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STRONGLY GORENSTEIN C-HOMOLOGICAL MODULES UNDER CHANGE OF RINGS

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Abstract. Some properties of strongly Gorenstein C-projective, C-injective and C-flat modules are studied, mainly considering these properties under change of rings. Specifically, the completions of rings, the localizations and the polynomial rings are considered.

1. Introduction

Unless stated otherwise, throughout this paper R is a commutative and noetherian ring with identity and C is a semidualizing R-module. Let R be a ring and we denote the category of left R-module by R-Mod. By $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ denote the classes of all C-projective, C-flat and C-injective R-modules, respectively. For any R-module M, $pd_R(M)$ denotes the projective dimension of M.

When R is two-sided noetherian, Auslander and Bridger [1] introduced the G-dimension, G-$\dim_R(M)$ for every finitely generated R-module M. They proved the inequality G-$\dim_R(M) \leq pd_R(M)$, with equality G-$\dim_R(M) = pd_R(M)$ when $pd_R(M)$ is finite. Several decades later, Enochs and Jenda [4, 5] extended the ideas of Auslander and Bridger and introduced the Gorenstein projective, injective and flat dimensions. Bennis and Mahdou [3] studied a particular case of Gorenstein projective, injective and flat modules, which they called strongly Gorenstein projective, injective and flat modules. Yang and Liu [16] discussed some connections between strongly Gorenstein projective, injective and flat modules, and they considered these properties under change of rings. Specifically, they considered completions of rings and localizations.

The nation of a "semidualizing module" is one of the most central notions in the relative homological algebra. This notion was first introduced by Foxby [7]. This notion has been investigated by many authors from different points

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of view. A semidualizing R-module C gives rise to several distinguished classes of modules. For instance, one has the classes $\mathcal{P}_C(R)$, $\mathcal{I}_C(R)$ and $\mathcal{F}_C(R)$ of C-projective, C-injective and C-flat R-modules. Over a commutative noetherian ring, Holm and Jørgensen [8] introduced G_C-projective, G_C-injective and G_C-flat modules. For Gorenstein homological theory with respect to semidualizing modules, Wang [14] gave another definition. An R-module M is called Gorenstein C-projective if there exists an exact sequence of R-modules

$$\mathbf{P} = \cdots \xrightarrow{f_1} C \otimes_R P_0 \xrightarrow{f_0} C \otimes_R P^0 \xrightarrow{f^0} C \otimes_R P^1 \xrightarrow{f^1} \cdots ,$$

where each P_i and P^i is projective with $M \cong \mathrm{Ker} f^0$, such that the complex $\mathrm{Hom}_R(\mathbf{P}, Q)$ is exact for each C-projective R-module Q.

Dually, Gorenstein C-injective modules are defined.

An R-module M is said to be Gorenstein C-flat, if there exists an exact sequence of C-flat R-module

$$\mathbf{F} = \cdots \xrightarrow{f_1} C \otimes_R F_0 \xrightarrow{f_0} C \otimes_R F^0 \xrightarrow{f^0} C \otimes_R F^1 \xrightarrow{f^1} \cdots ,$$

with $M \cong \mathrm{Ker} f^0$, such that complex $E \otimes_R \mathbf{F}$ is exact for each C-injective R-modules E.

Recently, Zhang, Liu, and Yang [17] introduced the concepts of strongly $\mathcal{W}_P$-Gorenstein, $\mathcal{W}_I$-Gorenstein and $\mathcal{W}_F$-Gorenstein modules and discussed the basic properties of these modules. Some results related to strongly Gorenstein projective, injective and flat modules were extended to strongly $\mathcal{W}_P$-Gorenstein, $\mathcal{W}_I$-Gorenstein and $\mathcal{W}_F$-Gorenstein modules. In this paper, we introduce the concepts of strongly Gorenstein projective, injective, flat modules with respect to a semidualizing module, which are different from the definition above and those in [17] and mainly consider these properties under change of rings. Specifically, the completions of rings, the localizations and the polynomial rings are considered.

2. Preliminaries

In this section, we recall some definitions and known facts needed in the sequel.

Definition ([17])**.** Let R be a ring. A finitely generated R-module C is called semidualizing if the following conditions are satisfied:

(1) The natural homothety morphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism,

(2) $\mathrm{Ext}_R^{i \geq 1}(C, C) = 0$.

Definition ([17])**.** An R-module is C-flat (resp., C-projective) if it has the form $C \otimes_R F$ for some flat (resp., projective) R-module F. An R-module is C-injective if it has the form $\mathrm{Hom}_R(C, I)$ for some injective R-module I. Set

$$\mathcal{F}_C(R) = \{C \otimes_R F \mid F \text{ is a flat } R\text{-module}\},$$

$$\mathcal{P}_C(R) = \{C \otimes_R P \mid P \text{ is a projective } R\text{-module}\},$$

$$\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C, I) \mid I \text{ is an injective } R\text{-module}\}.$$

Then $\mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$. Clearly, if $C = R$, then $\mathcal{P}_C(R), \mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ are just the classes of ordinary projective, flat and injective R-modules.

Definition. An R-module M is said to be strongly Gorenstein C-projective, if there exists an exact sequence of C-projective R-modules

$$\mathbf{P} = \cdots \xrightarrow{f} C \otimes_R P \xrightarrow{f} C \otimes_R P \xrightarrow{f} C \otimes_R P \xrightarrow{f} \cdots,$$

with $M \cong \mathrm{Ker} f$, such that the complex $\mathrm{Hom}_R(\mathbf{P}, Q)$ is exact for each C-projective R-module Q.

An R-module M is said to be strongly Gorenstein C-injective, if there exists an exact sequence of C-injective R-module

$$\mathbf{I} = \cdots \xrightarrow{f} \mathrm{Hom}_R(C, I) \xrightarrow{f} \mathrm{Hom}_R(C, I) \xrightarrow{f} \mathrm{Hom}_R(C, I) \xrightarrow{f} \cdots,$$

with $M \cong \mathrm{Ker} f$, such that the complex $\mathrm{Hom}_R(E, \mathbf{I})$ is exact for each C-injective R-module E.

An R-module M is said to be strongly Gorenstein C-flat, if there exists an exact sequence of C-flat R-module

$$\mathbf{F} = \cdots \xrightarrow{f} C \otimes_R F \xrightarrow{f} C \otimes_R F \xrightarrow{f} C \otimes_R F \xrightarrow{f} \cdots,$$

with $M \cong \mathrm{Ker} f$, such that complex $E \otimes_R \mathbf{F}$ is exact for each C-flat R-modules E.

By $\mathcal{SGP}_C(R)$, $\mathcal{SGI}_C(R)$, $\mathcal{SGF}_C(R)$ we denote the classes of all strongly Gorenstein C-projective, C-injective and C-flat R-modules, respectively.

It is easy to see that, when $C = R$, the definitions correspond to those of strongly Gorenstein projective, injective and flat modules. In particular, for more similarly details about strongly Gorenstein C-projective, C-injective and C-flat R-modules, the reader may consult [3, 16, 17].

We have the following simple facts.

Proposition 2.1. *Let C be a semidualizing R-module. For any module M, the following are equivalent*:

(1) *M is a strongly Gorenstein C-projective R-module*;

(2) *there exists a short exact sequence $0 \to M \to C \otimes_R P \to M \to 0$, where P is a projective R-module, and $\mathrm{Ext}_R^{i \geq 1}(M, Q) = 0$ for any C-projective R-module Q*;

(3) *there exists a short exact sequence $0 \to M \to C \otimes_R P \to M \to 0$, where P is a projective R-module, and $\mathrm{Ext}_R^{i \geq 1}(M, L) = 0$ for any module L with finite C-projective dimension*;

(4) *there exists a short exact sequence $0 \to M \to C \otimes_R P \to M \to 0$, where P is a projective R-module, such that the sequence $0 \to \mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(C \otimes_R P, Q) \to \mathrm{Hom}_R(M, Q) \to 0$ is exact for any C-projective R-module Q.*

Proof. (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (4) are obvious.

(2) $\Rightarrow$ (3) holds by dimension shifting.

(3) $\Rightarrow$ (2) is clear. □

3. Main results

Let (R, m) be a commutative local noetherian ring with the maximal ideal m and its residue field k and let $E(k)$ be the injective envelope of k. $\widehat{R}, \widehat{M}$ will denote the m-adic completion of a ring R and an R-module M, respectively. In [16], Yang and Liu discussed some properties of strongly Gorenstein projective (injective, flat) R-modules in the completions of the rings. In the following, we give some properties of strongly Gorenstein C-projective (C-injective, C-flat) R-modules in the completion of the ring.

Lemma 3.1. *Let (R,m) be a local ring. If C be a semidualizing R-module, then $\widehat{C}$ is a semidualizing module of $\widehat{R}$.*

Proof. This follows from [6, Theorem 2.5.14] and [6, Theorem 3.2.5]. □

Next are key lemmas, which play an important part.

Lemma 3.2. *Let (R, m) be a local ring.*

(1) *If P is a projective R-module, then $P \otimes_R \widehat{R}$ is a projective $\widehat{R}$-module.*

(2) *Assume that $\widehat{R}$ is a projective R-module. If $\overline{P}$ is a projective $\widehat{R}$-module, then $\overline{P}$ is a projective R-module.*

Proof. (1) Since $\mathrm{Ext}^{i\geq 1}_{\widehat{R}}(\widehat{R} \otimes_R P, -) = \mathrm{Hom}_R(P, \mathrm{Ext}^{i\geq 1}_{\widehat{R}}(\widehat{R}, -)) = 0$ by [12, p. 258, 9.20], $\mathrm{Ext}^{i\geq 1}_{\widehat{R}}(\widehat{R} \otimes_R P, -) = 0$. Hence $\widehat{R} \otimes_R P$ is a projective $\widehat{R}$-module.

(2) Since $\overline{P}$ is isomorphic to a summand of $\widehat{R}^{(X)}$ for some set X and $\widehat{R}^{(X)}$ is a projective R-module, it follows that $\overline{P}$ is a projective R-module. □

Lemma 3.3. *Let (R, m) be a local ring.*

(1) *If E is an injective R-module, then $E \otimes_R \widehat{R}$ is an injective $\widehat{R}$-module.*

(2) *If $\overline{E}$ is an injective $\widehat{R}$-module, then $\overline{E}$ is an injective R-module.*

Proof. (1) Let E be any injective R-module. Then E is isomorphic to a summand of $E(k)^X$ for some set X, and hence $E \otimes_R \widehat{R}$ is isomorphic to a summand of $E(k)^X \otimes_R \widehat{R} = E_{\widehat{R}}(\widehat{R}/\widehat{m})^X \otimes_R \widehat{R}$ by [6, Theorem 3.4.1]. Since $E_{\widehat{R}}(\widehat{R}/\widehat{m})^X \otimes_R \widehat{R}$ is an injective $\widehat{R}$-module, $E \otimes_R \widehat{R}$ is an injective $\widehat{R}$-module.

(2) Since $\widehat{R}$ is a faithfully flat R-module by [6, Theorem 2.5.18]. Let M be a finitely generated R-module. Then $\mathrm{Ext}^{i\geq 1}_R(M, \overline{E}) \otimes_R \widehat{R} = \mathrm{Ext}^{i\geq 1}_{\widehat{R}}(M \otimes_R \widehat{R}, \overline{E} \otimes_R \widehat{R}) = 0$ by [6, Theorem 3.2.5]. Since $\overline{E} \otimes_R \widehat{R}$ is an injective $\widehat{R}$-module by [6, Theorem 3.2.16], $\mathrm{Ext}^{i\geq 1}_R(M, \overline{E}) = 0$. Hence $\overline{E}$ is an injective R-module. □

Lemma 3.4. *Let (R, m) be a local ring.*

(1) *If F is a flat R-module, then $F \otimes_R \widehat{R}$ is a flat $\widehat{R}$-module.*

(2) *If $\overline{F}$ is a flat $\widehat{R}$-module, then $\overline{F}$ is a flat R-module.*

Proof. (1) It is clear by [6, p. 43 Exercise 9].

(2) Let M be any R-module. Then $\mathrm{Tor}^R_{i\geq 1}(M,\overline{F})\otimes_R \widehat{R} \cong \mathrm{Tor}^{\widehat{R}}_{i\geq 1}(M\otimes_R \widehat{R},\overline{F}\otimes_R \widehat{R})$ by [6, Theorem 2.1.11]. Since $\widehat{R}$ is a faithfully flat R-module by [6, Theorem 2.5.18], $\mathrm{Tor}^{\widehat{R}}_{i\geq 1}(M\otimes_R \widehat{R},\overline{F}\otimes_R \widehat{R})=0$. Hence $\mathrm{Tor}^R_{i\geq 1}(M,\overline{F})=0$. It follows that $\overline{F}$ is a flat R-module. □

Proposition 3.5. *Let (R,m) be a local ring.*

(1) *If $\widehat{R}$ is a projective R-module and $M\in\mathcal{SGP}_C(R)$, then $M\otimes_R\widehat{R}\in\mathcal{SGP}_{\widehat{C}}(\widehat{R})$.*

(2) *Let M be a finitely generated R-module. If $\widehat{M}\in\mathcal{SGP}_{\widehat{C}}(\widehat{R})$, then $\widehat{M}\in\mathcal{SGP}_C(R)$.*

Proof. (1) There is an exact sequence $0\to M\to C\otimes_R P\to M\to 0$ in R-Mod with P projective and $\mathrm{Ext}^{i\geq 1}_R(M,Q)=0$ for any C-projective R-module Q. Then

$$0\to M\otimes_R\widehat{R}\to (C\otimes_R P)\otimes_R\widehat{R}\to M\otimes_R\widehat{R}\to 0$$

is exact. Since $C\otimes_R P\otimes_R\widehat{R}\cong (C\otimes_R\widehat{R})\otimes_{\widehat{R}}(\widehat{R}\otimes_R P)\cong \widehat{C}\otimes_{\widehat{R}}(\widehat{R}\otimes_R P)$, $\widehat{R}\otimes_R P$ is a projective $\widehat{R}$-module by Lemma 3.2(1). Then

$$0\to M\otimes_R\widehat{R}\to \widehat{C}\otimes_{\widehat{R}}(\widehat{R}\otimes_R P)\to M\otimes_R\widehat{R}\to 0$$

is exact. Let $\overline{P}$ be any projective $\widehat{R}$-module. Then $\overline{P}$ is a projective R-module by Lemma 3.2(2). So

$$\begin{aligned}\mathrm{Ext}^{i\geq 1}_{\widehat{R}}(M\otimes_R\widehat{R},\widehat{C}\otimes_{\widehat{R}}\overline{P}) &\cong \mathrm{Ext}^{i\geq 1}_{\widehat{R}}(M\otimes_R\widehat{R},C\otimes_R\widehat{R}\otimes_{\widehat{R}}\overline{P})\\ &\cong \mathrm{Ext}^{i\geq 1}_{\widehat{R}}(M\otimes_R\widehat{R},C\otimes_R\overline{P})\\ &\cong \mathrm{Ext}^{i\geq 1}_{R}(M,\mathrm{Hom}_{\widehat{R}}(\widehat{R},C\otimes_R\overline{P}))\\ &\cong \mathrm{Ext}^{i\geq 1}_{R}(M,C\otimes_R\overline{P})=0\end{aligned}$$

by [12, p. 258, 9.20]. Hence $\widehat{M}\in\mathcal{SGP}_{\widehat{C}}(\widehat{R})$.

(2) There is an exact sequence $0\to\widehat{M}\to\widehat{C}\otimes_{\widehat{R}}\overline{P}\to\widehat{M}\to 0$ in $\widehat{R}$-Mod with $\overline{P}$ projective. Since $\widehat{C}\otimes_{\widehat{R}}\overline{P}\cong C\otimes_R\widehat{R}\otimes_{\widehat{R}}\overline{P}\cong C\otimes_R(\widehat{R}\otimes_{\widehat{R}}\overline{P})\cong C\otimes_R\overline{P}$ with $\overline{P}$ projective in R-Mod by Lemma 3.2(2). The following sequence

$$0\to\widehat{M}\to C\otimes_R\overline{P}\to\widehat{M}\to 0$$

is exact in R-Mod. Let P be any projective R-module. Then

$$\begin{aligned}\mathrm{Ext}^{i\geq 1}_R(M,C\otimes_R P)\otimes_R\widehat{R} &\cong \mathrm{Ext}^{i\geq 1}_{\widehat{R}}(M\otimes_R\widehat{R},C\otimes_R P\otimes_R\widehat{R})\\ &\cong \mathrm{Ext}^{i\geq 1}_{\widehat{R}}(\widehat{M},\widehat{C}\otimes_{\widehat{R}}(\widehat{R}\otimes_R P))=0\end{aligned}$$

by [6, Theorem 3.2.5] and Lemma 3.2(1). Since $\widehat{R}$ is a faithfully flat R-module, $\mathrm{Ext}_R^{i\geq 1}(M, C\otimes_R P)=0$. Since

$$\begin{aligned}\mathrm{Ext}_R^{i\geq 1}(\widehat{M}, C\otimes_R P) &\cong \mathrm{Ext}_R^{i\geq 1}(M\otimes_R \widehat{R}, C\otimes_R P)\\ &\cong \mathrm{Hom}_R(\widehat{R}, \mathrm{Ext}_R^{i\geq 1}(M, C\otimes_R P))\\ &= 0\end{aligned}$$

by [12, p. 258, 9.20], we have $\mathrm{Ext}_R^{i\geq 1}(\widehat{M}, C\otimes_R P)=0$. Hence $\widehat{M}\in \mathcal{SGP}_C(R)$. □

Dually, we discuss the properties of strongly Gorenstein C-injective R-modules.

Proposition 3.6. *Let (R,m) be a local ring. If $\widehat{R}$ is a projective R-module, then the following statements hold.*

(1) *If $M\in\mathcal{SGI}_C(R)$, then $\mathrm{Hom}_R(\widehat{R},M)\in\mathcal{SGI}_{\widehat{C}}(\widehat{R})$.*

(2) *If $\mathrm{Hom}_R(\widehat{R},M)\in\mathcal{SGI}_{\widehat{C}}(\widehat{R})$, then $\mathrm{Hom}_R(\widehat{R},M)\in\mathcal{SGI}_C(R)$.*

Proof. (1) There is an exact sequence $0\to M\to \mathrm{Hom}_R(C,E)\to M\to 0$ in R-Mod with E injective and $\mathrm{Ext}_R^{i\geq 1}(I,M)=0$ for any C-injective R-module I. Then

$$0\to \mathrm{Hom}_R(\widehat{R},M)\to \mathrm{Hom}_R(\widehat{R},\mathrm{Hom}_R(C,E))\to \mathrm{Hom}_R(\widehat{R},M)\to 0$$

is exact. Since

$$\begin{aligned}\mathrm{Hom}_R(\widehat{R},\mathrm{Hom}_R(C,E)) \cong \mathrm{Hom}_R(C\otimes_R \widehat{R},E) &\cong \mathrm{Hom}_R(C\otimes_R \widehat{R}\otimes_{\widehat{R}}\widehat{R},E)\\ &\cong \mathrm{Hom}_R(\widehat{C}\otimes_{\widehat{R}}\widehat{R},E)\\ &\cong \mathrm{Hom}_{\widehat{R}}(\widehat{C},\mathrm{Hom}_R(\widehat{R},E))\end{aligned}$$

by [12, p. 258, 9.20] and [12, p. 258, 9.21], the sequence

$$0\to \mathrm{Hom}_R(\widehat{R},M)\to \mathrm{Hom}_{\widehat{R}}(\widehat{C},\mathrm{Hom}_R(\widehat{R},E))\to \mathrm{Hom}_R(\widehat{R},M)\to 0$$

is exact in $\widehat{R}$-Mod with $\mathrm{Hom}_R(\widehat{R},E)$ injective in $\widehat{R}$-module.

Let $\overline{E}$ be any injective $\widehat{R}$-module. Then $\overline{E}$ is an injective R-module by Lemma 3.3(2). Since

$$\begin{aligned}\mathrm{Hom}_{\widehat{R}}(\widehat{C},\overline{E}) \cong \mathrm{Hom}_{\widehat{R}}(C\otimes_R\widehat{R},\overline{E}) &\cong \mathrm{Hom}_R(C,\mathrm{Hom}_{\widehat{R}}(\widehat{R},\overline{E}))\\ &\cong \mathrm{Hom}_R(C,\overline{E})\end{aligned}$$

by [12, p. 258, 9.21],

$$\begin{aligned}\mathrm{Ext}_{\widehat{R}}^{i\geq 1}(\mathrm{Hom}_{\widehat{R}}(\widehat{C},\overline{E}),\mathrm{Hom}_R(\widehat{R},M)) &\cong \mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_{\widehat{R}}(\widehat{C},\overline{E})\otimes_{\widehat{R}}\widehat{R},M)\\ &\cong \mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_{\widehat{R}}(\widehat{C},\overline{E}),M)\\ &\cong \mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_R(C,\overline{E}),M)=0\end{aligned}$$

by [12, p. 258, 9.21] and [6, Lemma 3.2.4]. Therefore $\mathrm{Hom}_R(\widehat{R},M)\in\mathcal{SGI}_{\widehat{C}}(\widehat{R})$.

(2) There is an exact sequence

$$0 \to \mathrm{Hom}_R(\widehat{R}, M) \to \mathrm{Hom}_{\widehat{R}}(\widehat{C}, \overline{E}) \to \mathrm{Hom}_R(\widehat{R}, M) \to 0$$

in R-Mod with $\overline{E}$ injective in $\widehat{R}$-Mod. Note that

$$\mathrm{Hom}_{\widehat{R}}(\widehat{C}, \overline{E}) \cong \mathrm{Hom}_{\widehat{R}}(C \otimes_R \widehat{R}, \overline{E}) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_{\widehat{R}}(\widehat{R}, \overline{E})) \cong \mathrm{Hom}_R(C, \overline{E})$$

by [12, p. 258, 9.21]. Let I be any injective R-module. Then $I \otimes_R \widehat{R}$ is an injective $\widehat{R}$-module by Lemma 3.3(1), and hence

$$\begin{aligned}
&\mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_R(C, I), \mathrm{Hom}_R(\widehat{R}, M)) \\
&\cong \mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_R(C, I), \mathrm{Hom}_{\widehat{R}}(\widehat{R}, \mathrm{Hom}_R(\widehat{R}, M))) \\
&\cong \mathrm{Ext}_{\widehat{R}}^{i\geq 1}(\widehat{R} \otimes_R \mathrm{Hom}_R(C, I), \mathrm{Hom}_R(\widehat{R}, M)) \\
&\cong \mathrm{Ext}_{\widehat{R}}^{i\geq 1}(\mathrm{Hom}_{\widehat{R}}(\widehat{C}, I \otimes_R \widehat{R}), \mathrm{Hom}_R(\widehat{R}, M)) = 0.
\end{aligned}$$

So $\mathrm{Hom}_R(\widehat{R}, M) \in \mathcal{SGI}_C(R)$. □

Theorem 3.7. *Let (R, m) be a local ring and M a finitely generated R-module. Consider the following statements*:

(1) $M \in \mathcal{SGF}_C(R)$;

(2) $\widehat{M} \in \mathcal{SGF}_{\widehat{C}}(\widehat{R})$;

(3) $\widehat{M} \in \mathcal{SGF}_C(R)$.

Then (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1). *If $\widehat{R}$ is a finitely generated projective R-module, then* (2) $\Rightarrow$ (3).

Proof.

$$\begin{aligned}
\mathrm{Tor}_{i\geq 1}^{\widehat{R}}(\mathrm{Hom}_{\widehat{R}}(\widehat{C}, E(k)), \widehat{M}) &\cong \mathrm{Tor}_{i\geq 1}^{\widehat{R}}(\mathrm{Hom}_R(C, E(k)) \otimes_R \widehat{R}, \widehat{M}) \\
&\cong \mathrm{Tor}_{i\geq 1}^{R}(\mathrm{Hom}_R(C, E(k)), M) \otimes_R \widehat{R}.
\end{aligned}$$

Since $\widehat{R}$ is a faithfully flat R-module,

$$\mathrm{Tor}_{i\geq 1}^{\widehat{R}}(\mathrm{Hom}_{\widehat{R}}(\widehat{C}, E(k)), \widehat{M}) = 0 \Leftrightarrow \mathrm{Tor}_{i\geq 1}^{R}(\mathrm{Hom}_R(C, E(k)), M) = 0.$$

(1) $\Rightarrow$ (2) There is an exact sequence $0 \to M \to C \otimes_R F \to M \to 0$ in R-Mod with F flat. Then

$$0 \to \widehat{M} \to \widehat{C} \otimes_{\widehat{R}} (F \otimes_R \widehat{R}) \to \widehat{M} \to 0$$

is exact with $(C \otimes_R F) \otimes_{\widehat{R}} \widehat{R} \cong \widehat{C} \otimes_{\widehat{R}} (F \otimes_R \widehat{R})$. Then $F \otimes_R \widehat{R}$ is a flat $\widehat{R}$-module by Lemma 3.4(1). Let $\overline{I}$ be any injective $\widehat{R}$-module. Then $\overline{I}$ is an injective R-module by Lemma 3.3(2). Hence $\overline{I}$ is isomorphic to a summand of $E(k)^X$ for some set X. Since $\widehat{R}$ is a faithfully flat R-module by [6, Theorem 3.2.26],

$$\begin{aligned}
\mathrm{Tor}_{i\geq 1}^{\widehat{R}}(\mathrm{Hom}_{\widehat{R}}(\widehat{C}, E(k)^X), \widehat{M}) &\cong \mathrm{Tor}_{i\geq 1}^{\widehat{R}}(\mathrm{Hom}_{\widehat{R}}(\widehat{C}, E(k))^X, \widehat{M}) \\
&\cong \mathrm{Tor}_{i\geq 1}^{\widehat{R}}(\mathrm{Hom}_{\widehat{R}}(\widehat{C}, E(k)), \widehat{M})^X = 0.
\end{aligned}$$

Thus $\mathrm{Tor}^{\widehat{R}}_{i\geq 1}(\mathrm{Hom}_{\widehat{R}}(\widehat{C},\overline{I}),\widehat{M})=0$. So $\widehat{M}\in\mathcal{SGF}_{\widehat{C}}(\widehat{R})$.

(2) $\Rightarrow$ (1) There is an exact sequence $0\to\widehat{M}\to\widehat{C}\otimes_{\widehat{R}}\overline{F}\to\widehat{M}\to 0$ in $\widehat{R}$-Mod with $\overline{F}$ flat. Then $\overline{F}$ is a flat R-module by Lemma 3.4(2). Since

$$\mathrm{Hom}_R(\widehat{R}\otimes_R\widehat{R},E(k))\cong\mathrm{Hom}_R(\widehat{R},\mathrm{Hom}_R(\widehat{R},E(k)))\cong\mathrm{Hom}_R(\widehat{R},E(k))$$

by the proof of [13, Corollary 2.5], we have $\widehat{R}\otimes_R\widehat{R}\cong\widehat{R}$. Since $\overline{F}\cong\overline{F}\otimes_{\widehat{R}}\widehat{R}\cong\overline{F}\otimes_{\widehat{R}}(\widehat{R}\otimes_R\widehat{R})\cong(\overline{F}\otimes_{\widehat{R}}\widehat{R})\otimes_R\widehat{R}\cong\overline{F}\otimes_R\widehat{R}$, we have $\widehat{C}\otimes_{\widehat{R}}\overline{F}\cong(\widehat{C}\otimes_R\overline{F})\otimes_R\widehat{R}$. Since $\widehat{R}$ is a faithfully flat R-module, it follows that

$$0\to M\to C\otimes_R\overline{F}\to M\to 0$$

is exact in R-Mod. Let I be any injective R-module. Since I is isomorphic to a summand of $E(k)^X$ for some set X, then

$$\mathrm{Tor}^R_{i\geq 1}(\mathrm{Hom}_R(C,I),M)\cong\mathrm{Tor}^R_{i\geq 1}(\mathrm{Hom}_R(C,E(k)),M)^X=0$$

by [6, Theorem 3.2.26]. So $\mathrm{Tor}^R_{i\geq 1}(\mathrm{Hom}_R(C,I),M)=0$. It follows that $M\in\mathcal{SGF}_C(R)$.

(3)$\Rightarrow$ (2) There is an exact sequence $0\to\widehat{M}\to C\otimes_R F\to\widehat{M}\to 0$ in R-Mod with F flat. Since $\widehat{R}$ is a flat R-module,

$$0\to\widehat{M}\otimes_R\widehat{R}\to(C\otimes_R F)\otimes_R\widehat{R}\to\widehat{M}\otimes_R\widehat{R}\to 0$$

is exact with $C\otimes_R F\otimes_R\widehat{R}\cong(C\otimes_R\widehat{R})\otimes_{\widehat{R}}(\widehat{R}\otimes_R F)\cong\widehat{C}\otimes_{\widehat{R}}(\widehat{R}\otimes_R F)$. Since $\widehat{R}\otimes_R F$ is a flat $\widehat{R}$-module by Lemma 3.4(1),

$$0\to\widehat{M}\to\widehat{C}\otimes_{\widehat{R}}(\widehat{R}\otimes_R F)\to\widehat{M}\to 0$$

is exact. Since $\widehat{R}\otimes_R\widehat{R}\cong\widehat{R}$,

$$\begin{aligned}\mathrm{Tor}^{\widehat{R}}_{i\geq 1}(\mathrm{Hom}_{\widehat{R}}(\widehat{C},E(k)),\widehat{M})&\cong\mathrm{Tor}^{\widehat{R}}_{i\geq 1}(\mathrm{Hom}_R(C,E(k))\otimes_R\widehat{R},M\otimes_R\widehat{R}\otimes_R\widehat{R})\\&\cong\mathrm{Tor}^R_{i\geq 1}(\mathrm{Hom}_R(C,E(k)),M\otimes_R\widehat{R})\otimes_R\widehat{R}\\&\cong\mathrm{Tor}^R_{i\geq 1}(\mathrm{Hom}_R(C,E(k)),\widehat{M})\otimes_R\widehat{R}.\end{aligned}$$

Since $\mathrm{Tor}^R_{i\geq 1}(\mathrm{Hom}_R(C,E(k)),\widehat{M})=0$, $\mathrm{Tor}^{\widehat{R}}_{i\geq 1}(\mathrm{Hom}_R(C,E(k)),\widehat{M})=0$. It follows that $\widehat{M}\in\mathcal{SGF}_{\widehat{C}}(\widehat{R})$.

(2)$\Rightarrow$ (3) There is an exact sequence $0\to\widehat{M}\to\widehat{C}\otimes_{\widehat{R}}\overline{F}\to\widehat{M}\to 0$ in $\widehat{R}$-Mod with $\overline{F}$ flat. Since

$$\widehat{C}\otimes_{\widehat{R}}\overline{F}\cong C\otimes_R\widehat{R}\otimes_{\widehat{R}}\overline{F}\cong C\otimes_R(\widehat{R}\otimes_{\widehat{R}}\overline{F})\cong C\otimes_R\overline{F},$$

$$0\to\widehat{M}\to C\otimes_R\overline{F}\to\widehat{M}\to 0$$

is exact in R-Mod. Since $\widehat{R}\otimes_R\widehat{R}\cong\widehat{R}$,

$$\begin{aligned}0&=\mathrm{Tor}^{\widehat{R}}_{i\geq 1}(\mathrm{Hom}_{\widehat{R}}(\widehat{C},E(k)),\widehat{M})\\&\cong\mathrm{Tor}^{\widehat{R}}_{i\geq 1}(\mathrm{Hom}_R(C,E(k))\otimes_R\widehat{R},M\otimes_R\widehat{R}\otimes_R\widehat{R})\end{aligned}$$

$$\cong \mathrm{Tor}^R_{i\geq 1}(\mathrm{Hom}_R(C, E(k)), M \otimes_R \widehat{R}) \otimes_R \widehat{R}$$
$$\cong \mathrm{Tor}^R_{i\geq 1}(\mathrm{Hom}_R(C, E(k)), \widehat{M}) \otimes_R \widehat{R}.$$

Hence $\mathrm{Tor}^R_{i\geq 1}(\mathrm{Hom}_R(C, E(k)), \widehat{M}) = 0$, and thus $\widehat{M} \in \mathcal{SGF}_C(R)$. □

Proposition 3.8. *Let (R, m) be a local ring. Assume that $\widehat{R}$ is a projective R-module. If M is a strongly Gorenstein C-flat R-module, then $\widehat{R}\otimes_R M$ is a strongly Gorenstein $\widehat{C}$-flat $\widehat{R}$-module.*

Proof. There exists an exact sequence of C-flat R-module

$$\mathfrak{F} = \cdots \xrightarrow{f} C \otimes_R F \xrightarrow{f} C \otimes_R F \xrightarrow{f} C \otimes_R F \xrightarrow{f} \cdots$$

with $M \cong \mathrm{Ker} f$. Then

$$\mathfrak{LF} = \cdots \xrightarrow{\overline{f}} \widehat{R} \otimes_R C \otimes_R F \xrightarrow{\overline{f}} \widehat{R} \otimes_R C \otimes_R F \xrightarrow{\overline{f}} \widehat{R} \otimes_R C \otimes_R F \xrightarrow{\overline{f}} \cdots$$

is exact. So

$$\mathfrak{F} = \cdots \xrightarrow{\overline{f}} (\widehat{R} \otimes_R C) \otimes_{\widehat{R}} (\widehat{R} \otimes_R F) \xrightarrow{\overline{f}} (\widehat{R} \otimes_R C) \otimes_{\widehat{R}} (\widehat{R} \otimes_R F) \xrightarrow{\overline{f}} (\widehat{R} \otimes_R C) \otimes_{\widehat{R}} (\widehat{R} \otimes_R F) \xrightarrow{\overline{f}} \cdots$$

is exact with $\widehat{R} \otimes_R M \cong \mathrm{Ker}\overline{f}$. Then

$$\cdots \xrightarrow{\overline{f}} \widehat{C} \otimes_{\widehat{R}} (\widehat{R} \otimes_R F) \xrightarrow{\overline{f}} \widehat{C} \otimes_{\widehat{R}} (\widehat{R} \otimes_R F) \xrightarrow{\overline{f}} \widehat{C} \otimes_{\widehat{R}} (\widehat{R} \otimes_R F) \xrightarrow{\overline{f}} \cdots$$

is exact in $\widehat{R}$-Mod with $\widehat{R} \otimes_R F$ flat in $\widehat{R}$-Mod. Let $\overline{E}$ be any injective $\widehat{R}$-module. Then $\overline{E}$ is an injective R-module by Lemma 3.3(2).

$$\begin{aligned}
\mathrm{Hom}_{\widehat{R}}(\widehat{C}, \overline{E}) \otimes_{\widehat{R}} (\widehat{C} \otimes_{\widehat{R}} (\widehat{R} \otimes_R F)) &\cong \mathrm{Hom}_{\widehat{R}}(\widehat{C}, \overline{E}) \otimes_{\widehat{R}} (C \otimes_R \widehat{R} \otimes_R F) \\
&\cong \mathrm{Hom}_{\widehat{R}}(\widehat{C}, \overline{E}) \otimes_{\widehat{R}} \widehat{R} \otimes_R (C \otimes_R F) \\
&\cong \mathrm{Hom}_{\widehat{R}}(\widehat{C}, \overline{E}) \otimes_R (C \otimes_R F) \\
&\cong \mathrm{Hom}_R(C, \mathrm{Hom}_{\widehat{R}}(\widehat{R}, \overline{E})) \otimes_R (C \otimes_R F) \\
&\cong \mathrm{Hom}_R(C, \overline{E}) \otimes_R (C \otimes_R F).
\end{aligned}$$

Sine $\mathrm{Hom}_R(C, \overline{E}) \otimes_R \mathfrak{F}$ is exact, then $\mathrm{Hom}_R(C, \overline{E}) \otimes_R \mathfrak{LF}$ is exact. So $\widehat{R}\otimes_R M$ is a strongly Gorenstein $\widehat{C}$-flat $\widehat{R}$-module. □

In this part, we consider these properties under localizations of rings.

Let R be a commutative ring and S a multiplicatively closed set of R. Then $S^{-1}R = (R \times S)/\sim = \{a/s \mid a \in R, s \in S\}$ is a ring and $S^{-1}M = M \times S/\sim = \{x/s \mid x \in M, s \in S\}$ is an $S^{-1}R$-module. If p is a prime ideal of R and $S = R - p$, then we will denote $S^{-1}M, S^{-1}R$ by M_p, R_p, respectively.

Lemma 3.9. *Let R be a commutative ring and S a multiplicatively closed set of R. If C is a semidualizing R-module, then $S^{-1}C$ is a semidualizing module of $S^{-1}R$.*

Proof. Since $\mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}C) \cong \mathrm{Hom}_{S^{-1}R}(C \otimes_R S^{-1}R, C \otimes_R S^{-1}R) \cong \mathrm{Hom}_R(C, C) \otimes_R S^{-1}R = R \otimes_R S^{-1}R = S^{-1}R$ by [6, Proposition 2.2.4] and [6, Theorem 3.2.5], $S^{-1}C$ is a semidualizing module of $S^{-1}R$. □

Lemma 3.10 ([16, Lemma 3.16])**.** *Let R be a commutative ring and S a multiplicatively closed set of R and $\overline{A} \in S^{-1}R$-Mod. If $S^{-1}R$ is a projective R-module, then $\overline{A}$ is a projective R-module if and only if $\overline{A}$ is a projective $S^{-1}R$-module.*

Theorem 3.11. *Let R be a commutative ring and S a multiplicatively closed set of R, and $S^{-1}R$ a projective R-module. Then the following statements hold.*

(1) *If A is a strongly Gorenstein C-projective R-module, then $S^{-1}A$ is a strongly Gorenstein $S^{-1}C$-projective $S^{-1}R$-module.*

(2) *If $S^{-1}R$ is a finitely generated R-module, then $\overline{B}$ is a strongly Gorenstein C-projective R-module if and only if $\overline{B}$ is a strongly Gorenstein $S^{-1}C$-projective $S^{-1}R$-module for any $\overline{B} \in S^{-1}R$-Mod.*

Proof. (1) There is an exact sequence $0 \to A \to C \otimes_R P \to A \to 0$ in R-Mod with P projective. Then

$$0 \to S^{-1}A \to S^{-1}(C \otimes_R P) \to S^{-1}A \to 0$$

is exact in $S^{-1}R$-Mod. Since $S^{-1}(C \otimes_R P) \cong S^{-1}C \otimes_{S^{-1}R} S^{-1}P$ by [10, Proposition 5.17], $S^{-1}P$ is a projective $S^{-1}R$-module by Lemma 3.10. Let $\overline{Q}$ be any projective $S^{-1}R$-module. Since $\overline{Q}$ is a projective R-module by Lemma 3.10,

$$\begin{aligned}
&\mathrm{Ext}^{i\geq 1}_{S^{-1}R}(S^{-1}A, S^{-1}C \otimes_{S^{-1}R} \overline{Q})\\
&\cong \mathrm{Ext}^{i\geq 1}_{S^{-1}R}(A \otimes_R S^{-1}R, S^{-1}C \otimes_{S^{-1}R} \overline{Q})\\
&\cong \mathrm{Ext}^{i\geq 1}_{R}(A, \mathrm{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}C \otimes_{S^{-1}R} \overline{Q}))\\
&\cong \mathrm{Ext}^{i\geq 1}_{R}(A, S^{-1}C \otimes_{S^{-1}R} \overline{Q})\\
&\cong \mathrm{Ext}^{i\geq 1}_{R}(A, C \otimes_R S^{-1}R \otimes_{S^{-1}R} \overline{Q})\\
&\cong \mathrm{Ext}^{i\geq 1}_{R}(A, C \otimes_R \overline{Q}) = 0.
\end{aligned}$$

So $S^{-1}A$ is a strongly Gorenstein $S^{-1}C$-projective $S^{-1}R$-module.

(2) $\Rightarrow$) Since $\overline{B} \cong S^{-1}\overline{B}$ by [10, Proposition 5.17], the result is true by (1).

$\Leftarrow$) There is an exact sequence $0 \to \overline{B} \to S^{-1}C \otimes_{S^{-1}R} \overline{P} \to \overline{B} \to 0$ in $S^{-1}R$-module with $\overline{P}$ projective. Then $\overline{P}$ is a projective R-module by Lemma 3.10 and

$$S^{-1}C \otimes_{S^{-1}R} \overline{P} \cong C \otimes_R S^{-1}R \otimes_{S^{-1}R} \overline{P} \cong C \otimes_R \overline{P}.$$

Then

$$0 \to \overline{B} \to C \otimes_R \overline{P} \to \overline{B} \to 0$$

is exact. Let Q be any projective R-module. Then $S^{-1}Q$ is a projective $S^{-1}R$-module. Since $S^{-1}R$ is a finitely generated projective R-module, it follows that

$$\begin{aligned}\operatorname{Ext}_R^{i\geq 1}(\overline{B}, C\otimes_R Q) &\cong \operatorname{Ext}_R^{i\geq 1}(\overline{B}\otimes_{S^{-1}R} S^{-1}R, C\otimes_R Q)\\ &\cong \operatorname{Ext}_{S^{-1}R}^{i\geq 1}(\overline{B}, \operatorname{Hom}_R(S^{-1}R, C\otimes_R Q))\\ &\cong \operatorname{Ext}_{S^{-1}R}^{i\geq 1}(\overline{B}, S^{-1}\operatorname{Hom}_R(S^{-1}R, C\otimes_R Q))\\ &\cong \operatorname{Ext}_{S^{-1}R}^{i\geq 1}(\overline{B}, \operatorname{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}C\otimes_{S^{-1}R} S^{-1}Q))\\ &\cong \operatorname{Ext}_{S^{-1}R}^{i\geq 1}(\overline{B}, S^{-1}C\otimes_{S^{-1}R} S^{-1}Q) = 0\end{aligned}$$

by [10, Proposition 5.17], [12, p. 258, 9.21] and [12, Theorem 3.84]. Thus $\overline{B}$ is a strongly Gorenstein C-projective R-module. □

Proposition 3.12. *Let R be a commutative ring and S a multiplicatively closed set of R. Assume that $S^{-1}R$ is a faithfully flat R-module. If $\overline{B}$ is a finitely generated strongly Gorenstein $S^{-1}C$-projective $S^{-1}R$-module, then $\overline{B}$ is a strongly Gorenstein C-flat R-module.*

Proof. There is an exact sequence $0\to\overline{B}\to S^{-1}C\otimes_{S^{-1}R}\overline{P}\to\overline{B}\to 0$ in $S^{-1}R$-Mod with $\overline{P}$ projective. Then $\overline{P}$ is a flat R-module by [10, Theorem 5.18] and

$$S^{-1}C\otimes_{S^{-1}R}\overline{P}\cong C\otimes_R S^{-1}R\otimes_{S^{-1}R}\overline{P}\cong C\otimes_R\overline{P}.$$

Let I be any injective R-module. Then $S^{-1}I$ is an injective $S^{-1}R$-module. Since $S^{-1}R$ is a noetherian ring by [9, Theorem 85],

$$\begin{aligned}0 &= \operatorname{Hom}_{S^{-1}R}(\operatorname{Ext}_{S^{-1}R}^{i\geq 1}(\overline{B}, S^{-1}C), S^{-1}I)\\ &\cong \operatorname{Tor}_{i\geq 1}^{S^{-1}R}(\operatorname{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I), \overline{B})\\ &\cong \operatorname{Tor}_{i\geq 1}^{S^{-1}R}(S^{-1}\operatorname{Hom}_R(C, I), \overline{B})\\ &\cong S^{-1}\operatorname{Tor}_{i\geq 1}^{R}(\operatorname{Hom}_R(C, I), \overline{B})\\ &\cong S^{-1}R\otimes_R \operatorname{Tor}_{i\geq 1}^{R}(\operatorname{Hom}_R(C, I), \overline{B})\end{aligned}$$

by [6, Theorem 3.2.13] and [12, Theorem 9.49]. Since $S^{-1}R$ is a faithfully flat R-module, $\operatorname{Tor}_{i\geq 1}^{R}(\operatorname{Hom}_R(C, I), \overline{B}) = 0$, and hence $\overline{B}$ is a strongly Gorenstein C-flat R-module. □

Lemma 3.13. *Let R be a commutative ring and S a multiplicatively closed set of R. Assume that $S^{-1}R$ is a finitely generated R-module. If I is an injective R-module, then $\operatorname{Hom}_R(S^{-1}R, \operatorname{Hom}_R(C, I))$ is an $S^{-1}C$-injective $S^{-1}R$-module.*

Proof. Since $S^{-1}R$ is a finitely generated R-module,

$$\begin{aligned}\operatorname{Hom}_R(S^{-1}R, \operatorname{Hom}_R(C, I)) &\cong S^{-1}\operatorname{Hom}_R(S^{-1}R, \operatorname{Hom}_R(C, I))\\ &\cong \operatorname{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}\operatorname{Hom}_R(C, I))\\ &\cong \operatorname{Hom}_{S^{-1}R}(S^{-1}R, \operatorname{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I))\end{aligned}$$

$$\cong \mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I)$$

by [10, Proposition 5.17] and [12, Theorem 3.84]. Since I is injective in R-Mod, $S^{-1}I$ is injective in $S^{-1}R$-Mod by [2, Lemma 1.2]. Hence $\mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}_R(C, I))$ is an $S^{-1}C$-injective $S^{-1}R$-module. □

Proposition 3.14. *Let R be a commutative ring and S a multiplicatively closed set of R. If $S^{-1}R$ is a finitely generated projective R-module, then the following statements hold.*

(1) *If A is a strongly Gorenstein C-injective R-module, then* $\mathrm{Hom}_R(S^{-1}R, A)$ *is a strongly Gorenstein $S^{-1}C$-injective $S^{-1}R$-module.*

(2) *For any $B \in R$-Mod,* $\mathrm{Hom}_R(S^{-1}R, B)$ *is a strongly Gorenstein C-injective R-module if and only if* $\mathrm{Hom}_R(S^{-1}R, B)$ *is a strongly Gorenstein $S^{-1}C$-injective $S^{-1}R$-module.*

Proof. (1) There is an exact sequence $0 \to A \to \mathrm{Hom}_R(C, E) \to A \to 0$ in R-Mod with E injective. So

$$0 \to \mathrm{Hom}_R(S^{-1}R, A) \to \mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}_R(C, E)) \to \mathrm{Hom}_R(S^{-1}R, A) \to 0$$

is exact in $S^{-1}R$-Mod, $\mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}_R(C, E))$ is an $S^{-1}C$-injective $S^{-1}R$-module by Lemma 3.13. Let $\overline{E}$ be any injective $S^{-1}R$-module. Then $\overline{E}$ is an injective R-module by [2, Lemma 1.2].

$$\begin{aligned}
&\mathrm{Ext}^{i\geq 1}_{S^{-1}R}(\mathrm{Hom}_{S^{-1}R}(S^{-1}C, \overline{E}), \mathrm{Hom}_R(S^{-1}R, A)) \\
\cong\ &\mathrm{Ext}^{i\geq 1}_{R}(S^{-1}R \otimes_{S^{-1}R} \mathrm{Hom}_{S^{-1}R}(S^{-1}C, \overline{E}), A) \\
\cong\ &\mathrm{Ext}^{i\geq 1}_{R}(\mathrm{Hom}_{S^{-1}R}(S^{-1}C, \overline{E}), A) \\
\cong\ &\mathrm{Ext}^{i\geq 1}_{R}(\mathrm{Hom}_R(C, \mathrm{Hom}_{S^{-1}R}(S^{-1}R, \overline{E})), A) \\
\cong\ &\mathrm{Ext}^{i\geq 1}_{R}(\mathrm{Hom}_R(C, \overline{E}), A) = 0
\end{aligned}$$

by [12, p. 258, 9.21]. So $\mathrm{Hom}_R(S^{-1}R, A)$ is a strongly Gorenstein $S^{-1}C$-injective $S^{-1}R$-module.

(2) $\Rightarrow$) There is an exact sequence

$$0 \to \mathrm{Hom}_R(S^{-1}R, B) \to \mathrm{Hom}_R(C, E) \to \mathrm{Hom}_R(S^{-1}R, B) \to 0$$

in R-Mod with E injective. Then

$$0 \to S^{-1}\mathrm{Hom}_R(S^{-1}R, B) \to S^{-1}\mathrm{Hom}_R(C, E) \to S^{-1}\mathrm{Hom}_R(S^{-1}R, B) \to 0$$

is exact in $S^{-1}R$-Mod and $S^{-1}\mathrm{Hom}_R(S^{-1}R, B) \cong \mathrm{Hom}_R(S^{-1}R, B)$ by [10, Proposition 5.17]. Since $S^{-1}\mathrm{Hom}_R(C, E) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}E)$ by [12, Theorem 3.84],

$$0 \to \mathrm{Hom}_R(S^{-1}R, B) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}E) \to \mathrm{Hom}_R(S^{-1}R, B) \to 0$$

is exact in $S^{-1}R$-Mod and $S^{-1}E$ is an injective $S^{-1}R$-module. Let $\overline{E}$ be any injective $S^{-1}R$-module. Then $\overline{E}$ is an injective R-module by [2, Lemma 1.2]. So

$$\mathrm{Ext}^{i\geq 1}_{S^{-1}R}(\mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}\overline{E}), \mathrm{Hom}_R(S^{-1}R, B))$$

$$\begin{aligned}
&\cong \mathrm{Ext}^{i\geq 1}_{S^{-1}R}(S^{-1}\mathrm{Hom}_R(C,\overline{E}),\mathrm{Hom}_R(S^{-1}R,B))\\
&\cong \mathrm{Ext}^{i\geq 1}_{R}(\mathrm{Hom}_R(C,\overline{E}),\mathrm{Hom}_{S^{-1}R}(S^{-1}R,\mathrm{Hom}_R(S^{-1}R,B)))\\
&\cong \mathrm{Ext}^{i\geq 1}_{R}(\mathrm{Hom}_R(C,\overline{E}),\mathrm{Hom}_R(S^{-1}R,B)) = 0
\end{aligned}$$

by [12, p. 258, 9.21] and [12, Theorem 3.84]. It follows that $\mathrm{Hom}_R(S^{-1}R,B)$ is a strongly Gorenstein $S^{-1}C$-injective $S^{-1}R$-module.

$\Leftarrow$) There is an exact sequence

$$0 \to \mathrm{Hom}_R(S^{-1}R,B) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}C,\overline{E}) \to \mathrm{Hom}_R(S^{-1}R,B) \to 0$$

in $S^{-1}R$-Mod and $\overline{E}$ is an injective $S^{-1}R$-module. Then $\overline{E}$ is an injective R-module by [2, Lemma 1.2]. Since

$$\begin{aligned}
\mathrm{Hom}_{S^{-1}R}(S^{-1}C,\overline{E}) &\cong \mathrm{Hom}_{S^{-1}R}(S^{-1}R\otimes_R C,\overline{E})\\
&\cong \mathrm{Hom}_R(C,\mathrm{Hom}_{S^{-1}R}(S^{-1}R,\overline{E}))\\
&\cong \mathrm{Hom}_R(C,\overline{E}),
\end{aligned}$$

$\mathrm{Hom}_{S^{-1}R}(S^{-1}C,\overline{E})$ is an C-injective R-module. Let E be any injective R-module. Then $S^{-1}E$ is an injective $S^{-1}R$-module.

$$\begin{aligned}
&\mathrm{Ext}^{i\geq 1}_{R}(\mathrm{Hom}_R(C,E),\mathrm{Hom}_R(S^{-1}R,B))\\
&\cong \mathrm{Ext}^{i\geq 1}_{R}(\mathrm{Hom}_R(C,E),\mathrm{Hom}_{S^{-1}R}(S^{-1}R,\mathrm{Hom}_R(S^{-1}R,B)))\\
&\cong \mathrm{Ext}^{i\geq 1}_{S^{-1}R}(S^{-1}R\otimes_R \mathrm{Hom}_R(C,E),\mathrm{Hom}_R(S^{-1}R,B))\\
&\cong \mathrm{Ext}^{i\geq 1}_{S^{-1}R}(\mathrm{Hom}_{S^{-1}R}(S^{-1}C,S^{-1}E),\mathrm{Hom}_R(S^{-1}R,B))\\
&= 0
\end{aligned}$$

by [12, p. 258, 9.21] and [12, Theorem 9.50]. Hence $\mathrm{Hom}_R(S^{-1}R,B)$ is a strongly Gorenstein C-injective R-module. □

Lemma 3.15. *Let $\overline{F}$ be an $S^{-1}R$-module. If $\overline{F}$ is a flat $S^{-1}R$-module, then $\overline{F}$ is a flat R-module.*

Proof. There is an exact sequence $0 \to M \to N \to Q \to 0$ in R-Mod. Then

$$0 \to S^{-1}M \to S^{-1}N \to S^{-1}Q \to 0$$

is exact in $S^{-1}R$-Mod. Since $\overline{F}$ is a flat $S^{-1}R$-module, the sequence

$$0 \to S^{-1}M \otimes_{S^{-1}R} \overline{F} \to S^{-1}N \otimes_{S^{-1}R} \overline{F} \to S^{-1}Q \otimes_{S^{-1}R} \overline{F} \to 0$$

is exact. So

$$0 \to M \otimes_R S^{-1}\overline{F} \to N \otimes_R S^{-1}\overline{F} \to Q \otimes_R S^{-1}\overline{F} \to 0$$

is exact by [10, Proposition 5.17]. Since $S^{-1}\overline{F} \cong \overline{F}$, the sequence

$$0 \to M \otimes_R \overline{F} \to N \otimes_R \overline{F} \to Q \otimes_R \overline{F} \to 0$$

is exact in R-Mod. Hence $\overline{F}$ is a flat R-module. □

Proposition 3.16. *Let R be a commutative ring and S a multiplicatively closed set of R.*

(1) *If A is a strongly Gorenstein C-flat R-module, then $S^{-1}A$ is a strongly Gorenstein C-flat R-module.*

(2) *If A is a strongly Gorenstein C-flat R-module, then $S^{-1}A$ is a strongly Gorenstein $S^{-1}C$-flat $S^{-1}R$-module.*

(3) *Let $\overline{B}$ be an $S^{-1}R$-module. Then $\overline{B}$ is a strongly Gorenstein C-flat R-module if and only if $\overline{B}$ is a Gorenstein $S^{-1}C$-flat $S^{-1}R$-module.*

Proof. (1) There is a complete C-flat R-module

$$\mathfrak{F} = \cdots \xrightarrow{f} C\otimes_R F \xrightarrow{f} C\otimes_R F \xrightarrow{f} C\otimes_R F \xrightarrow{f} \cdots$$

in R-Mod with $A \cong \mathrm{Ker} f$. Then

$$\mathfrak{S}^{-1}\mathfrak{F} = \cdots \xrightarrow{f} S^{-1}(C\otimes_R F) \xrightarrow{S^{-1}f} S^{-1}(C\otimes_R F) \xrightarrow{S^{-1}f} S^{-1}(C\otimes_R F) \xrightarrow{S^{-1}f} \cdots$$

is exact in $S^{-1}R$-Mod with $S^{-1}A \cong \mathrm{Ker} S^{-1}f$. Since $S^{-1}(C\otimes_R F) \cong C\otimes_R S^{-1}F$ by [10, Proposition 5.17],

$$\mathfrak{C}\otimes\mathfrak{S}^{-1}\mathfrak{F} = \cdots \xrightarrow{f} C\otimes_R S^{-1}F \xrightarrow{S^{-1}f} C\otimes_R S^{-1}F \xrightarrow{S^{-1}f} C\otimes_R S^{-1}F \xrightarrow{S^{-1}f} \cdots$$

is exact in R-Mod. Since $S^{-1}F$ is a flat $S^{-1}R$-module, $S^{-1}F$ is a flat R-module by Lemma 3.15. Let I be any injective R-module. Then $\mathrm{Hom}_R(C,I)\otimes_R \mathfrak{F}$ is exact, and so $S^{-1}(\mathrm{Hom}_R(C,I)\otimes_R \mathfrak{F})$ is exact.

$$\begin{aligned} S^{-1}(\mathrm{Hom}_R(C,I)\otimes_R \mathfrak{F}) &\cong \mathrm{Hom}_R(C,E)\otimes_R S^{-1}\mathfrak{F} \\ &\cong \mathrm{Hom}_R(C,E)\otimes_R (\mathfrak{C}\otimes\mathfrak{S}^{-1}\mathfrak{F}) \end{aligned}$$

by [10, Proposition 5.17], and hence $\mathrm{Hom}_R(C,E)\otimes_R (\mathfrak{C}\otimes\mathfrak{S}^{-1}\mathfrak{F})$ is exact. It follows that $S^{-1}A$ is a strongly Gorenstein C-flat R-module.

(2) There is an exact sequence $0\to A\to C\otimes_R F\to A\to 0$ in R-Mod with F flat. Then

$$0\to S^{-1}A\to S^{-1}(C\otimes_R F)\to S^{-1}A\to 0$$

is exact in $S^{-1}R$-Mod. $S^{-1}(C\otimes_R F)\cong S^{-1}C\otimes_{S^{-1}R} S^{-1}F$ by [10, Proposition 5.17], and so

$$0\to S^{-1}A\to S^{-1}C\otimes_{S^{-1}R} S^{-1}F\to S^{-1}A\to 0$$

is exact in $S^{-1}R$-Mod with $S^{-1}F$ flat in $S^{-1}R$-Mod. Let $\overline{E}$ be any injective $S^{-1}R$-module. Then $\overline{E}$ is an injective R-module by [2, Lemma 1.2]. So

$$\begin{aligned} \mathrm{Tor}_{i\geq 1}^{S^{-1}R}(\mathrm{Hom}_{S^{-1}R}(S^{-1}C,E),S^{-1}A) &\cong \mathrm{Tor}_{i\geq 1}^{S^{-1}R}(S^{-1}\mathrm{Hom}_R(C,E),S^{-1}A) \\ &\cong S^{-1}\mathrm{Tor}_{i\geq 1}^{R}(\mathrm{Hom}_R(C,E),A) \\ &= 0 \end{aligned}$$

by [12, Theorem 9.49]. Hence $S^{-1}A$ is a strongly Gorenstein $S^{-1}C$-flat $S^{-1}R$-module.

(3) $\Rightarrow$) It is clear by (2).

$\Leftarrow$) There is an exact sequence of $S^{-1}R$-modules

$$\mathfrak{F} = \cdots \xrightarrow{f} S^{-1}C \otimes_{S^{-1}R} \overline{F} \xrightarrow{f} S^{-1}C \otimes_{S^{-1}R} \overline{F} \xrightarrow{f} S^{-1}C \otimes_{S^{-1}R} \overline{F} \xrightarrow{f} \cdots ,$$

where $\overline{F}$ is a flat $S^{-1}R$-module with $\overline{B} \cong \mathrm{Ker} f$. Since

$$S^{-1}C \otimes_{S^{-1}R} \overline{F} \cong C \otimes_R S^{-1}\overline{F} \cong C \otimes_R \overline{F}$$

by [10, Proposition 5.17]. Then

$$\mathfrak{L}\mathfrak{F} = \cdots \xrightarrow{f} C \otimes_R \overline{F} \xrightarrow{f} C \otimes_R \overline{F} \xrightarrow{f} C \otimes_R \overline{F} \xrightarrow{f} \cdots$$

is exact. Since $\overline{F}$ is a flat $S^{-1}R$-module, $\overline{F}$ is a flat R-module by Lemma 3.15. Let E be any injective R-module. Then $S^{-1}E$ is an injective $S^{-1}R$-module by [2, Lemma 1.2].

$$\begin{aligned}
&\mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}E) \otimes_{S^{-1}R} (S^{-1}C \otimes_{S^{-1}R} \overline{F}) \\
&\cong S^{-1}\mathrm{Hom}_R(C, E) \otimes_{S^{-1}R} (S^{-1}C \otimes_{S^{-1}R} \overline{F}) \\
&\cong \mathrm{Hom}_R(C, E) \otimes_R S^{-1}(C \otimes_R \overline{F}) \\
&\cong \mathrm{Hom}_R(C, E) \otimes_R (C \otimes_R S^{-1}\overline{F}) \\
&\cong \mathrm{Hom}_R(C, E) \otimes_R (C \otimes_R \overline{F})
\end{aligned}$$

by [10, Lemma 5.17]. Since $\mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}E) \otimes_{S^{-1}R} \mathfrak{F}$ is exact,

$$\mathrm{Hom}_R(C, E) \otimes_R \mathfrak{L}\mathfrak{F}$$

is exact. So $\overline{B}$ is a strongly Gorenstein C-flat R-module. □

If R is a ring, then $R[x]$ is the polynomial ring. If M is a left R-module, write $M[x] = R[x] \otimes M$. Since $R[x]$ is a free R-module and since tensor product commutes with sums, we may regard the elements of $M[x]$ as 'Vectors' $(x^i \otimes_R m_i), i \geq 0, M_i \in M$ with almost all $m_i = 0$. $M[[x^{-1}]]$ is the $R[x]$-module such that $x(m_0 + m_1x^{-1} + \cdots) = m_1 + m_2x^{-1} + \cdots$ and $r(m_0 + m_1^{-1} + \cdots) = rm_0 + rm_1^{-1} + \cdots$, where $r \in R$.

Lemma 3.17. *If C is a semidualizing R-module, then $C[x]$ is a semidualizing R-module of $R[x]$.*

Proof. This follows from [6, Theorem 3.2.15]. □

Proposition 3.18. *Let m be an R-module. Then the following statements hold.*

(1) *If M is a strongly Gorenstein C-projective R-module, then $M[x]$ is a strongly Gorenstein $C[x]$-projective $R[x]$-module.*

(2) *If $M[x]$ is a strongly Gorenstein $C[x]$-projective $R[x]$-module, then $M[x]$ is a strongly Gorenstein C-projective R-module.*

Proof. (1) There is an exact sequence $0 \to M \to C \otimes_R P \to M \to 0$ in R-Mod with P projective and $\mathrm{Ext}_R^{i\geq 1}(M, Q) = 0$ for any C-projective R-module Q.

$$0 \to M \otimes_R R[x] \to (C \otimes_R P) \otimes_R R[x] \to M \otimes_R R[x] \to 0$$

is exact. Since $(C \otimes_R P) \otimes_R R[x] \cong (C \otimes_R R[x]) \otimes_{R[x]} (R[x] \otimes_R P) \cong C[x] \otimes_{R[x]} (P[x])$. $P[x]$ is a projective $R[x]$-module by [10, Proposition 5.11], then

$$0 \to M[x] \to C[x] \otimes_{R[x]} P[x] \to M[x] \to 0$$

is exact. Let $\overline{Q}$ be any projective $R[x]$-module, $\overline{Q}[x] \cong R[x] \otimes_R \overline{Q} \cong \overline{Q}^{(N)}$, then $\overline{Q}^{(N)}$ is a projective $R[x]$-module, and so $\overline{Q}$ is a projective R-module by [10, Proposition 5.11]. Thus

$$\begin{aligned}\mathrm{Ext}_{R[x]}^{i\geq 1}(M[x], \overline{Q}) \cong \mathrm{Ext}_{R[x]}^{i\geq 1}(R[x] \otimes_R M, \overline{Q}) &\cong \mathrm{Ext}_R^{i\geq 1}(M, \mathrm{Hom}_{R[x]}(R[x], \overline{Q})) \\ &\cong \mathrm{Ext}_R^{i\geq 1}(M, \overline{Q}) = 0\end{aligned}$$

by [12, p. 258, 9.21], and hence $M[x]$ is a strongly Gorenstein $C[x]$-projective $R[x]$-module.

(2) There is an exact sequence $0 \to M[x] \to C[x] \otimes_{R[x]} \overline{P} \to M[x] \to 0$ in $R[x]$-Mod with $\overline{P}$ projective. Since $C[x] \otimes_{R[x]} \overline{P} \cong C \otimes_R R[x] \otimes_{R[x]} \overline{P} \cong C \otimes_R (R[x] \otimes_{R[x]} \overline{P}) \cong C \otimes_R \overline{P}$ with $\overline{P}$ projective in R-Mod by [10, Proposition 5.11], we have the following exact sequence

$$0 \to M[x] \to C \otimes_R \overline{P} \to M[x] \to 0.$$

Let P be any projective R-module. Then $P[x]$ is a projective $R[x]$-module by [10, Proposition 5.11].

$$0 = \mathrm{Ext}_{R[x]}^{i\geq 1}(M[x], C[x] \otimes_{R[x]} P[x]) \cong \mathrm{Ext}_{R[x]}^{i\geq 1}(M[x], C \otimes_R P[x])$$

by [12, p. 258, 9.20], Since P is isomorphic to a summand of $P[x]$, it follows that $\mathrm{Ext}_R^{i\geq 1}(M, C \otimes_R P) = 0$.

$$\begin{aligned}\mathrm{Ext}_R^{i\geq 1}(M[x], C \otimes_R P) &\cong \mathrm{Ext}_R^{i\geq 1}(M \otimes_R R[x], C \otimes_R P) \\ &\cong \mathrm{Hom}_R(R[x], \mathrm{Ext}_R^{i\geq 1}(M, C \otimes_R P))\end{aligned}$$

by [12, p. 258, 9.20]. Since $\mathrm{Ext}_R^{i\geq 1}(M, C \otimes_R P) = 0$, $\mathrm{Ext}_R^{i\geq 1}(M[x], C \otimes_R P) = 0$. Hence $M[x]$ is a strongly Gorenstein C-projective R-module. □

Proposition 3.19. *Let m be an R-module. Then the following statements hold.*

(1) *If M is a strongly Gorenstein C-injective R-module, then $M[[x^{-1}]]$ is a strongly Gorenstein $C[x]$-injective $R[x]$-module.*

(2) *If $M[[x^{-1}]]$ is a strongly Gorenstein $C[x]$-injective $R[x]$-module, then $M[[x^{-1}]]$ is a strongly Gorenstein C-injective R-module.*

Proof. (1) There is an exact sequence $0 \to M \to \mathrm{Hom}_R(C, E) \to M \to 0$ in R-Mod with E injective and $\mathrm{Ext}_R^{i\geq 1}(I, M) = 0$ for any C-injective R-module I. So

$$0 \to \mathrm{Hom}_R(R[x], M) \to \mathrm{Hom}_R(R[x], \mathrm{Hom}_R(C, E)) \to \mathrm{Hom}_R(R[x], M) \to 0$$

is exact in $R[x]$-module. Since

$$\begin{aligned}\mathrm{Hom}_R(R[x], \mathrm{Hom}_R(C, E)) &\cong \mathrm{Hom}_R(C \otimes_R R[x], E)\\ &\cong \mathrm{Hom}_R(C \otimes_R R[x] \otimes_{R[x]} R[x], E)\\ &\cong \mathrm{Hom}_R(C[x] \otimes_{R[x]} R[x], E)\\ &\cong \mathrm{Hom}_{R[x]}(C[x], \mathrm{Hom}_R(R[x], E)),\end{aligned}$$

the sequence

$$0 \to \mathrm{Hom}_R(R[x], M) \to \mathrm{Hom}_{R[x]}(C[x], \mathrm{Hom}_R(R[x], E)) \to \mathrm{Hom}_R(R[x], M) \to 0$$

is exact with $\mathrm{Hom}_R(R[x], E)$ injective in $R[x]$-Mod by [11, Lemma 1.2]. Let $\overline{E}$ be any injective $R[x]$-module. Then $\overline{E}$ is an injective R-module.

$$\begin{aligned}&\mathrm{Ext}_{R[x]}^{i\geq 1}(\mathrm{Hom}_{R[x]}(C[x], \overline{E}), \mathrm{Hom}_R(R[x], M))\\ \cong\ &\mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_{R[x]}(C[x], \overline{E}) \otimes_{R[x]} R[x], M)\\ \cong\ &\mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_{R[x]}(C[x], \overline{E}), M)\\ \cong\ &\mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_R(C, \overline{E}), M) = 0\end{aligned}$$

by [12, p. 258, 9.21] and [6, Lemma 3.2.4]. Since $M[[x^{-1}]] \cong \mathrm{Hom}_R(R[x], M)$ by [11, Lemma 1.2]. Thus $M[[x^{-1}]]$ is a strongly Gorenstein $C[x]$-injective $R[x]$-module.

(2) There is an exact sequence $0 \to M[[x^{-1}]] \to \mathrm{Hom}_{R[x]}(C[x], \overline{E}) \to M[[x^{-1}]] \to 0$ in $R[x]$-Mod with $\overline{E}$ injective. Since

$$\begin{aligned}\mathrm{Hom}_{R[x]}(C[x], \overline{E}) \cong \mathrm{Hom}_{R[x]}(C \otimes_R R[x], \overline{E}) &\cong \mathrm{Hom}_R(C, \mathrm{Hom}_{R[x]}(R[x], \overline{E}))\\ &\cong \mathrm{Hom}_R(C, \overline{E}),\end{aligned}$$

the sequence

$$0 \to M[[x^{-1}]] \to \mathrm{Hom}_R(C, \overline{E}) \to M[[x^{-1}]] \to 0$$

is exact. Let E be any injective R-module. Then $\mathrm{Hom}_R(R[x], E)$ is an injective $R[x]$-module.

$$\begin{aligned}&\mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_R(C, \mathrm{Hom}_R(R[x], E)), M[[x^{-1}]])\\ \cong\ &\mathrm{Ext}_{R[x]}^{i\geq 1}(\mathrm{Hom}_{R[x]}(C[x], \mathrm{Hom}_R(R[x], E)), M[[x^{-1}]]) = 0\end{aligned}$$

by [12, p. 258, 9.21]. Since E is isomorphic to a summand of $\mathrm{Hom}_R(R[x], E)$,

$$\mathrm{Ext}_R^{i\geq 1}(\mathrm{Hom}_R(C, E), M[[x^{-1}]]) = 0.$$

Thus $M[[x^{-1}]]$ is a strongly Gorenstein C-injective R-module. □

Proposition 3.20. *Let m be an R-module. Then the following statements hold.*

(1) *If M is a strongly Gorenstein C-flat R-module, then $M[x]$ is a strongly Gorenstein $C[x]$-flat $R[x]$-module.*

(2) *If $M[x]$ is a strongly Gorenstein $C[x]$-flat $R[x]$-module, then $M[x]$ is a strongly Gorenstein C-flat R-module.*

Proof. (1) There is an exact sequence $0 \to M \to C \otimes_R F \to M \to 0$ in R-Mod with F flat. Then

$$0 \to M \otimes_R R[x] \to C \otimes_R F \otimes_R R[x]) \to M \otimes_R R[x] \to 0$$

is exact with $(C \otimes_R F) \otimes_{R[x]} R[x] \cong C \otimes_R R[x] \otimes_{R[x]} (F \otimes_R R[x]) \cong C[x] \otimes_{R[x]} F[x]$. $F[x]$ is a flat $R[x]$-module by [15, Theorem, 3.8.21]. Let $\overline{I}$ be any injective $R[x]$-module. Then $\overline{I}$ is an injective R-module.

$$\begin{aligned}\mathrm{Tor}_{i\geq 1}^{R[x]}(\mathrm{Hom}_{R[x]}(C[x], \overline{I}), M[x])^+ &\cong \mathrm{Tor}_{i\geq 1}^{R[x]}(\mathrm{Hom}_{R[x]}(C \otimes_R R[x], \overline{I}), M[x])^+ \\ &\cong \mathrm{Tor}_{i\geq 1}^{R[x]}(\mathrm{Hom}_R(C, \overline{I}), M[x])^+ \\ &\cong \mathrm{Ext}_{R[x]}^{i\geq 1}(M[x], \mathrm{Hom}_R(C, \overline{I})^+) \\ &\cong \mathrm{Ext}_{R[x]}^{i\geq 1}(M \otimes_R R[x], \mathrm{Hom}_R(C, \overline{I})^+) \\ &\cong \mathrm{Ext}_R^{i\geq 1}(M, \mathrm{Hom}_R(C, \overline{I})^+) \\ &\cong \mathrm{Tor}_{i\geq 1}^R(\mathrm{Hom}_R(C, \overline{I}), M)^+ = 0\end{aligned}$$

by [12, p. 258, 9.21] and [6, Lemma 3.2.1]. Thus $M[x]$ is a strongly Gorenstein $C[x]$-flat $R[x]$-module.

(2) There is an exact sequence $0 \to M[x] \to C[x] \otimes_{R[x]} \overline{F} \to M[x] \to 0$ in $R[x]$-Mod with $\overline{F}$ flat. $C[x] \otimes_{R[x]} \overline{F} \cong C \otimes_R (R[x] \otimes_{R[x]} \overline{F} \cong C \otimes_R (R[x] \otimes_{R[x]} \overline{F}) \cong C \otimes_R \overline{F}$. Then

$$0 \to M[x] \to C \otimes_R \overline{F} \to M[x] \to 0$$

is exact. Let E be any injective R-module. Then $\mathrm{Hom}_R(R[x], E)$ is an injective $R[x]$-module.

$$\begin{aligned}&\mathrm{Tor}_{i\geq 1}^{R[x]}(M[x], \mathrm{Hom}_{R[x]}(C[x], \mathrm{Hom}_R(R[x], E)))^+ \\ \cong\ &\mathrm{Tor}_{i\geq 1}^{R[x]}(M[x], \mathrm{Hom}_{R[x]}(C \otimes_R R[x], \mathrm{Hom}_R(R[x], E)))^+ \\ \cong\ &\mathrm{Tor}_{i\geq 1}^{R[x]}(M[x], \mathrm{Hom}_R(C, \mathrm{Hom}_{R[x]}(R[x], \mathrm{Hom}_R(R[x], E))))^+ \\ \cong\ &\mathrm{Tor}_{i\geq 1}^{R[x]}(M[x], \mathrm{Hom}_R(C, \mathrm{Hom}_R(R[x], E)))^+ \\ \cong\ &\mathrm{Ext}_R^{i\geq 1}(M[x], \mathrm{Hom}_R(C, \mathrm{Hom}_R(R[x], E))^+) \\ \cong\ &\mathrm{Tor}_{i\geq 1}^R(M[x], \mathrm{Hom}_R(C, \mathrm{Hom}_R(R[x], E)))^+\end{aligned}$$

by [12, p. 258, 9.21] and [6, Lemma 3.2.1]. Thus $M[x]$ is a strongly Gorenstein $C[x]$-flat $R[x]$-module. Since E is isomorphic to a summand of $\mathrm{Hom}_R(R[x], E)$,

$$\mathrm{Tor}^R_{i\geq 1}(M[x], \mathrm{Hom}_R(C, E)) = 0.$$

Thus $M[x]$ is a strongly Gorenstein C-flat R-module. □

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