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ON THE EXISTENCE OF SOLUTIONS OF FERMAT-TYPE DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. We investigate the non-existence of finite order transcendental entire solutions of Fermat-type differential-difference equations

$$\left[f(z)f'(z)\right]^n + P^2(z)f^m(z+\eta) = Q(z)$$

and

$$\left[f(z)f'(z)\right]^n + P(z)[\Delta_\eta f(z)]^m = Q(z),$$

where $P(z)$ and $Q(z)$ are non-zero polynomials, m and n are positive integers, and $\eta \in \mathbb{C} \setminus \{0\}$. In addition, we discuss transcendental entire solutions of finite order of the following Fermat-type differential-difference equation

$$P^2(z)\left[f^{(k)}(z)\right]^2 + [\alpha f(z+\eta) - \beta f(z)]^2 = e^{r(z)},$$

where $P(z) \not\equiv 0$ is a polynomial, $r(z)$ is a non-constant polynomial, $\alpha \neq 0$ and β are constants, k is a positive integer, and $\eta \in \mathbb{C} \setminus \{0\}$. Our results generalize some previous results.

1. Introduction

Let $\mathbb{C}$ denote the complex plane and suppose that $f(z)$ is a meromorphic function in $\mathbb{C}$. Here and in the sequel it is assumed that the reader is familiar with the Nevanlinna theory and standard notations (see [7,8]) such as $T(r,f)$, $m(r,f)$, $N(r,f)$ and $S(r,f)$. If a meromorphic function $a(z)$ ($\not\equiv \infty$) satisfies

$$T(r,a) = o(T(r,f)) = S(r,f), \quad r \to \infty,$$

outside possibly an exceptional set of finite logarithmic measure, then $a(z)$ is called a small function of $f(z)$. And we define the order ρ of growth of $f(z)$ by

$$\rho := \rho(f(z)) = \limsup_{r\to\infty} \frac{\log T(r,f(z))}{\log r}.$$

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For a non-zero complex constant η, we will define its shift by $f_\eta(z) = f(z+\eta)$ and its difference operators by

$$\Delta_\eta f(z) = f(z+\eta) - f(z), \quad \Delta_\eta^n f(z) = \Delta_\eta^{n-1}(\Delta_\eta f(z)),\ n \in \mathbb{N},\ n \geq 2.$$

In 1966, Gross [3,4] studied the Fermat-type functional equation $f^n(z) + g^n(z) = 1$, and many famous results have been obtained since then (see [6,9, 11,13,14,16,18]).

In 1970, Yang [14] studied the Fermat-type functional equation

$$a(z)f^n(z) + b(z)g^m(z) = 1, \tag{1.1}$$

where $a(z)$ and $b(z)$ are small functions with respect to $f(z)$, and obtained the following result.

Theorem A (see [14]). *Let m, n be positive integers satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Then there are no non-constant entire solutions $f(z)$ and $g(z)$ that satisfy* (1.1).

In recent years, it is an interesting and quite difficult question to study the solvability and existence of entire or meromorphic solutions of non-linear difference (or differential, or differential-difference) equations in the complex domain. Many authors have investigated this question since Halburd and Korhonen [5], Chiang and Feng [2] successfully proved the lemma analogue of the logarithmic derivative, and many remarkable results have been rapidly obtained (see [1,6,9–13,15,18]). For instance, in 2012, Liu et al. [9] obtained the following results.

Theorem B (see [9]). *If $n \neq m$, then the equation*

$$f'^n(z) + f^m(z+\eta) = 1 \tag{1.2}$$

has no finite order transcendental entire solutions, where m and n are positive integers, $\eta \in \mathbb{C} \setminus \{0\}$.

Theorem C (see [9]). *If $m \neq n > 1$, then the equation*

$$f'^n(z) + [\Delta_\eta f(z)]^m = 1 \tag{1.3}$$

has no finite order transcendental entire solutions, where m and n are positive integers, $\eta \in \mathbb{C} \setminus \{0\}$.

Firstly, we consider the non-existence of finite order transcendental entire solutions of Fermat-type differential-difference equations

$$[f(z)f'(z)]^n + P^2(z)f^m(z+\eta) = Q(z), \tag{1.4}$$

$$[f(z)f'(z)]^n + P(z)[\Delta_\eta f(z)]^m = Q(z), \tag{1.5}$$

and prove the following results.

Theorem 1.1. *If $m = n$, then the equation* (1.4) *has no finite order transcendental entire solutions, where $P(z)$ and $Q(z)$ are non-zero polynomials, m and n are positive integers, and $\eta \in \mathbb{C} \setminus \{0\}$.*

Remark 1.1. We shall give an example below to show that (1.4) may have a finite order transcendental entire solution f for $n = 1$, $m = 2$.

Example 1.1. If $n = 1$, $m = 2$, $P(z) \equiv 1$, and $Q(z) \equiv 4$ in (1.4), then

$$f(z)f'(z) + f^2(z+\eta) = 4$$

has a solution $f(z) = 2e^{i(z+b)} + e^{-i(z+b)} = 2\sqrt{2}\cos z$, where $e^{ib} = \frac{\sqrt{2}}{2}$, $\eta = \frac{3\pi}{4} + k\pi$, k is an integer, and b is a constant.

Theorem 1.2. *If $m \neq n$, $n > 2$, then the equation* (1.5) *has no finite order transcendental entire solutions, where $P(z)$ and $Q(z)$ are non-zero polynomials, m and n are positive integers, and $\eta \in \mathbb{C} \setminus \{0\}$.*

Remark 1.2. We shall give an example below to show that (1.5) may have a finite order transcendental entire solution f for $n = 1$, $m = 2$.

Example 1.2. If $n = 1$, $m = 2$, $P(z) \equiv 1$, and $Q(z) \equiv -4$ in (1.5), then

$$f(z)f'(z) + [\Delta_\eta f(z)]^2 = -4$$

has a solution $f(z) = e^{2iz} - e^{-2iz} = 2i\sin 2z$, where $\eta = \frac{\pi}{4} + k\pi$, k is an integer.

From the beginning of Theorem A, the case of $m > 2$, $n > 2$ in (1.3) is obvious. If $m = n = 2$ in (1.3), then a result can be stated as follows.

Theorem D (see [9]). *The finite order transcendental entire solutions of the differential-difference equation*

$$[f'(z)]^2 + [\Delta_\eta f(z)]^2 = 1 \tag{1.6}$$

must satisfy $f(z) = \frac{1}{2}\sin(2z + Bi)$, where $\eta = n\pi + \frac{\pi}{2}$, n is an integer, and B is a constant.

In 2019, Zeng et al. [18] generalized the complex differential-difference equation (1.6) in Theorem D as

$$[f^{(k)}(z)]^2 + [\alpha f(z+\eta) - \beta f(z)]^2 = 1, \tag{1.7}$$

where $\alpha \neq 0$ and β are constants, k is a positive integer, and $\eta \in \mathbb{C} \setminus \{0\}$. Then they obtained the following result.

Theorem E (see [18]). *If $f(z)$ is a finite order entire solution of* (1.7)*, then there exist two cases*:

(I) *if $f(z)$ is a transcendental solution of* (1.7)*, then either*

(I.i) *when k is an odd number, $f(z)$ must satisfy the form that*

$$f(z) = \frac{e^{az+b} - e^{-az-b}}{2a^k} + d,$$

where a, b, d are constants, n is an integer. In this case, (I.i.i) *if $\alpha = \beta$, then $a^k = -2\alpha i$, $\eta = \frac{(2n+1)\pi}{a} i$;* (I.i.ii) *if $\alpha = -\beta$, then $a^k = 2\alpha i$, $\eta = \frac{2n\pi}{a} i$, $d = 0$;* (I.i.iii) *if $\alpha \neq \pm\beta$, then $a^k = -(\alpha+\beta)i$, $\eta = \frac{(2n+1)\pi}{a} i$ or $a^k = (\alpha - \beta)i$, $\eta = \frac{2n\pi}{a} i$, $d = 0$; or*

(I.ii) *when k is an even number and $\alpha = \pm\beta$, the equation* (1.7) *does not have transcendental entire solutions*; *when k is an even number and $\alpha \neq \pm\beta$, $f(z)$ must satisfy the form that*

$$f(z) = \frac{e^{az+b} + e^{-az-b}}{2a^k},$$

where a and b are constants satisfying $a^k = \sqrt{\alpha^2 - \beta^2}$; $a^k = -\sqrt{\alpha^2 - \beta^2}$ and $\eta = \frac{\ln(\frac{i\beta + a^k}{i\alpha}) + 2n\pi i}{a}$, where n is an integer.

(II) *if $f(z)$ is a polynomial solution of* (1.7)*, then either*

(II.i) *if $\alpha = \beta$, then $f(z) = Bz + C$, where B and C are constants satisfying $B^2(1 + \alpha^2\eta^2) = 1$ for $k = 1$; $(\alpha B\eta)^2 = 1$ for $k \geq 2$; or*

(II.ii) *if $\alpha \neq \beta$, then $f(z) \equiv \pm\frac{1}{\alpha - \beta}$.*

In the following, we study the finite order transcendental entire solutions of the Fermat-type differential-difference equation

$$P^2(z)\left[f^{(k)}(z)\right]^2 + [\alpha f(z+\eta) - \beta f(z)]^2 = e^{r(z)}, \tag{1.8}$$

where $P(z) \not\equiv 0$ is a polynomial, $r(z)$ is a non-constant polynomial, $\alpha \neq 0$ and β are constants, k is a positive integer, and $\eta \in \mathbb{C} \setminus \{0\}$. Then we will prove the following result.

Theorem 1.3. *If $f(z)$ is a finite order transcendental entire solution of* (1.8)*, then there exist three cases*:

(I) *$P(z)$ reduces to a constant, $r(z)$ is a polynomial with degree* 1*, and*

$$f(z) = \frac{e^{az+a_0}}{2Aa^k} + \frac{e^{bz+b_0}}{2Ab^k} + d, \quad a \neq \pm b,$$

where $A \neq 0$, $a \neq 0$, $b \neq 0$, d, a_0, b_0 are constants. In this case, (I.i) *if $\alpha = \beta$, then $a^k = \frac{i\alpha(e^{a\eta}-1)}{A}$, $b^k = \frac{i\alpha(1-e^{b\eta})}{A}$;* (I.ii) *if $\alpha = -\beta$, then $a^k = \frac{i\alpha(e^{a\eta}+1)}{A}$, $b^k = \frac{-i\alpha(1+e^{b\eta})}{A}$, $d = 0$;* (I.iii) *if $\alpha \neq \pm\beta$, then $a^k = \frac{i\alpha e^{a\eta} - i\beta}{A}$, $b^k = \frac{i\beta - i\alpha e^{b\eta}}{A}$, $d = 0$.*

(II) *$P(z)$ reduces to a constant, $r(z)$ is a polynomial with degree* 1*, and*

$$f(z) = \frac{e^{az+a_0}}{2Aa^k} + \frac{e^{az+b_0}}{2Aa^k} + d, \quad b_0 \neq a_0 + 2n\pi i,$$

where $A \neq 0$, $a \neq 0$, d, a_0, b_0 are constants, n is an integer. In this case, (II.i) *if $\alpha = \beta$, then $a^k = \frac{i\alpha(e^{a\eta}-1)(1+e^{b_0-a_0})}{A(1-e^{b_0-a_0})}$;* (II.ii) *if $\alpha = -\beta$, then $a^k = \frac{i\alpha(e^{a\eta}+1)(1+e^{b_0-a_0})}{A(1-e^{b_0-a_0})}$, $d = 0$;* (II.iii) *if $\alpha \neq \pm\beta$, then $a^k = \frac{(i\alpha e^{a\eta} - i\beta)(1+e^{b_0-a_0})}{A(1-e^{b_0-a_0})}$, $d = 0$.*

(III) *$P(z)$ reduces to a constant, $r(z)$ is a polynomial with degree* 1*, and*

$$f(z) = \frac{e^{az+a_0}}{Aa^k} + d, \quad b_0 = a_0 + 2n\pi i,$$

where $A \neq 0$, $a \neq 0$, d, a_0, b_0 are constants, n is an integer. In this case, (III.i) *if $e^{a\eta} = \frac{\beta}{\alpha}$, $\alpha = \beta$, then $e^{a\eta} = 1$;* (III.ii) *if $e^{a\eta} = \frac{\beta}{\alpha}$, $\alpha \neq \beta$, then $e^{a\eta} \neq 1$, $d = 0$.*

Remark 1.3. We give the following Examples 1.3-1.5 to show that Case (I) above does exist.

Example 1.3. If $k = 1$, $\alpha = \beta = \frac{5-\sqrt{7}i}{16i}$, $P(z) \equiv 1$, and $r(z) = -\frac{z}{2}$ in (1.8), then

$$f'^2(z) + \left[\frac{5-\sqrt{7}i}{16i} f(z+\eta) - \frac{5-\sqrt{7}i}{16i} f(z)\right]^2 = e^{-\frac{z}{2}}$$

has a solution $f(z) = -\frac{1}{2}e^{-z} + e^{\frac{z}{2}} + 1$, where $A = 1$, $a = -1$, $b = \frac{1}{2}$, $e^{-\eta} = \frac{-11-\sqrt{7}i}{5-\sqrt{7}i}$, and $e^{\frac{\eta}{2}} = \frac{-3-\sqrt{7}i}{5-\sqrt{7}i}$. Clearly, $a \neq \pm b$, $a^k = -1 = \frac{i\alpha(e^{a\eta}-1)}{A}$, and $b^k = \frac{1}{2} = \frac{i\alpha(1-e^{b\eta})}{A}$.

Example 1.4. If $k = 1$, $\alpha = \frac{-3i-\sqrt{15}}{24}$, $\beta = \frac{3i+\sqrt{15}}{24}$, $P(z) \equiv -1$, and $r(z) = 2z$ in (1.8), then

$$f'^2(z) + \left[\frac{-3i-\sqrt{15}}{24} f(z+\eta) + \frac{-3i-\sqrt{15}}{24} f(z)\right]^2 = e^{2z}$$

has a solution $f(z) = -\frac{1}{3}e^{\frac{3}{2}z} - e^{\frac{z}{2}}$, where $A = -1$, $a = \frac{3}{2}$, $b = \frac{1}{2}$, $e^{\frac{\eta}{2}} = \frac{9+\sqrt{15}i}{3-\sqrt{15}i}$, and $e^{\frac{3\eta}{2}} = \frac{\sqrt{15}i-39}{3-\sqrt{15}i}$. Clearly, $a \neq \pm b$, $a^k = \frac{3}{2} = \frac{i\alpha(e^{a\eta}+1)}{A}$, and $b^k = \frac{1}{2} = \frac{-i\alpha(1+e^{b\eta})}{A}$.

Example 1.5. If $k = 1$, $\alpha = 1$, $\beta = \frac{(2i+1)+\sqrt{12i+1}}{2}$, $P(z) \equiv 1$, and $r(z) = 3z$ in (1.8), then

$$f'^2(z) + \left[f(z+\eta) - \frac{(2i+1)+\sqrt{12i+1}}{2} f(z)\right]^2 = e^{3z}$$

has a solution $f(z) = \frac{1}{2}e^{z} + \frac{1}{4}e^{2z}$, where $A = 1$, $a = 1$, $b = 2$, $e^{\eta} = \frac{1+\sqrt{12i+1}}{2}$, and $e^{2\eta} = \frac{i-6+i\sqrt{12i+1}}{2i}$. Clearly, $a \neq \pm b$, $a^k = 1 = \frac{i\alpha e^{a\eta}-i\beta}{A}$, and $b^k = 2 = \frac{i\beta - i\alpha e^{b\eta}}{A}$.

Remark 1.4. We give the following Examples 1.6-1.8 to show that Case (II) above does exist.

Example 1.6. If $k = 1$, $\alpha = \beta = 1$, $P(z) \equiv 1$, and $r(z) = 2z+1$ in (1.8), then

$$f'^2(z) + [f(z+\eta) - f(z)]^2 = e^{2z+1}$$

has a solution $f(z) = \frac{(e+1)}{2}e^{z} + 1$, where $A = 1$, $a = b = 1$, $a_0 = 0$, $b_0 = 1$, $e^{\eta} = 1 + \frac{1-e}{i(1+e)}$, and n is an integer. Clearly, $1 = b_0 \neq a_0 + 2n\pi i = 2n\pi i$, $a^k = 1 = \frac{i\alpha(e^{a\eta}-1)(1+e^{b_0-a_0})}{A(1-e^{b_0-a_0})}$.

Example 1.7. If $k = 1$, $\alpha = 1$, $\beta = -1$, $P(z) \equiv 1$, and $r(z) = 2z + 1$ in (1.8), then

$$f'^2(z) + [f(z+\eta) + f(z)]^2 = e^{2z+1}$$

has a solution $f(z) = \frac{(e+1)}{2}e^z$, where $A = 1$, $a = b = 1$, $a_0 = 0$, $b_0 = 1$, $e^\eta = \frac{i(e-1)}{1+e} - 1$, and n is an integer. Clearly, $1 = b_0 \neq a_0 + 2n\pi i = 2n\pi i$, $a^k = 1 = \frac{i\alpha(e^{a\eta}+1)(1+e^{b_0-a_0})}{A(1-e^{b_0-a_0})}$.

Example 1.8. If $k = 1$, $\alpha = -i$, $\beta = -2i$, $P(z) \equiv 1$, and $r(z) = 2z + 1$ in (1.8), then

$$f'^2(z) + [-if(z+\eta) + 2if(z)]^2 = e^{2z+1}$$

has a solution $f(z) = \frac{(e+1)}{2}e^z$, where $A = 1$, $a = b = 1$, $a_0 = 0$, $b_0 = 1$, $e^\eta = \frac{3+e}{1+e}$, and n is an integer. Clearly, $1 = b_0 \neq a_0 + 2n\pi i = 2n\pi i$, $a^k = 1 = \frac{(i\alpha e^{a\eta}-i\beta)(1+e^{b_0-a_0})}{A(1-e^{b_0-a_0})}$.

Remark 1.5. We give the following Examples 1.9 and 1.10 to show that Case (III) above does exist.

Example 1.9. If $k = 1$, $\alpha = \beta = 1$, $P(z) \equiv 1$, and $r(z) = 2z$ in (1.8), then

$$f'^2(z) + [f(z+2\pi i) - f(z)]^2 = e^{2z}$$

has a solution $f(z) = e^z + 1$, where $a = b = 1$, $a_0 = b_0 = n = 0$, and $\eta = 2\pi i$. Clearly, $b_0 = 0 = a_0 + 2n\pi i$, $e^{a\eta} = 1$.

Example 1.10. If $k = 1$, $\alpha = 1$, $\beta = e$, $P(z) \equiv 1$, and $r(z) = 2z$ in (1.8), then

$$f'^2(z) + [f(z+1) - ef(z)]^2 = e^{2z}$$

has a solution $f(z) = e^z$, where $a = b = 1$, $a_0 = b_0 = n = 0$, and $\eta = 1$. Clearly, $b_0 = 0 = a_0 + 2n\pi i$, $e^{a\eta} = e \neq 1$.

Remark 1.6. Actually, if the non-constant polynomial $r(z)$ reduces to a constant in Theorem 1.3, then by using the similar method as in the proof of Theorem E, we can get Corollary 1.1 immediately. Therefore, the proof is omitted.

Corollary 1.1. *Let $P(z) \not\equiv 0$ be a polynomial, $\alpha \neq 0$ and β be constants, k be a positive integer, $\eta \in \mathbb{C} \setminus \{0\}$. If the equation*

$$P^2(z)[f^{(k)}(z)]^2 + [\alpha f(z+\eta) - \beta f(z)]^2 = 1 \tag{1.9}$$

admits an entire solution of finite order, then there exist two cases:

(I) *if $f(z)$ is a transcendental solution of* (1.9)*, then either*

(I.i) *when k is an odd number, $f(z)$ must satisfy the form that*

$$f(z) = \frac{e^{az+b} - e^{-az-b}}{2a^k A} + d,$$

where A, a, b, d are constants, n is an integer. In this case, (I.i.i) *if $\alpha = \beta$, then $a^k = \frac{-2\alpha i}{A}$, $\eta = \frac{(2n+1)\pi}{a}i$;* (I.i.ii) *if $\alpha = -\beta$, then $a^k = \frac{2\alpha i}{A}$, $\eta = \frac{2n\pi}{a}i$,*

$d = 0$; (I.i.iii) *if* $\alpha \neq \pm\beta$, *then* $a^k = \frac{-(\alpha+\beta)i}{A}$, $\eta = \frac{(2n+1)\pi}{a} i$ *or* $a^k = \frac{(\alpha-\beta)i}{A}$, $\eta = \frac{2n\pi}{a} i$, $d = 0$; *or*

(I.ii) *when* k *is an even number and* $\alpha = \pm\beta$, *the equation* (1.9) *does not have transcendental entire solutions*; *when* k *is an even number and* $\alpha \neq \pm\beta$, $f(z)$ *must satisfy the form that*

$$f(z) = \frac{e^{az+b} + e^{-az-b}}{2a^k A},$$

where A, a, b *are constants satisfying* $a^k = \frac{\sqrt{\alpha^2-\beta^2}}{A}$; $a^k = \frac{-\sqrt{\alpha^2-\beta^2}}{A}$ *and* $\eta = \frac{\ln(\frac{i\beta + a^k A}{i\alpha}) + 2n\pi i}{a}$, *where* n *is an integer.*

(II) *if* $f(z)$ *is a polynomial solution of* (1.9), *then either*

(II.i) *if* $\alpha = \beta$, *then* $f(z) = Bz + C$, *where* B *and* C *are constants satisfying* $B^2(1 + \alpha^2\eta^2) = 1$ *for* $k = 1$; $(\alpha B\eta)^2 = 1$ *for* $k \geq 2$; *or*

(II.ii) *if* $\alpha \neq \beta$, *then* $f(z) \equiv \pm\frac{1}{\alpha-\beta}$.

Remark 1.7. We give the following Examples 1.11-1.14 to show that Case (I.i) above does exist.

Example 1.11. If $k = 1$, $\alpha = \beta = \frac{\pi}{2}$, and $P(z) \equiv -1$ in (1.9), then

$$f'^2(z) + \left[\frac{\pi}{2} f(z + 2n + 1) - \frac{\pi}{2} f(z)\right]^2 = 1$$

has a solution $f(z) = -\frac{1}{\pi} \sin \pi z + 1$, where $A = -1$, $a = \pi i$, $\eta = 2n + 1$, and n is an integer. Clearly, $a^k = \pi i = \frac{-2\alpha i}{A}$, $\eta = 2n + 1 = \frac{(2n+1)\pi}{a} i$.

Example 1.12. If $k = 1$, $\alpha = \frac{1}{4}$, $\beta = -\frac{1}{4}$, and $P(z) \equiv \frac{1}{4}$ in (1.9), then

$$\frac{1}{16} f'^2(z) + \left[\frac{1}{4} f(z + n\pi) + \frac{1}{4} f(z)\right]^2 = 1$$

has a solution $f(z) = 2 \sin 2z$, where $A = \frac{1}{4}$, $a = 2i$, $\eta = n\pi$, and n is an integer. Clearly, $a^k = 2i = \frac{2\alpha i}{A}$, $\eta = n\pi = \frac{2n\pi}{a} i$.

Example 1.13. If $k = 1$, $\alpha = \frac{3}{4i}$, $\beta = \frac{1}{4i}$, and $P(z) \equiv -1$ in (1.9), then

$$f'^2(z) + \left[\frac{3}{4i} f(z + \pi i) - \frac{1}{4i} f(z)\right]^2 = 1$$

has a solution $f(z) = i \sin iz$, where $A = -1$, $a = 1$, $\eta = \pi i$, and $n = 0$. Clearly, $a^k = 1 = -\frac{(\alpha+\beta)i}{A}$, $\eta = \pi i = \frac{(2n+1)\pi}{a} i$.

Example 1.14. If $k = 1$, $\alpha = \frac{3\pi}{2i}$, $\beta = \frac{\pi}{2i}$, and $P(z) \equiv \frac{1}{2i}$ in (1.9), then

$$-\frac{1}{4} f'^2(z) + \left[\frac{3\pi}{2i} f(z + n) - \frac{\pi}{2i} f(z)\right]^2 = 1$$

has a solution $f(z) = \frac{i}{\pi} \sin 2\pi z$, where $A = \frac{1}{2i}$, $a = 2\pi i$, $\eta = n$, and n is an integer. Clearly, $a^k = 2\pi i = \frac{(\alpha-\beta)i}{A}$, $\eta = n = \frac{2n\pi}{a} i$.

Remark 1.8. We give the following Examples 1.15 and 1.16 to show that Case (I.ii) above does exist.

Example 1.15. If $k = 2$, $\alpha = 1$, $\beta = 0$, and $P(z) \equiv 1$ in (1.9), then

$$f''^2(z) + \left[f\left(z - \frac{\pi i}{2}\right)\right]^2 = 1$$

has a solution $f(z) = \cos iz$, where $A = 1$, $a = 1$, $\eta = -\frac{\pi i}{2}$, and $n = 0$. Clearly, $a^k = 1 = \frac{\sqrt{\alpha^2-\beta^2}}{A}$, $\eta = -\frac{\pi i}{2} = \frac{\ln(\frac{i\beta+a^k A}{i\alpha})+2n\pi i}{a}$.

Example 1.16. If $k = 2$, $\alpha = 1$, $\beta = 0$, and $P(z) \equiv 1$ in (1.9), then

$$f''^2(z) + \left[f\left(z + \frac{\pi}{2}\right)\right]^2 = 1$$

has a solution $f(z) = -\cos z$, where $A = 1$, $a = i$, $\eta = \frac{\pi}{2}$, and $n = 0$. Clearly, $a^k = -1 = \frac{-\sqrt{\alpha^2-\beta^2}}{A}$, $\eta = \frac{\pi}{2} = \frac{\ln(\frac{i\beta+a^k A}{i\alpha})+2n\pi i}{a}$.

Remark 1.9. We give the following Examples 1.17 and 1.18 to show that Case (II.i) above does exist.

Example 1.17. If $k = 1$, $\alpha = \beta = 1$, and $P(z) \equiv 1$ in (1.9), then

$$f'^2(z) + \left[f\left(z + \frac{\sqrt{2}}{2}i\right) - f(z)\right]^2 = 1$$

has a solution $f(z) = \sqrt{2}z$, where $B = \sqrt{2}$, $\eta = \frac{\sqrt{2}}{2}i$. Clearly, $B^2(1+\alpha^2\eta^2) = 1$.

Example 1.18. If $k = 2$, $\alpha = \beta = 1$, and $P(z) \equiv 1$ in (1.9), then

$$f''^2(z) + \left[f\left(z + \frac{\sqrt{2}}{2}\right) - f(z)\right]^2 = 1$$

has a solution $f(z) = \sqrt{2}z$, where $B = \sqrt{2}$, $\eta = \frac{\sqrt{2}}{2}$. Clearly, $(\alpha B\eta)^2 = 1$.

Remark 1.10. We give the following Example 1.19 to show that Case (II.ii) above does exist.

Example 1.19. If $k = 1$, $\alpha = 2$, $\beta = 1$, and $P(z) \equiv 1$ in (1.9), then

$$f'^2(z) + [2f(z+\eta) - f(z)]^2 = 1$$

has a solution $f(z) \equiv \pm 1$, where η is a non-zero constant. Clearly, $f(z) \equiv \pm 1 = \pm\frac{1}{\alpha-\beta}$.

2. Some lemmas

Lemma 2.1 (see [15]). *Let $f(z)$ be a finite order ρ transcendental meromorphic solution of the difference equation*

$$U(z,f)P(z,f)=Q(z,f),$$

where $U(z,f)$, $P(z,f)$, $Q(z,f)$ are difference polynomials in f such that the total degree of $U(z,f)$ in f and its shifts are n, and that the total degree of $Q(z,f)$ as a polynomial in f and its shifts are at most n. If $U(z,f)$ just contains one term of maximal total degree, then for each $\varepsilon>0$,

$$m(r,P(z,f))=O(r^{\rho-1+\varepsilon})+S(r,f)$$

holds possibly outside of an exceptional set with finite logarithmic measure.

Lemma 2.2 (see [17]). *Let $f(z)$ be an entire function of finite order ρ with zeros $\{z_1,z_2,\ldots\}\subset\mathbb{C}\backslash\{0\}$ and a k-fold zero at the origin. Then*

$$f(z)=z^kP(z)e^{Q(z)},$$

where $P(z)$ is the canonical product of $f(z)$ formed with the non-null zeros of $f(z)$, and $Q(z)$ is a polynomial of degree at most ρ.

Lemma 2.3 (see [17]). *If $f(z)$ is a transcendental meromorphic function in $\mathbb{C}$, then*

$$\lim_{r\to\infty}\frac{T(r,f)}{\log r}=\infty.$$

Lemma 2.4 (see [2]). *Let $f(z)$ be a meromorphic function such that the order $\rho<+\infty$, and $\eta\in\mathbb{C}\setminus\{0\}$. Then for any $\varepsilon>0$,*

$$T(r,f(z+\eta))=T(r,f)+O(r^{\rho-1+\varepsilon})+O(\log r). \tag{2.1}$$

Thus, if $f(z)$ is a finite order ρ transcendental meromorphic function, then we have

$$T(r,f(z+\eta))=T(r,f)+S(r,f). \tag{2.2}$$

Lemma 2.5 (see [17]). *If meromorphic functions $f_j(z)$ $(j=1,2,\ldots,n)$ $(n\geq 2)$ and entire functions $g_j(z)$ $(j=1,2,\ldots,n)$ $(n\geq 2)$ satisfy the following conditions*:

(1) $\sum_{j=1}^n f_je^{g_j}\equiv 0$;

(2) *g_j-g_l are not constants for $1\leq j<l\leq n$;*

(3) *$T(r,f_j)=o\left(T\left(r,e^{g_h-g_l}\right)\right)$ $(r\to\infty,\ r\notin E)$ for $1\leq j\leq n$, $1\leq h<l\leq n$,*

then we have $f_j\equiv 0$ $(j=1,2,\ldots,n)$.

Lemma 2.6 (see [17]). *Suppose that $f_j(z)\not\equiv 0$ $(j=1,2,\ldots,n)$ $(n\geq 3)$ are meromorphic functions such that $f_1(z),f_2(z),\ldots,f_{n-1}(z)$ are non-constants, $\sum_{j=1}^n f_j(z)\equiv 1$ and*

$$\sum_{j=1}^n N\left(r,\frac{1}{f_j}\right)+(n-1)\sum_{j=1}^n\overline{N}\left(r,f_j\right)<(\lambda+o(1))T\left(r,f_k\right),$$

where $\lambda < 1$ *and* $k = 1, 2, \ldots, n-1$. *Then* $f_n(z) \equiv 1$.

3. Proofs of Theorems

3.1. Proof of Theorem 1.1

First, we may assume that $f(z)$ is a finite order transcendental entire solution of (1.4). Next, We will discuss three cases step by step and give the relative contradictions.

Case 1. If $n = m = 1$, then (1.4) is changed into the following form

$$f(z)f'(z) + P^2(z)f(z+\eta) = Q(z). \tag{3.1}$$

Thus, (3.1) can be rewritten as

$$f(z)f'(z) = Q(z) - P^2(z)f(z+\eta).$$

Combining Lemma 2.1 with (2.2), we get

$$m(r, f'(z)) = S(r, f(z)),$$

and then

$$T(r, f'(z)) = m(r, f'(z)) = S(r, f(z)).$$

This contradicts our assumption that $f(z)$ is transcendental.

Case 2. If $n = m = 2$, then (1.4) is changed into the following form

$$[f(z)f'(z)]^2 + P^2(z)[f(z+\eta)]^2 = Q(z). \tag{3.2}$$

Thus, (3.2) can be rewritten as

$$[f(z)f'(z) + iP(z)f(z+\eta)][f(z)f'(z) - iP(z)f(z+\eta)] = Q(z). \tag{3.3}$$

It follows that $f(z)f'(z) + iP(z)f(z+\eta)$ and $f(z)f'(z) - iP(z)f(z+\eta)$ have finitely many zeros. Combining (3.3) with Lemma 2.2, we know that

$$f(z)f'(z) + iP(z)f(z+\eta) = Q_1(z)e^{h(z)}$$

and

$$f(z)f'(z) - iP(z)f(z+\eta) = Q_2(z)e^{-h(z)},$$

where $h(z)$ is a non-constant polynomial, $Q_1(z)$ and $Q_2(z)$ are non-zero polynomials such that $Q(z) = Q_1(z)Q_2(z)$. Thus, we get

$$f(z)f'(z) = \frac{Q_1(z)e^{h(z)} + Q_2(z)^{-h(z)}}{2} \tag{3.4}$$

and

$$f(z+\eta) = \frac{Q_1(z)e^{h(z)} - Q_2(z)^{-h(z)}}{2iP(z)}. \tag{3.5}$$

Differentiating (3.5) results in

$$f'(z+\eta) = \frac{P(z)Q_1'(z) - P'(z)Q_1(z) + P(z)Q_1(z)h'(z)}{2iP^2(z)}e^{h(z)}$$

$$- \frac{P(z)Q_2'(z) - P'(z)Q_2(z) - P(z)Q_2(z)h'(z)}{2iP^2(z)} e^{-h(z)},$$

and then

$$f(z+\eta)f'(z+\eta) = \frac{h_1(z)e^{2h(z)} - h_2(z) + h_3(z)e^{-2h(z)}}{-4P^3(z)}, \tag{3.6}$$

where

$$\begin{cases} h_1(z) = P(z)Q_1'(z)Q_1(z) - P'(z)Q_1^2(z) + P(z)Q_1^2(z)h'(z), \\ h_2(z) = P(z)(Q_1(z)Q_2(z))' - 2P'(z)Q_1(z)Q_2(z), \\ h_3(z) = P(z)Q_2'(z)Q_2(z) - P'(z)Q_2^2(z) - P(z)Q_2^2(z)h'(z) \end{cases}$$

for any $h_j(z)$ ($j = 1, 2, 3$) are polynomials. From (3.4) and (3.6), we have

$$\frac{h_1(z)e^{2h(z)} - h_2(z) + h_3(z)e^{-2h(z)}}{-4P^3(z)} \equiv \frac{Q_1(z+\eta)e^{h(z+\eta)} + Q_2(z+\eta)e^{-h(z+\eta)}}{2},$$

i.e.,

$$\begin{aligned} &h_1(z)e^{2h(z)+h(z+\eta)} - h_2(z)e^{h(z+\eta)} + h_3(z)e^{h(z+\eta)-2h(z)} \\ &+ 2P^3(z)Q_1(z+\eta)e^{2h(z+\eta)} + 2P^3(z)Q_2(z+\eta) \equiv 0. \end{aligned} \tag{3.7}$$

Note that $\deg(4h(z)) = \deg(2h(z)) = \deg(\pm h(z+\eta)) = \deg(2h(z+\eta)) \geq 1$, $\deg(2h(z) - h(z+\eta)) \geq 1$, $\deg[2h(z) + h(z+\eta)] \geq 1$, $\deg[-2h(z) - h(z+\eta)] \geq 1$ and $\deg[h(z+\eta) - 2h(z)] \geq 1$. By Lemma 2.5, we have $2P^3(z)Q_1(z+\eta) \equiv 2P^3(z)Q_2(z+\eta) \equiv 0$. This contradicts the assumption that $P(z)$, $Q_1(z)$, $Q_2(z)$ are non-zero polynomials.

Case 3. If $n = m > 2$, then (1.4) can be rewritten as $\frac{1}{Q(z)}[f(z)f'(z)]^n + \frac{P^2(z)}{Q(z)} f^n(z+\eta) = 1$. By Theorem A, we see that the above equation has no transcendental entire solutions of finite order. Hence we complete the proof of Theorem 1.1.

3.2. Proof of Theorem 1.2

From the beginning of Theorem A, we only need to prove that there is no transcendental entire solutions of the following differential-difference equation

$$[f(z)f'(z)]^n + P(z)[\Delta_\eta f(z)] = Q(z), \tag{3.8}$$

where $n > 2$. We may assume that (3.8) has a transcendental entire solution $f(z)$.

Differentiating (3.8) results in

$$\begin{aligned} &nf'^{n-1}(z)[f^{n-1}(z)f'^2(z) + f^n(z)f''(z)] \\ &= Q'(z) - P'(z)[\Delta_\eta f(z)] - P(z)[\Delta_\eta f'(z)]. \end{aligned} \tag{3.9}$$

Substituting (3.8) into (3.9) yields

$$f'^{n-1}(z)\left[nf^{n-1}(z)f'^2(z) + nf^n(z)f''(z) - \frac{P'(z)}{P(z)} f^n(z)f'(z)\right]$$

$$= Q'(z) - \frac{P'(z)}{P(z)}Q(z) - P(z)[\Delta_\eta f'(z)].$$

Denote $\varphi(z) = nf^{n-1}(z)f'^2(z) + nf^n(z)f''(z) - \frac{P'(z)}{P(z)}f^n(z)f'(z)$, $g(z) = f'(z)$, and then we rewrite the above equation as the form

$$g^{n-1}(z)\varphi(z) = Q'(z) - \frac{P'(z)}{P(z)}Q(z) - P(z)[\Delta_\eta g(z)]. \tag{3.10}$$

Noting that $n-1 \geq 2$, by Lemma 2.1, we have

$$m(r, \varphi(z)) = S(r, g(z)), \quad m(r, g(z)\varphi(z)) = S(r, g(z)).$$

We see that $\varphi(z) \not\equiv 0$, otherwise $(f^2(z))'^n = C_1P(z)$, where C_1 is a non-zero constant, a contradiction. And since $f(z)$ is a transcendental entire solution, $N(r, \varphi(z)) = S(r, g(z))$. Thus it follows that

$$\begin{aligned} T(r, g(z)) = m(r, g(z)) &\leq m(r, g(z)\varphi(z)) + m\left(r, \frac{1}{\varphi(z)}\right) \\ &\leq m(r, \varphi(z)) + N(r, \varphi(z)) + S(r, g(z)) = S(r, g(z)), \end{aligned}$$

i.e., $T(r, f'(z)) = T(r, g(z)) \leq S(r, g(z)) = S(r, f'(z))$, a contradiction. Hence we complete the proof of Theorem 1.2.

3.3. Proof of Theorem 1.3

Assume that $f(z)$ is a finite order transcendental entire solution satisfying (1.8). We rewrite (1.8) as

$$\begin{aligned} &[P(z)f^{(k)}(z) + i(\alpha f(z+\eta) - \beta f(z))] \\ &\times [P(z)f^{(k)}(z) - i(\alpha f(z+\eta) - \beta f(z))] = e^{r(z)}. \end{aligned} \tag{3.11}$$

It then follows that $P(z)f^{(k)}(z) + i(\alpha f(z+\eta) - \beta f(z))$ and $P(z)f^{(k)}(z) - i(\alpha f(z+\eta) - \beta f(z))$ have no zeros. By Lemma 2.2, we have

$$\begin{cases} P(z)f^{(k)}(z) + i(\alpha f(z+\eta) - \beta f(z)) = e^{r_1(z)}; \\ P(z)f^{(k)}(z) - i(\alpha f(z+\eta) - \beta f(z)) = e^{r_2(z)}, \end{cases} \tag{3.12}$$

where $r(z)$ is a non-constant polynomial such that $r(z) = r_1(z) + r_2(z)$. Thus

$$\begin{cases} f^{(k)}(z) = \frac{e^{r_1(z)} + e^{r_2(z)}}{2P(z)}; \\ \alpha f(z+\eta) - \beta f(z) = \frac{e^{r_1(z)} - e^{r_2(z)}}{2i}. \end{cases} \tag{3.13}$$

By mathematical induction, we can deduce that

$$(e^{r_1(z)})^{(k)} = e^{r_1(z)}[r_1^{(k)}(z) + \cdots + r_1'^k(z)] = e^{r_1(z)}M(z), \tag{3.14}$$

$$(-e^{r_2(z)})^{(k)} = e^{r_2(z)}[-(r_2^{(k)}(z) + \cdots + r_2'^k(z))] = e^{r_2(z)}N(z), \tag{3.15}$$

where $M(z) = r_1^{(k)}(z) + \cdots + r_1'^k(z)$, $N(z) = -[r_2^{(k)}(z) + \cdots + r_2'^k(z)]$. From (3.13)-(3.15), we get

$$\alpha f^{(k)}(z+\eta) - \beta f^{(k)}(z) \equiv \frac{M(z)e^{r_1(z)} + N(z)e^{r_2(z)}}{2i}. \tag{3.16}$$

Then, combining (3.13) with (3.16), we have

$$\begin{aligned} &\frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_1(z)-r_2(z+\eta)} \\ &+ \frac{i\beta P(z+\eta)+N(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_2(z)-r_2(z+\eta)} - e^{r_1(z+\eta)-r_2(z+\eta)} \equiv 1. \end{aligned} \tag{3.17}$$

Since $r(z)$ is a non-constant polynomial, $r_1(z)$ and $r_2(z)$ cannot be constants simultaneously. Otherwise, $r(z)$ is a constant. Now we assume that at least one of $r_1(z)$ and $r_2(z)$ is a non-constant polynomial. To this end, we divide our discussion into three cases.

Case 1. If $r_1(z)$ is a constant and $r_2(z)$ is a non-constant polynomial, then $M(z) \equiv 0$ and $r(z)$ is a non-constant polynomial that satisfies the assumption.

Now we claim that $\frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)} \not\equiv 0$, that is, $i\beta P(z+\eta) \not\equiv 0$. Otherwise, we may assume that $\frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)} \equiv 0$. By (3.17), we have

$$\frac{i\beta P(z+\eta) + N(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_2(z)-r_2(z+\eta)} - e^{r_1(z+\eta)-r_2(z+\eta)} \equiv 1.$$

We denote $g(z) = e^{r_1(z+\eta)-r_2(z+\eta)}$. Then, using the second main theorem of Nevanlinna theory and Lemma 2.3, we have

$$\begin{aligned} T(r,g) &\le \overline{N}(r,g) + \overline{N}(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{g+1}) + S(r,g) \\ &\le \overline{N}\left(\frac{1}{\frac{i\beta P(z+\eta)+N(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_2(z)-r_2(z+\eta)}}\right) + S(r,g) \\ &\le O(\log r) + S(r,g) = S(r,g), \end{aligned}$$

which is a contradiction to $e^{r_1(z+\eta)-r_2(z+\eta)}$ being transcendental entire function. Thus, the claim is proved. Note that $r_1(z)$ is a constant and $r_2(z)$ is not a constant. Then both $r_1(z) - r_2(z+\eta)$ and $r_1(z+\eta) - r_2(z+\eta)$ are not constants. Hence, $\frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_1(z)-r_2(z+\eta)}$ and $e^{r_1(z+\eta)-r_2(z+\eta)}$ are not constants. Combining (3.17) with Lemma 2.6, we have

$$\frac{i\beta P(z+\eta) + N(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_2(z)-r_2(z+\eta)} \equiv 1, \tag{3.18}$$

which implies that $r_2(z) - r_2(z+\eta)$ is a constant. Then $\deg(r_2(z)) = 1$. Assume that $r_2(z) = bz + b_0$, where $b \neq 0$ and b_0 are constants. Thus, we have

$N(z) \equiv -b^k$. Substituting it into (3.18) yields

$$\frac{i\beta P(z+\eta) - b^k P(z)P(z+\eta)}{i\alpha P(z)} e^{-b\eta} \equiv 1. \tag{3.19}$$

From the above identity, we can deduce that $P(z)$ is a constant, say A ($\neq 0$). And since $r_1(z)$ is a constant, we set $e^{r_1(z)} = c_1 \neq 0$. Then, it follows from (3.13) that

$$f^{(k)}(z) = \frac{c_1 + e^{bz+b_0}}{2A}, \quad f(z) = \frac{c_1}{2Ak!} z^k + \frac{e^{bz+b_0}}{2Ab^k} + S(z), \tag{3.20}$$

where $S(z)$ is a polynomial with $\deg(S(z)) \leq k-1$. Further, we obtain from (3.17)-(3.19) that $\alpha = \beta$, $e^{b\eta} = 1 - \frac{b^k A}{i\alpha}$. Combining (3.13) with (3.20), we have

$$f(z+\eta) - f(z) \equiv \frac{c_1 - e^{bz+b_0}}{2i\alpha};$$

$$\frac{c_1}{2Ak!}(z+\eta)^k - \frac{c_1}{2Ak!} z^k + S(z+\eta) - S(z) - \frac{c_1}{2i\alpha} \equiv (-\frac{1}{2i\alpha} - \frac{e^{b\eta}-1}{2Ab^k}) e^{bz+b_0}.$$

Then,

$$\frac{c_1}{2Ak!}(z+\eta)^k - \frac{c_1}{2Ak!} z^k - \frac{c_1}{2i\alpha} \equiv S(z) - S(z+\eta).$$

Since $c_1 \neq 0$ is a constant, the maximum degree of the left-hand side of the above identity is $k-1$ and the degree of the right-hand side of the above identity is at most $k-2$, a contradiction.

Case 2. If $r_1(z)$ is a non-constant polynomial and $r_2(z)$ is a constant, then $N(z) \equiv 0$ and $r(z)$ is a non-constant polynomial that satisfies the assumption. Thus (3.17) can be written as

$$\begin{aligned} &\frac{i\beta P(z+\eta) + M(z)P(z)P(z+\eta)}{i\alpha P(z)} e^{r_1(z) - r_2(z+\eta)} \\ &- e^{r_1(z+\eta) - r_2(z+\eta)} \equiv \frac{i\alpha P(z) - i\beta P(z+\eta)}{i\alpha P(z)}. \end{aligned} \tag{3.21}$$

Now we distinguish $i\alpha P(z) - i\beta P(z+\eta) \equiv 0$, $i\alpha P(z) - i\beta P(z+\eta) \not\equiv 0$ two subcases to get the contradictions.

Subcase 2.1. If $i\alpha P(z) - i\beta P(z+\eta) \equiv 0$, then by (3.21),

$$\frac{i\beta P(z+\eta) + M(z)P(z)P(z+\eta)}{i\alpha P(z)} e^{r_1(z) - r_1(z+\eta)} \equiv 1, \tag{3.22}$$

which implies that $r_1(z) - r_1(z+\eta)$ is a constant. Then $\deg(r_1(z)) = 1$. Assume that $r_1(z) = az + a_0$, where $a \neq 0$ and a_0 are constants. Thus, we have $M(z) \equiv a^k$. Substituting it into (3.22) yields

$$\frac{i\beta P(z+\eta) + a^k P(z)P(z+\eta)}{i\alpha P(z)} e^{-a\eta} \equiv 1. \tag{3.23}$$

From the above identity, we can deduce that $P(z)$ is a constant again, say A ($\neq 0$). And since $r_2(z)$ is a constant, we set $e^{r_2(z)} = c_2 \neq 0$. Then, it follows from (3.13) that

$$f^{(k)}(z) = \frac{e^{az+a_0} + c_2}{2A}, \quad f(z) = \frac{c_2}{2Ak!}z^k + \frac{e^{az+a_0}}{2Aa^k} + T(z), \tag{3.24}$$

where $T(z)$ is a polynomial with $\deg(T(z)) \leq k-1$. Further, we obtain from (3.21)-(3.23) that $\alpha = \beta$, $e^{a\eta} = 1 + \frac{a^k A}{i\alpha}$. Combining (3.13) with (3.24), we have

$$f(z+\eta) - f(z) \equiv \frac{e^{az+a_0} - c_2}{2i\alpha},$$

$$\frac{c_2}{2Ak!}(z+\eta)^k - \frac{c_2}{2Ak!}z^k + T(z+\eta) - T(z) + \frac{c_2}{2i\alpha} \equiv (\frac{1}{2i\alpha} + \frac{1-e^{a\eta}}{2Aa^k})e^{az+a_0}.$$

Then,

$$\frac{c_2}{2Ak!}(z+\eta)^k - \frac{c_2}{2Ak!}z^k + \frac{c_2}{2i\alpha} \equiv T(z) - T(z+\eta).$$

With the above identity, we can also get a similar contradiction as in Case 1.

Subcase 2.2. If $i\alpha P(z) - i\beta P(z+\eta) \not\equiv 0$, then by (3.21),

$$H_{12}(z)e^{r_1(z)} + H_{11}(z)e^{h_0(z)} \equiv 0, \tag{3.25}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} H_{12}(z) = \frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)c_2} - \frac{1}{c_2}e^{r_1(z+\eta)-r_1(z)}, \\ H_{11}(z) = -\frac{i\alpha P(z)-i\beta P(z+\eta)}{i\alpha P(z)}. \end{cases}$$

Noting that $\deg(r_1(z+\eta) - r_1(z)) = \deg(r_1(z)) - 1 < \deg(r_1(z))$ and using Lemma 2.5, we see that $H_{1j}(z) \equiv 0$ ($j = 1,\ 2$). By $H_{11}(z) \equiv 0$, we have $i\alpha P(z) - i\beta P(z+\eta) \equiv 0$, which contradicts that $i\alpha P(z) - i\beta P(z+\eta) \not\equiv 0$.

Case 3. If $r_1(z)$ and $r_2(z)$ are non-constant polynomials, then (3.17) can be written as

$$\begin{aligned} &\frac{i\beta P(z+\eta) + M(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_1(z)} \\ &+ \frac{i\beta P(z+\eta) + N(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_2(z)} - e^{r_1(z+\eta)} - e^{r_2(z+\eta)} \equiv 0. \end{aligned} \tag{3.26}$$

Next three subcases will be considered in the following.

Subcase 3.1. If $\deg(r_1(z)) > \deg(r_2(z)) \geq 1$, then $r(z)$ is a non-constant polynomial that satisfies the assumption. Thus (3.26) can be written as

$$H_{22}(z)e^{r_1(z)} + H_{21}(z)e^{h_0(z)} \equiv 0, \tag{3.27}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} H_{22}(z) = \frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)} - e^{r_1(z+\eta)-r_1(z)}, \\ H_{21}(z) = \frac{i\beta P(z+\eta)+N(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_2(z)} - e^{r_2(z+\eta)}. \end{cases}$$

Noting that $\deg(r_1(z)) > \deg(r_2(z))$, $\deg(r_1(z+\eta) - r_1(z)) = \deg(r_1(z)) - 1 < \deg(r_1(z))$ and using Lemma 2.5, we see that $H_{2j}(z) \equiv 0$ ($j = 1, 2$). By

$H_{22}(z) \equiv H_{21}(z) \equiv 0$, we have $\deg(r_1(z)) = \deg(r_2(z)) = 1$, which contradicts that $\deg(r_1(z)) > \deg(r_2(z)) \geq 1$.

Subcase 3.2. If $\deg(r_2(z)) > \deg(r_1(z)) \geq 1$, then $r(z)$ is a non-constant polynomial that satisfies the assumption. Thus (3.26) can be written as

$$H_{32}(z)e^{r_2(z)} + H_{31}(z)e^{h_0(z)} \equiv 0, \tag{3.28}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} H_{32}(z) = \frac{i\beta P(z+\eta)+N(z)P(z)P(z+\eta)}{i\alpha P(z)} - e^{r_2(z+\eta)-r_2(z)}, \\ H_{31}(z) = \frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)} e^{r_1(z)} - e^{r_1(z+\eta)}. \end{cases}$$

By the arguments similar to that in Subcase 3.1, we can get a contradiction.

Subcase 3.3. If $\deg(r_1(z)) = \deg(r_2(z)) = n \geq 1$, then we set $r_1(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, $r_2(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0$, where $a_n(\neq 0), a_{n-1}, \ldots, a_0, b_n(\neq 0), b_{n-1}, \ldots, b_0$ are constants, n is an integer. Thus, we have $r(z) = (a_n + b_n)z^n + (a_{n-1} + b_{n-1})z^{n-1} + \cdots + a_0 + b_0$. Since $r(z)$ is a non-constant polynomial, we see that, for all $j = 1, 2, \ldots, n$, at least one of $a_j \neq -b_j$.

To prove $\deg(r_1(z)) = \deg(r_2(z)) = 1$, we discuss two subcases $b_n \neq a_n$ and $b_n = a_n$.

Subcase 3.3.1. If $b_n \neq a_n$, then (3.26) can be written as

$$H_{42}(z)e^{r_1(z)} + H_{41}(z)e^{r_2(z)} \equiv 0, \tag{3.29}$$

where

$$\begin{cases} H_{42}(z) = \frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)} - e^{r_1(z+\eta)-r_1(z)}, \\ H_{41}(z) = \frac{i\beta P(z+\eta)+N(z)P(z)P(z+\eta)}{i\alpha P(z)} - e^{r_2(z+\eta)-r_2(z)}. \end{cases}$$

It is easy to get that $\deg(r_1(z+\eta) - r_1(z)) = \deg(r_2(z+\eta) - r_2(z)) = n-1$. Noting that $b_n \neq a_n$, we see that $\deg(r_1(z+\eta)-r_1(z)) = \deg(r_2(z+\eta)-r_2(z)) < \deg(r_1(z) - r_2(z)) = n$. Therefore, using Lemma 2.5, we have $H_{4j}(z) \equiv 0$. By $H_{42}(z) \equiv H_{41}(z) \equiv 0$, we see that $r_1(z+\eta) - r_1(z)$, $r_2(z+\eta) - r_2(z)$ must be constants. Then $\deg(r_1(z)) = \deg(r_2(z)) = 1$. Assume that $r_1(z) = az + a_0$, $r_2(z) = bz + b_0$, where $b \neq 0$, $a \neq 0$, a_0, b_0 are constants with $a \neq b$. Further, by the fact that, for all $j = 1, 2, \ldots, n$, at least one of $a_j \neq -b_j$, we have $a \neq -b$. Hence $a \neq \pm b$.

In the following, since $r_1(z) = az + a_0$, $r_2(z) = bz + b_0$, we get $M(z) \equiv a^k$, $N(z) \equiv -b^k$. Substituting it into $H_{42}(z) \equiv H_{41}(z) \equiv 0$ yields

$$i\beta P(z+\eta) + a^k P(z)P(z+\eta) \equiv i\alpha P(z)e^{a\eta} \tag{3.30}$$

and

$$i\beta P(z+\eta) - b^k P(z)P(z+\eta) \equiv i\alpha P(z)e^{b\eta}, \tag{3.31}$$

which imply that $P(z)$ must be a constant, say $A(\neq 0)$. Then (3.30), (3.31) can be written as

$$a^k = \frac{i\alpha e^{a\eta} - i\beta}{A}, \quad b^k = \frac{i\beta - i\alpha e^{b\eta}}{A}.$$

Also, it follows from (3.13) that

$$f^{(k)}(z)=\frac{e^{az+a_0}+e^{bz+b_0}}{2A},\quad f(z)=\frac{e^{az+a_0}}{2Aa^k}+\frac{e^{bz+b_0}}{2Ab^k}+U(z), \tag{3.32}$$

where $U(z)$ is a polynomial with $\deg(U(z))\leq k-1$.

If $\alpha=\beta$, then $a^k=\frac{i\alpha(e^{a\eta}-1)}{A}$, $b^k=\frac{i\alpha(1-e^{b\eta})}{A}$. Combining (3.13) with (3.32), we have $\alpha[U(z+\eta)-U(z)]\equiv 0$. By $\alpha\neq 0$, we get $U(z)\equiv d$, where d is a constant. Then,

$$f(z)=\frac{e^{az+a_0}}{2Aa^k}+\frac{e^{bz+b_0}}{2Ab^k}+d.$$

If $\alpha=-\beta$, then $a^k=\frac{i\alpha(e^{a\eta}+1)}{A}$, $b^k=\frac{-i\alpha(1+e^{b\eta})}{A}$. In the same way, we have $U(z)\equiv 0$. Then,

$$f(z)=\frac{e^{az+a_0}}{2Aa^k}+\frac{e^{bz+b_0}}{2Ab^k}.$$

If $\alpha\neq\pm\beta$, then $a^k=\frac{i\alpha e^{a\eta}-i\beta}{A}$, $b^k=\frac{i\beta-i\alpha e^{b\eta}}{A}$. Similarly, we have $U(z)\equiv 0$. Then,

$$f(z)=\frac{e^{az+a_0}}{2Aa^k}+\frac{e^{bz+b_0}}{2Ab^k}.$$

This belongs to Case (I) in Theorem 1.3.

Subcase 3.3.2. If $b_n=a_n$, then (3.26) can be written as

$$H_{51}(z)e^{r_1(z)}\equiv 0, \tag{3.33}$$

where

$$\begin{aligned}H_{51}(z)=&\frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)}\\&+\frac{i\beta P(z+\eta)+N(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_2(z)-r_1(z)}\\&-e^{r_1(z+\eta)-r_1(z)}-e^{r_2(z+\eta)-r_1(z)}.\end{aligned}$$

Since $e^{r_1(z)}\not\equiv 0$, we have $H_{51}(z)\equiv 0$. Then

$$\begin{aligned}&\frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)}\\&+\frac{i\beta P(z+\eta)+N(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_2(z)-r_1(z)}\\&-e^{r_1(z+\eta)-r_1(z)}-e^{r_2(z+\eta)-r_1(z)}\equiv 0.\end{aligned} \tag{3.34}$$

If $n\geq 2$, then $\deg(r_2(z)-r_2(z+\eta))=\deg(r_1(z+\eta)-r_1(z))=n-1\geq 1$. By $b_n=a_n$, we see that $\deg(r_2(z+i)-r_1(z+j))\leq n-1$ $(i,\,j=0,\,\eta)$. Further, if $\deg(r_2(z+i)-r_1(z+j))<n-1$ $(i,\,j=0,\,\eta)$, then (3.34) can be written as

$$\Bigg[\frac{i\beta P(z+\eta)+N(z)P(z)P(z+\eta)}{i\alpha P(z)}e^{r_2(z)-r_1(z)}-e^{r_2(z+\eta)-r_1(z)}$$

$$+\frac{i\beta P(z+\eta)+M(z)P(z)P(z+\eta)}{i\alpha P(z)}\Bigg] - e^{r_1(z+\eta)-r_1(z)} \equiv 0.$$

Thus, using Lemma 2.5, we have $-1 \equiv 0$, which is absurd; if $\deg(r_2(z+i) - r_1(z+j)) = n-1$ $(i, j = 0, \eta)$, then applying Lemma 2.5 to (3.34), we can also get a similar contradiction as above. Therefore, we have $n = 1$. Assume that $r_1(z) = az + a_0$, $r_2(z) = az + b_0$, where $a \neq 0$, a_0, b_0 are constants. It can be seen that $r(z)$ is a non-constant polynomial that satisfies the assumption.

In the following, since $r_1(z) = az + a_0$, $r_2(z) = az + b_0$, we get that $M(z) \equiv a^k$, $N(z) \equiv -a^k$. Substituting it into (3.34) yields

$$\begin{aligned}&(i\beta + i\beta e^{b_0-a_0})P(z+\eta) - i\alpha e^{a\eta}[1+e^{b_0-a_0}]P(z)\\ \equiv\ &-(a^k - a^k e^{b_0-a_0})P(z)P(z+\eta).\end{aligned} \tag{3.35}$$

Next we will divide our argument into two subcases respectively.

Subcase 3.3.2.1. When $b_0 \neq a_0 + 2n\pi i$ for some integer n, it follows by (3.35) that $P(z)$ must be constant, say $A(\neq 0)$. Then (3.35) can be rewritten as

$$a^k = \frac{(i\alpha e^{a\eta} - i\beta)(1+e^{b_0-a_0})}{A(1-e^{b_0-a_0})}. \tag{3.36}$$

From (3.13), we have

$$f^{(k)}(z) = \frac{e^{az+a_0}+e^{az+b_0}}{2A}, \quad f(z) = \frac{e^{az+a_0}}{2Aa^k} + \frac{e^{az+b_0}}{2Aa^k} + W(z), \tag{3.37}$$

where $W(z)$ is a polynomial with $\deg(W(z)) \leq k-1$.

If $\alpha = \beta$, then $a^k = \frac{i\alpha(e^{a\eta}-1)(1+e^{b_0-a_0})}{A(1-e^{b_0-a_0})}$. From (3.13), (3.36) and (3.37), we have $\alpha[W(z+\eta) - W(z)] \equiv 0$. By $\alpha \neq 0$, we get $W(z) \equiv d$, where d is a constant. Then,

$$f(z) = \frac{e^{az+a_0}}{2Aa^k} + \frac{e^{az+b_0}}{2Aa^k} + d.$$

If $\alpha = -\beta$, then $a^k = \frac{i\alpha(e^{a\eta}+1)(1+e^{b_0-a_0})}{A(1-e^{b_0-a_0})}$. In the same way, we have $W(z) \equiv 0$. Then,

$$f(z) = \frac{e^{az+a_0}}{2Aa^k} + \frac{e^{az+b_0}}{2Aa^k}.$$

If $\alpha \neq \pm\beta$, then $a^k = \frac{(i\alpha e^{a\eta}-i\beta)(1+e^{b_0-a_0})}{A(1-e^{b_0-a_0})}$. Similarly, we have $W(z) \equiv 0$. Then,

$$f(z) = \frac{e^{az+a_0}}{2Aa^k} + \frac{e^{az+b_0}}{2Aa^k}.$$

This belongs to Case (II) in Theorem 1.3.

Subcase 3.3.2.2. When $b_0 = a_0 + 2n\pi i$ for some integer n, (3.35) can be written as

$$2i\beta P(z+\eta) - 2i\alpha e^{a\eta}P(z) \equiv 0. \tag{3.38}$$

If $\alpha e^{a\eta} = -\beta$ or $\alpha e^{a\eta} \neq \pm\beta$, then by (3.38), it follows that $P(z) \equiv 0$, which contradicts that $P(z) \not\equiv 0$. This means that $\alpha e^{a\eta} = \beta$, that is, $e^{a\eta} = \frac{\beta}{\alpha}$ and thus we deduce from (3.38) that $P(z)$ must be constant, say A ($\neq 0$). It follows from (3.13) that

$$f^{(k)}(z) = \frac{e^{az+a_0}}{A}, \quad f(z) = \frac{e^{az+a_0}}{Aa^k} + V(z), \tag{3.39}$$

where $V(z)$ is a polynomial with $\deg(V(z)) \leq k-1$.

If $\alpha = \beta$, then $e^{a\eta} = 1$. Combining (3.13) with (3.39), we have $\alpha[V(z+\eta) - V(z)] \equiv 0$. By $\alpha \neq 0$, we get $V(z) \equiv d$, where d is a constant. Then,

$$f(z) = \frac{e^{az+a_0}}{Aa^k} + d.$$

If $\alpha \neq \beta$, then $e^{a\eta} \neq 1$. In the same way, we have $V(z) \equiv 0$. Then,

$$f(z) = \frac{e^{az+a_0}}{Aa^k}.$$

This belongs to Case (III) in Theorem 1.3. Hence we complete the proof of Theorem 1.3.

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