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STRONG COMMUTATIVITY PRESERVING MAPS OF UPPER TRIANGULAR MATRIX LIE ALGEBRAS OVER A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with identity 1, $n \geq 3$, and let $\mathcal{T}_n(R)$ be the linear Lie algebra of all upper triangular $n \times n$ matrices over R. A linear map φ on $\mathcal{T}_n(R)$ is called to be strong commutativity preserving if $[\varphi(x), \varphi(y)] = [x, y]$ for any $x, y \in \mathcal{T}_n(R)$. We show that an invertible linear map φ preserves strong commutativity on $\mathcal{T}_n(R)$ if and only if it is a composition of an idempotent scalar multiplication, an extremal inner automorphism and a linear map induced by a linear function on $\mathcal{T}_n(R)$.

1. Introduction

Let M be a matrix space over a field $\mathbb{F}$. A linear map φ on M is said to be commutativity preserving if $\varphi(A)$ commutes with $\varphi(B)$ for every pair of commuting elements $A, B \in M$. It is one of the linear preserver problems to classify commutativity preserving linear maps on matrix spaces. Several authors have classified commutativity preserving linear maps on a number of variations of matrix spaces, see [2, 6, 8, 11]. Mathematicians similarly study maps that preserve commutativity on rings. Let R be a ring with center $Z(R)$. Then R is a Lie ring under the Lie product $[A, B] = AB - BA$. Similarly, a map $\varphi : R \to R$ preserves commutativity if $[\varphi(A), \varphi(B)] = 0$ whenever $[A, B] = 0$ for all $A, B \in R$. The problem of characterizing linear (or additive) bijective maps preserving commutativity had been studied intensively on various rings and algebras (see [3–5] and the references therein). The authors in [17, 18] also determine the linear bijective maps preserving commutativity on finite-dimensional simple Lie algebras and their Borel subalgebras.

In [1], Bell and Daif gave the conception of strong commutativity preserving maps. Let S be a subset of a Lie ring R. A bijective map $\varphi : S \to R$ is said to be strong commutativity preserving on S if $[\varphi(x), \varphi(y)] = [x, y]$

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for all $x, y \in S$. Note that a strong commutativity preserving map must be commutativity preserving, but the inverse is not true generally. Bell and Daif [1] proved that R must be commutative if R is a prime ring and R admits a derivation or a nonidentity endomorphism which is strong commutativity preserving on a right ideal of R. Brešar and Miers in [4] proved that every additive map φ which is strong commutativity preserving on a semiprime ring R has the form $\varphi(x) = \lambda x + \mu(x)$, where $\lambda \in C$, C is the extended centroid of R, $\lambda^2 = 1$ and $\mu : R \to R$ is an additive map. Deng and Ashraf [9] proved that if R is a prime ring of characteristic not 2 and there exists a nonidentity endomorphism φ of R such that $[\varphi(x), \varphi(y)] - [x, y] \in Z(R)$ for all x, y in some essential right ideal of R, then R is commutative. Let L be a noncentral Lie ideal of a prime ring R. Recently, Lin and Liu in [13] have proved that every additive map $\varphi : L \to R$ which is strong commutativity preserving has the form $\varphi(x) = \lambda x + \mu(x)$, where $\lambda \in C$ with $\lambda^2 = 1$ and $\mu : R \to Z(R)$ is an additive map, unless char $R = 2$ and R satisfies the standard identity of degree 4. Recently, the authors in [7] determined the invertible linear maps preserving strong commutativity on the Lie algebra $N(\mathbb{F})$ of the strictly upper triangular matrices over a field. There are other results about strong commutativity preserving maps of associative rings or Lie algebras, see [10,12–15] for example.

In this paper, we consider the invertible linear maps preserving strong commutativity on the Lie ring $\mathcal{T}$ of all upper triangular matrices over a commutative ring. Note that the ring $\mathcal{T}$ is not semiprime. So our result about invertible linear maps preserving strong commutativity on $\mathcal{T}$ is new. We have tried to determine such linear maps on $\mathcal{T}$ through the form of linear maps preserving commutativity obtained in [16], but it is difficult. In this paper, we determine the concrete forms of such linear maps through their actions on the basis elements of $\mathcal{T}$. In the following, we always assume that R is a commutative ring with identity 1, $n > 2$, and R^* is the set of all invertible elements in R. Let $\mathcal{T} = \mathcal{T}_n(R)$ be the Lie ring consisting of all upper triangular $n \times n$ matrices over R, and the Lie multiplication $[-, -]$ is defined by $[X, Y] = XY - YX$. Denote by $\mathcal{D}$ the set of the diagonal matrices in $\mathcal{T}$. Let I be the identity matrix of $\mathcal{T}$, and E_{ij} the matrix in $\mathcal{T}$ whose sole nonzero entry is 1 in the (i, j) position, $1 \le i \le j \le n$. The center of $\mathcal{T}$ is $Z(\mathcal{T}) = \{X \in \mathcal{T} \mid [X, Y] = 0 \text{ for all } Y \in \mathcal{T}\}$. It is known that $Z(\mathcal{T}) = RI$. Set δ_{ij} to be the Kronecker delta function defined by $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \ne j$.

2. Certain linear maps preserving strong commutativity

A bijective map φ on a Lie algebra $\mathfrak{g}$ is called strong commutativity preserving if $[\varphi(x), \varphi(y)] = [x, y]$ for any $x, y \in \mathfrak{g}$. In this section, we construct certain maps preserving strong commutativity on $\mathcal{T}$, which will be used to describe arbitrary maps preserving strong commutativity.

(A) Extremal inner automorphisms.

Let $a \in R$. Denote $S_a = I + aE_{1n}$. Then S_a is an invertible matrix, and $S_a^{-1} = I - aE_{1n}$. Recall that the map $\varphi_a : \mathcal{T} \to \mathcal{T}$ defined by $\varphi_a(X) = S_a^{-1} X S_a$ is an inner automorphism of $\mathcal{T}$. We call φ_a an *extremal inner automorphism* of $\mathcal{T}$.

Lemma 2.1. *For $\forall a \in R$, $\varphi_a : \mathcal{T} \to \mathcal{T}$ preserves strong commutativity.*

Proof. For any $X = \sum_{1 \le i \le j \le n} x_{ij} E_{ij} \in \mathcal{T}$, where $x_{ij} \in R$, $1 \le i \le j \le n$, we have $\varphi_a(X) = S_a^{-1} X S_a = (I - aE_{1n}) X (I + aE_{1n}) = X + a(x_{11} - x_{nn}) E_{1n}$. For any $Y = \sum_{1 \le i \le j \le n} y_{ij} E_{ij}$, $[\varphi_a(X), \varphi_a(Y)] = [X + a(x_{11} - x_{nn}) E_{1n}, Y + a(y_{11} - y_{nn}) E_{1n}] = [X, Y] + a(x_{11} - x_{nn})(y_{nn} - y_{11}) E_{1n} + a(y_{11} - y_{nn})(x_{11} - x_{nn}) E_{1n} = [X, Y]$. So the lemma holds. □

(B) Idempotent scalar multiplications.

Let

$$\mathcal{U} = \{ r \in R \,|\, r^2 = 1 \}.$$

For $r \in \mathcal{U}$, the map $\eta_r : \mathcal{T} \to \mathcal{T}$ defined by $\eta_r(X) = rX$ is called an idempotent scalar multiplication. It is easy to see that η_r is an invertible linear map preserving strong commutativity, and $\eta_r^{-1} = \eta_r$.

(C) Linear maps induced by a linear function on $\mathcal{T}$.

Let $f : \mathcal{T} \to R$ be a linear function satisfying that $1 + f(I) \in R^*$. Define a map $\theta_f : \mathcal{T} \to \mathcal{T}$ by

$$\theta_f(X) = X + f(X) I, \quad \forall X \in \mathcal{T}.$$

Then θ_f is linear and invertible, and $\theta_f^{-1}(X) = X - \frac{f(X)}{1 + f(I)} I$. It is easy to see that θ_f preserves strong commutativity.

3. Strong commutativity preserving maps on $\mathcal{T}$

Lemma 3.1. *Let φ be an invertible linear map preserving strong commutativity on $\mathcal{T}$. Then there exists $a \in R^*$ such that $\varphi(I) = aI$.*

Proof. For any $X \in \mathcal{T}$, there exists $\bar{X} \in \mathcal{T}$ such that $X = \varphi(\bar{X})$. Then $[\varphi(I), X] = [\varphi(I), \varphi(\bar{X})] = [I, \bar{X}] = 0$, and so $\varphi(I) \in Z(\mathcal{T}) = RI$. Then there exists $a \in R$ such that $\varphi(I) = aI$. Let φ^{-1} be the inverse map of φ. Then φ is linear, and for any $X, Y \in \mathcal{T}$, $[\varphi^{-1}(X), \varphi^{-1}(Y)] = [\varphi\varphi^{-1}(X), \varphi\varphi^{-1}(Y)] = [X, Y]$. So for any $X \in \mathcal{T}$, we can set $\varphi^{-1}(I) = bI$, $b \in R$. Thus $I = \varphi\varphi^{-1}(I) = \varphi(bI) = abI$, and so $ab = 1$. Therefore, $a \in R^*$. □

Lemma 3.2. *Let φ be an invertible linear map preserving strong commutativity on $\mathcal{T}$, $n \ge 3$. For any $1 \le i < j \le n$, there exist elements $b_{ij}, c_{ij} \in R$ such that*

$$E_{ij} = c_{ij} I + b_{ij} \varphi(E_{ij}).$$

Proof. Since φ is linear and invertible, then the set $\varphi(\mathcal{D}) \cup \{\varphi(E_{kl}) \,|\, 1 \le k < l \le n\}$ spans $\mathcal{T}$. Assume that

$$E_{ij} = \varphi(D_{ij}) + \sum_{1 \le k < l \le n} a_{kl}^{(ij)} \varphi(E_{kl}), \text{ where } D_{ij} \in \mathcal{D}, a_{kl}^{(ij)} \in R.$$

At first we prove that $a_{kl}^{(ij)} = 0$ for any $(k, l) \neq (i, j)$. For given $(k, l) \neq (i, j)$, $1 \leq k < l \leq n$, we can choose an integer p such that $p = k$ or l, but p, i, j are distinct. In fact, if k, l, i, j are distinct, we can choose $p = k$; if $k = i$ or $k = j$, we can choose $p = l$; if $l = i$ or $l = j$, we can choose $p = k$; then p meets the requirements. On one hand,

$$\begin{aligned}[\varphi(E_{pp}), E_{ij}] &= [\varphi(E_{pp}), [E_{ii}, E_{ij}]] \\ &= [\varphi(E_{pp}), [\varphi(E_{ii}), \varphi(E_{ij})]] \\ &= [[\varphi(E_{pp}), \varphi(E_{ii})], \varphi(E_{ij})] + [\varphi(E_{ii}), [\varphi(E_{pp}), \varphi(E_{ij})]] \\ &= [[E_{pp}, E_{ii}], \varphi(E_{ij})] + [\varphi(E_{ii}), [E_{pp}, E_{ij}]] \\ &= 0.\end{aligned} \tag{3.1}$$

On the other hand,

$$\begin{aligned}[\varphi(E_{pp}), E_{ij}] &= [\varphi(E_{pp}), \varphi(D_{ij}) + \sum_{1 \leq s < t \leq n} a_{st}^{(ij)} \varphi(E_{st})] \\ &= \sum_{1 \leq s < t \leq n} a_{st}^{(ij)} [\varphi(E_{pp}), \varphi(E_{st})] \\ &= \sum_{1 \leq s < t \leq n} a_{st}^{(ij)} [E_{pp}, E_{st}] \\ &= \sum_{t=p+1}^{n} a_{pt}^{(ij)} E_{pl} - \sum_{s=1}^{p-1} a_{sp}^{(ij)} E_{sp}.\end{aligned} \tag{3.2}$$

Then $a_{pt}^{(ij)} = 0$ for any $t \in \{p+1, \ldots, n\}$, and $a_{sp}^{(ij)} = 0$ for any $s \in \{1, 2, \ldots, p-1\}$. In particular, if $p = k$, then $a_{kt}^{(ij)} = 0$ for any $t > k$; if $p = l$, then $a_{s1}^{(ij)} = 0$ for any $s < l$. Thus for any $k < l$ with $(k, l) \neq (i, j)$, we have $a_{kl}^{(ij)} = 0$. So

$$E_{ij} = \varphi(D_{ij}) + a_{ij}^{(ij)} \varphi(E_{ij}).$$

Next we prove that $\varphi(D_{ij}) = c_{ij} I$ for some $c_{ij} \in R$.

For $i < j$, we choose $(k, l) \neq (i, j)$, then i, k, l are distinct or j, k, l are distinct. Assume that j, k, l are distinct. On one hand,

$$\begin{aligned}[\varphi(E_{kl}), E_{ij}] &= -[\varphi(E_{kl}), [E_{jj}, E_{ij}]] \\ &= -[\varphi(E_{kl}), [\varphi(E_{jj}), \varphi(E_{ij})]] \\ &= -[[\varphi(E_{kl}), \varphi(E_{jj})], \varphi(E_{ij})] - [\varphi(E_{jj}), [\varphi(E_{kl}), \varphi(E_{ij})]] \\ &= -[[E_{kl}, E_{jj}], \varphi(E_{ij})] - [\varphi(E_{jj}), [E_{kl}, E_{ij}]] \\ &= -[\varphi(E_{jj}), \delta_{li} E_{kj}] \\ &= -[\varphi(E_{jj}), \delta_{li}(\varphi(D_{kj}) + b_{kj} \varphi(E_{kj}))] \\ &= -\delta_{li}[\varphi(E_{jj}), \varphi(D_{kj})] - \delta_{li}[\varphi(E_{jj}), b_{kj} \varphi(E_{kj})] \\ &= -\delta_{li}[E_{jj}, D_{kj}] - \delta_{li} b_{kj} [E_{jj}, E_{kj}] \\ &= \delta_{li} b_{kj} E_{kj}.\end{aligned} \tag{3.3}$$

On the other hand,

$$
\begin{aligned}
[\varphi(E_{kl}), E_{ij}] &= [\varphi(E_{kl}), \varphi(D_{ij}) + b_{ij}\varphi(E_{ij})] \\
&= [E_{kl}, D_{ij}] + b_{ij}[E_{kl}, E_{ij}] \\
&= [E_{kl}, D_{ij}] + \delta_{li} b_{ij} E_{kj}.
\end{aligned} \tag{3.4}
$$

Thus $[E_{kl}, D_{ij}] = 0$. Similarly, if i, k, l are distinct, we can obtain that $[E_{kl}, D_{ij}] = 0$. So $[E_{kl}, D_{ij}] = 0$ for any $(k, l) \neq (i, j)$. Since D_{ij} is a diagonal matrix, it is easy to see that $D_{ij} \in Z(\mathcal{T}) = RI$. Then $D_{ij} = r_{ij}I$ for some $r_{ij} \in R$. By Lemma 3.1, $\varphi(D_{ij}) = \varphi(r_{ij}I) = ar_{ij}I$. Set

$$c_{ij} = ar_{ij}, b_{ij} = a_{ij}^{(ij)}.$$

Then the lemma holds. $\square$

Lemma 3.3. *Let φ be an invertible map preserving strong commutativity on $\mathcal{T}$, $n \geq 3$. If $E_{ij} = c_{ij}I + b_{ij}\varphi(E_{ij})$, $1 \leq i < j \leq n$, then all b_{ij} are equal.*

Proof. At first we prove that

$$b_{ij} = b_{kj}, \quad \forall 1 \leq k \leq i-1. \tag{3.5}$$

On one hand,

$$
\begin{aligned}
[\varphi(E_{ki}), E_{ij}] &= -[\varphi(E_{ki}), [E_{jj}, E_{ij}]] \\
&= -[\varphi(E_{ki}), [\varphi(E_{jj}), \varphi(E_{ij})]] \\
&= -[[\varphi(E_{ki}), \varphi(E_{jj})], \varphi(E_{ij})] - [\varphi(E_{jj}), [\varphi(E_{ki}), \varphi(E_{ij})]] \\
&= -[[E_{ki}, E_{jj}], \varphi(E_{ij})] - [\varphi(E_{jj}), [E_{ki}, E_{ij}]] \\
&= -[\varphi(E_{jj}), E_{kj}] \\
&= -[\varphi(E_{jj}), c_{kj}I + b_{kj}\varphi(E_{kj})] \\
&= -b_{kj}[E_{jj}, E_{kj}] \\
&= b_{kj}E_{kj}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
[\varphi(E_{ki}), E_{ij}] &= [\varphi(E_{ki}), c_{ij}I + b_{ij}\varphi(E_{ij})] \\
&= b_{ij}[\varphi(E_{ki}), \varphi(E_{ij})] \\
&= b_{ij}[E_{ki}, E_{ij}] \\
&= b_{ij}E_{kj}.
\end{aligned}
$$

Then $b_{ij} = b_{kj}$.

Similarly, we can prove that

$$b_{ij} = b_{il}, \quad \forall l > j. \tag{3.6}$$

Set $b_{12} = r$. By (3.6), $r = b_{13} = \cdots = b_{1n}$. For any (i, j), $2 \leq i < j \leq n$, we have $b_{ij} = b_{i-1,j} = \cdots = b_{1j} = r$ by (3.5). So the lemma holds. $\square$

Theorem 3.4. *Let R be a commutative ring with identity 1, $n \geq 3$, let $\mathcal{T}_n(R)$ be the linear Lie algebra of all upper triangular $n \times n$ matrices over R, and let $\varphi : \mathcal{T} \to \mathcal{T}$ be a map. Then φ is an invertible linear map preserving strong commutativity if and only if φ is a composition of an idempotent scalar multiplication η_r, an extremal inner automorphism φ_q and a linear map θ_f induced by a linear function f, i.e., there exist $q \in R$, $r \in R^*$ with $r^2 = 1$, and a linear function $f : \mathcal{T} \to R$ with $1 + f(I) \in R^*$, such that*

$$\varphi = \varphi_q \cdot \eta_r \cdot \theta_f.$$

Proof. The sufficient direction is obvious. We prove the necessity. By Lemmas 3.1-3.3, there exist $r \in R$ and $c_{ij} \in R$, $1 \leq i < j \leq n$, such that

$$E_{ij} = c_{ij} I + r\varphi(E_{ij}).$$

By computations, $E_{13} = [E_{12}, E_{23}] = [c_{12}I + r\varphi(E_{12}), c_{23}I + r\varphi(E_{23})] = r^2[\varphi(E_{12}), \varphi(E_{23})] = r^2[E_{12}, E_{23}] = r^2 E_{13}$. Then $r^2 = 1$, i.e., $r \in \mathcal{U}$. Furthermore, $rE_{ij} = rc_{ij}I + \varphi(E_{ij})$, and so

$$\varphi(E_{ij}) = rE_{ij} - rc_{ij}I, \quad 1 \leq i < j \leq n.$$

Next we will prove that for any $1 \leq k \leq n$, $\varphi(E_{kk}) = rE_{kk} + b_k I + a_k E_{1n}$ for some $b_k, a_k \in R$. Fix $k \in \{1, 2, \ldots, n\}$. Assume that

$$\varphi(E_{kk}) = \sum_{p=1}^{n} c_{pp}^{(k)} E_{pp} + \sum_{1 \leq s < t \leq n} c_{st}^{(k)} E_{st}.$$

For any $1 \leq i \leq n-1$,

$$\begin{aligned}
[\varphi(E_{kk}), E_{i,i+1}] &= [\sum_{p=1}^{n} c_{pp}^{(k)} E_{pp} + \sum_{1 \leq s < t \leq n} c_{st}^{(k)} E_{st}, E_{i,i+1}] \\
&= (c_{ii}^{(k)} - c_{i+1,i+1}^{(k)}) E_{i,i+1} + \sum_{1 \leq s < i} c_{si}^{(k)} E_{s,i+1} - \sum_{i+1 < t \leq n} c_{i+1,t}^{(k)} E_{it}. \qquad (3.7)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[\varphi(E_{kk}), E_{i,i+1}] &= [\varphi(E_{kk}), c_{i,i+1}I + r\varphi(E_{i,i+1})] \\
&= r[\varphi(E_{kk}), \varphi(E_{i,i+1})] \qquad (3.8) \\
&= r[E_{kk}, E_{i,i+1}] \\
&= r(\delta_{ki} - \delta_{k,i+1}) E_{i,i+1}.
\end{aligned}$$

By the equalities (3.7) and (3.8), we have

$$c_{si}^{(k)} = 0, \quad \forall 1 \leq s < i, \qquad (3.9)$$

$$c_{i+1,t}^{(k)} = 0, \quad \forall i+1 < t \leq n. \qquad (3.10)$$

Assume that $(s,t) \neq (1,n)$, $s < t$. If $s > 1$, then by the equality (3.10), $c_{st}^{(k)} = c_{s-1+1,t}^{(k)} = 0$, where $s - 1 \in \{1, 2, \ldots, n-1\}$. If $s = 1$, then $t \neq n$, and by the equality (3.9), $c_{st}^{(k)} = 0$. Thus $c_{st}^{(k)} = 0$ for any $(s,t) \neq (1,n)$.

Moreover, by the equalities (3.7) and (3.8), $c_{ii}^{(k)} - c_{i+1,i+1}^{(k)} = r(\delta_{ki} - \delta_{k,i+1})$, $1 \le i \le n-1$. So for $i > k$, $c_{ii}^{(k)} = c_{i-1,i-1}^{(k)} = \cdots = c_{k+1,k+1}^{(k)} = c_{kk}^{(k)} - r$, and for $i < k$, $c_{ii}^{(k)} = c_{i+1,i+1}^{(k)} = \cdots = c_{k-1,k-1}^{(k)} = c_{kk}^{(k)} - r$. Thus $\varphi(E_{kk}) = \sum\limits_{p<k}(c_{kk}^{(k)} - r)E_{pp} + c_{kk}^{(k)}E_{kk} + \sum\limits_{p>k}(c_{kk}^{(k)} - r)E_{pp} + c_{1n}^{(k)}E_{1n}$. Set $b_k = c_{kk}^{(k)} - r, a_k = c_{1n}^{(k)}$. Then

$$\varphi(E_{kk}) = rE_{kk} + b_k I + a_k E_{1n}, \quad 1 \le k \le n.$$

For any $k \ne 1$ or n, $[\varphi(E_{11}), \varphi(E_{kk})] = [b_1 I + rE_{11} + a_1 E_{1n}, b_k I + rE_{kk} + a_k E_{1n}] = [rE_{11} + a_1 E_{1n}, rE_{kk} + a_k E_{1n}] = ra_k E_{1n}$. On the other hand, $[\varphi(E_{11}), \varphi(E_{kk})] = [E_{11}, E_{kk}] = 0$. Thus $ra_k = 0$. Since $r^2 = 1$, then $a_k = 0$. Furthermore, $[\varphi(E_{11}), \varphi(E_{nn})] = [b_1 I + rE_{11} + a_1 E_{1n}, b_n I + rE_{nn} + a_n E_{1n}] = r(a_1 + a_n)E_{1n} = 0$, and so $a_1 = -a_n$. Set $a_1 = a$. Thus

$$\varphi(E_{11}) = b_1 I + rE_{11} + aE_{1n},$$
$$\varphi(E_{nn}) = b_n I + rE_{nn} - aE_{1n},$$
$$\varphi(E_{kk}) = b_k I + rE_{kk}, \forall k \ne 1, n.$$

Since $\varphi(I) = \varphi(\sum\limits_{k=1}^{n} E_{kk}) = (r + \sum\limits_{k=1}^{n} b_k)I \ne 0$, then $r + \sum\limits_{k=1}^{n} b_k \ne 0$, which implies that

$$1 + r\sum_{k=1}^{n} b_k \ne 0.$$

Define $\eta_r : \mathcal{T} \to \mathcal{T}$ by $\eta_r(X) = rX$. Then η_r is an idempotent scalar multiplication, and η_r preserves strong commutativity. Set $q = ra$, and define $\varphi_{-q} : \mathcal{T} \to \mathcal{T}$ by $\varphi_{-q}(X) = S_{-q}^{-1} X S_q$, where $S_{-q} = I - qE_{1n}$. Then φ_{-q} is an inner automorphism preserving strong commutativity, and so $\varphi_{-q}\eta_r\varphi$ is also an invertible linear map preserving strong commutativity. Next we will prove that $\varphi_{-q}\eta_r\varphi$ is a linear map induced by a linear function.

By computations,

$$\begin{aligned}(\varphi_{-q}\eta_r\varphi)(E_{11}) &= \varphi_{-q}(rb_1 I + E_{11} + raE_{1n})\\ &= S_{-q}^{-1}(rb_1 I)S_{-q} + S_{-q}^{-1}(E_{11})S_{-q} + S_q^{-1}(qE_{1n})S_{-q}\\ &= rb_1 I + (I + qE_{1n})E_{11}(I - qE_{1n}) + (I + qE_{1n})qE_{1n}(I - qE_{1n})\\ &= E_{11} + rb_1 I.\end{aligned}$$

Similarly, by computations, we have $(\varphi_{-q}\eta_r\varphi)(E_{kk}) = E_{kk} + rb_k I$ for $k \ne 1$. And for any $1 \le i < j \le n$, $(\varphi_{-q}\eta_r\varphi)(E_{ij}) = \varphi_{-q}(E_{ij} - c_{ij}I) = (I + qE_{1n})E_{ij}(I - qE_{1n}) - (I + qE_{1n})c_{ij}I(I - qE_{1n}) = E_{ij} - c_{ij}I$.

Define a linear function $f : \mathcal{T} \to R$ defined by $f(E_{ij}) = -c_{ij}$, $1 \le i < j \le n$, and $f(E_{kk}) = rb_k$, $1 \le k \le n$. Then $f(I) = r(\sum\limits_{k=1}^{n} b_k) \ne -1$, and so the linear map $\theta_f : \mathcal{T} \to \mathcal{T}$ defined by $\theta_f(X) = X + f(X)I$ is an invertible map preserving strong commutativity. Since the linear maps $\varphi_{-q}\eta_r\varphi$ and θ_f have the same actions on the basis $\{E_{ij} \mid 1 \le i < j \le n\} \cup \{E_{kk} \mid 1 \le k \le n\}$, then

$\varphi_{-q}\eta_r\varphi = \theta_f$. Thus $\varphi = \eta_r^{-1}\varphi_{-q}^{-1}\theta_f = \eta_r\varphi_q\theta_f$, i.e., φ is a composition of the idempotent scalar multiplication η_r, an extremal inner automorphism φ_q and a linear map θ_f induced by the linear function f on $\mathcal{T}$. $\square$

Remark 3.5. Theorem 3.4 does not hold for $n = 2$. Here we give a counterexample. Let a be an invertible element in R, and $r \in R$ such that $r \neq a^{-1}$. Define a linear map $\varphi : \mathcal{T} \to \mathcal{T}$ such that $\varphi(E_{11}) = aE_{11}, \varphi(E_{22}) = aE_{22}$, $\varphi(E_{12}) = a^{-1}E_{12} + rI$. Then φ is invertible, and φ^{-1} is the linear map defined by $\varphi^{-1}(E_{11}) = a^{-1}E_{11}$, $\varphi^{-1}(E_{22}) = a^{-1}E_{22}$, $\varphi^{-1}(E_{12}) = aE_{12} - rI$. It is easy to see that $[\varphi(E_{ij}), \varphi(E_{kl})] = [E_{ij}, E_{kl}]$ for any $1 \leq i \leq j \leq 2, 1 \leq k \leq l \leq 2$, then φ preserves strong commutativity. However, for any $q, r \in R$, and any linear function $f : \mathcal{T} \to R$, we have

$$(\eta_r\varphi_q\theta_f)(E_{12}) = rS_q^{-1}(f(E_{12})I + E_{12})S_q = rf(E_{12})I + rE_{12}.$$

Since $r \neq a^{-1}$, then $(\eta_r\varphi_q\theta_f)(E_{12}) \neq \varphi(E_{12})$, and so $\eta_r\varphi_q\theta_f \neq \varphi$.

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