕ-flat by Theorem 1.4. Thus $R_{\mathfrak{m}}$ is a ϕ-CR by [18, Theorem 4.3]. □

Corollary 3.4. *Suppose R is a ϕ-ring. Then R is a ϕ-*Krull *ring if and only if R is both a ϕ-*PvMR *and a ϕ-*SM *ring.*

Proof. By [7, Theorem 2.4] a ϕ-ring R is a ϕ-SM ring if and only if $R/\mathrm{Nil}(R)$ is an SM domain. A ϕ-ring R is a ϕ-Krull ring if and only if $R/\mathrm{Nil}(R)$ is a Krull domain (see [2, Theorem 3.1]). Since R is a Krull domain if and only if R is an SM PvMD (see [8, Theorem 7.9.3]), the equivalence holds by Theorem 3.3. □

Corollary 3.5. *Suppose R is a strongly ϕ-ring. Then R is a ϕ-*PvMR *if and only if R is a* PvMR.

Proof. Suppose R is a ϕ-PvMR and let I be a finitely generated regular ideal of R. Then $\overline{I}$ is a finitely generated regular ideal of $\overline{R}$. By Theorem 3.3, $\overline{R}$ is a PvMD. Then $(\overline{I}\overline{I^{-1}})_{\overline{w}} = \overline{R}$. Thus there is a GV-ideal $\overline{J}$ of $\overline{R}$ with J finitely generated over R such that $\overline{J} \subseteq \overline{I}\overline{I^{-1}}$. Since R is a strongly ϕ-ring, J is a GV-ideal of R by [7, Lemma 2.11]. Since $J \subseteq II^{-1}$ in R, $(II^{-1})_w = R$. Assume R is a PvMR. Since R is a strongly ϕ-ring, $\phi(R) = R$ is a PvMR. Thus R is a ϕ-PvMR by Theorem 3.3. □

The condition that R is a strongly ϕ-ring in Corollary 3.5 can't be removed by the following example.

Example 3.6. Let D be an integral domain which is not a PvMD and K its quotient field. Since K/D is a divisible D-module, the ring $R = D(+)K/D$ is a ϕ-ring but not a strongly ϕ-ring (see [2, Remark 1]). Since $\mathrm{Nil}(R) = 0(+)K/D$, we have $R/\mathrm{Nil}(R) \cong D$ is not a PvMD. Thus R is not a ϕ-PvMR by Theorem 3.3. Denote by $U(R)$ and $U(D)$ the sets of unit elements of R and D respectively. Since $\mathrm{Z}(R) = \{(r, m) \,|\, r \in \mathrm{Z}(D) \cup \mathrm{Z}(K/D)\} = R - \mathrm{U}(D)(+)K/D = R - \mathrm{U}(R)$, R is a PvMR obviously.

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