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ALGEBRAIC RICCI SOLITONS IN THE FINSLERIAN CASE

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Abstract. In this paper, we study algebraic Ricci solitons in the Finslerian case. We show that any simply connected Finslerian algebraic Ricci soliton is a Finslerian Ricci soliton. Furthermore, we study Randers algebraic Ricci solitons. It turns out that a shrinking, steady, or expanding Randers algebraic Ricci soliton with vanishing S-curvature is Einstein, locally Minkowskian, or Riemannian, respectively.

1. Introduction

The problem to find a distinguished metric on a smooth manifold is important in differential geometry. The Einstein metrics and Ricci soliton metrics are candidates [5]. A Ricci soliton g on a manifold M is a Riemannian metric satisfying

$$\mathrm{ric} = cg + L_X g,$$

where ric is the Ricci curvature of (M, g), $c \in \mathbb{R}$, and X is a smooth vector field on M [6]. In particular, they correspond to self-similar solutions of the famous Hamilton's Ricci flow

$$\frac{\partial}{\partial t} g(t)_{ij} = -2[\mathrm{ric}_{g(t)}]_{ij}.$$

That is, g is the initial value of a solution to the Ricci flow of the form $g(t) = c(t)\varphi_t^* g$, where $c(t)$ is a scaling parameter, and φ_t is a diffeomorphism of M. In 1982, Hamilton [10] showed that a closed three-manifold with positive Ricci curvature is diffeomorphic to S^3. Since then, the study of the Ricci flow has been one of the central problems in differential geometry. For instance, Perelman [18] proved Thurston's geometrization conjecture.

In this paper, we study algebraic Ricci solitons on homogeneous Finsler spaces. The concept of an algebraic Ricci soliton was first introduced by Lauret in the Riemannian case [15]. Lauret proved that left invariant Ricci solitons on

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homogenous nilmanifolds are algebraic Ricci solitons. In the literature, these algebraic Ricci solitons are called Ricci nilsolitons.

In the general homogeneous Riemannian case, (semi-)algebraic Ricci solitons have been studied extensively by Jablonski [12,13], defined as follows. Let G be a Lie group with a compact subgroup H, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$, where $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. We say a G-invariant Riemannian metric g on G/H is a G-semi-algebraic Ricci soliton if it's Ricci operator Ric satisfies

$$\mathrm{Ric} = c\mathrm{Id} + \frac{1}{2}(D_{\mathfrak{m}} + D_{\mathfrak{m}}^t) \tag{1}$$

for some derivation $D \in \mathrm{Der}(\mathfrak{g})$ satisfying $D(\mathfrak{h}) \subset \mathfrak{h}$. Here $D_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ is the map induced by D, and $D_{\mathfrak{m}}^t$ is the adjoint map of $D_{\mathfrak{m}}$ with respect to the inner product $g|_{\mathfrak{m}}$ induced by g on the tangent space $\mathfrak{m} = T_{eH}(G/H)$. Furthermore, it is said to be a G-algebraic Ricci soliton if $D_{\mathfrak{m}} = D_{\mathfrak{m}}^t$. Notice that the definition of (semi-)algebraic is relative to a choice of transitive group G. Jablonski proved the following important results.

(1) A connected homogeneous Ricci solition (M, g) is a semi-algebraic Ricci solition relative to the full isometry group $G = \mathrm{Iso}(M, g)$ [13].

(2) Every G-semi-algebraic Ricci solition is necessarily G-algebraic [12].

Now we turn to the Finslerian case. The concept of Ricci flow in Finsler geometry is first considered by Bao [2] in the following sense,

$$\frac{\partial}{\partial t} \ln F_t = -\frac{1}{F_t^2} \cdot \mathrm{ric}_{F_t}, \tag{2}$$

where ric_{F_t} denotes the Ricci curvature of Finsler space (M, F_t). As in the Riemannian case, we say F is a Finslerian Ricci soliton if there exists diffeomorphisms φ_t of M such that $F_t = c(t)\varphi_t^* F$ is a solution to the Ricci flow (2) starting at $F_0 = F$ for some scaling function $c(t) > 0$.

In the present work, we introduce algebraic Ricci solitons in the Finslerian case. We show that any simply connected Finslerian algebraic Ricci soliton is a Finslerian Ricci soliton. Furthermore, we study Randers algebraic Ricci solitons with vanishing S-curvature, and obtain a complete description of such metrics.

This paper is organized as follows. In Section 2, we recall some fundamental facts about Finsler spaces. In Section 3, we introduce algebraic Ricci solitons in the Finslerian case. In Section 4, we study Randers algebraic Ricci solitons.

2. Finsler spaces

In this section, we review some fundamental facts about Finsler spaces, for details, see [3]. Throughout this paper, manifolds are always assumed to be connected and smooth.

Recall that a Minkowski norm on V is a real function F on V which is smooth on $V\backslash\{0\}$ and satisfies the following conditions:

(1) $F(y) \geq 0, \forall y \in V$;
(2) $F(\lambda y) = \lambda F(y), \forall \lambda > 0$;
(3) For any basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of V, write $F(y) = F(y^1, y^2, \ldots, y^n)$ for $y = y^i \varepsilon_i$. Then the Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive-definite at any point of $V \backslash \{0\}$.

For any Minkowski norm F on a real vector space V, one defines

$$C_{ijk} = \frac{1}{4} [F^2]_{y^i y^j y^k}.$$

Then for any $y \neq 0$, we can define two tensors on V, namely, the fundamental tensor and the Cartan tensor. They are defined respectively as

$$g_y(u, v) = g_{ij}(y) u^i v^j,$$

$$C_y(u, v, w) = C_{ijk}(y) u^i v^j w^k.$$

By the homogeneity of F, one easily sees that

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(y + su + tv) \Big|_{s=t=0},$$

and that

$$C_y(u, v, w) = \frac{1}{4} \frac{\partial^3}{\partial r \partial s \partial t} F^2(y + ru + sv + tw) \Big|_{r=s=t=0}.$$

In particular, one has

$$F^2(y) = g_y(y, y), \quad C_y(y, u, v) = 0,$$

and

$$\frac{d}{dt} F^2(y + tu) \Big|_{t=0} = 2 g_y(y, u).$$

A Finsler metric on an n-dimensional smooth manifold M is a function $F : TM \to [0, +\infty)$ which is C^∞ on the slit tangent bundle $TM \backslash \{0\}$ and whose restriction to any tangent space $T_x M$, $x \in M$ is a Minkowski norm.

The notion of Riemann curvature for Riemannian metrics can be extended to Finsler metrics. For a nonzero vector $y \in T_x M \backslash \{0\}$, the Riemann curvature $R_y : T_x M \to T_x M$ is a linear map defined by

$$R_y(u) = R^i_k(y) u^k \frac{\partial}{\partial x^i}, \quad u = u^i \frac{\partial}{\partial x^i},$$

where

$$R^i_k(y) = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2 G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i \partial G^j}{\partial y^j \partial y^k},$$

$$G^i(y) = \frac{1}{4} g^{il} \left[\frac{\partial^2 (F^2)}{\partial x^k \partial y^l} y^k - \frac{\partial (F^2)}{\partial x^l} \right],$$

and (g^{ij}) is the inverse matrix of (g_{ij}). The trace of the Riemann curvature R_y is a scalar function ric on TM

$$\mathrm{ric}(y) = \mathrm{tr}(R_y),$$

which is called the Ricci curvature of (M, F). We say that (M, F) is Einstein if $\mathrm{ric}(y) = cF^2(y)$ for some constant c.

We now recall the notion of S-curvature of a Finsler space. It is a quantity to measure the rate of change of the volume form of a Finsler space along geodesics [20]. S-curvature is a non-Riemannian quantity, or in other words, any Riemannian manifold has vanishing S-curvature. Let V be an n-dimensional real vector space and F be a Minkowski norm on V. For a basis $\{v_i\}$ of V, let

$$\sigma_F = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n \mid F(y^i v_i) < 1\}},$$

where Vol means the volume of a subset in the standard Euclidean space $\mathbb{R}^n$ and B^n is the open ball of radius 1. This quantity is generally dependent on the choice of the basis $\{v_i\}$. But it is easily seen that

$$\tau(y) = \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma_F}, \quad y \in V \backslash \{0\}$$

is independent of the choice of the basis. The quantity $\tau = \tau(y)$ is called the distortion of (V, F). For any $y \in T_x M \backslash \{0\}$, let $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Then the quantity

$$S(x, y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]\Big|_{t=0}$$

is called the S-curvature of the Finsler space (M, F).

3. Algebraic Ricci solitons

Recall that the group of isometries of a Finsler space (M, F) is a Lie transformation group of M [8]. A Finsler space (M, F) is called homogeneous if its isometry group acts transitively on M. A homogeneous Finsler space can be expressed as $(G/H, F)$, where G is a connected Lie group, H is a compact subgroup of G and F is invariant under the action of G. Moveover, the action of G on G/H is almost effective and the Lie algebra $\mathfrak{g}$ of G has a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ satisfying $\mathrm{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m}, \forall h \in H$. We identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ of G/H at the origin $o = eH$ through the mapping $X \mapsto \frac{d}{dt}(\exp(tX)H)|_{t=0}$, $X \in \mathfrak{m}$. Under this identification, G-invariant Finsler metric on G/H is in one-to-one correspondence with H-invariant Minkowski norm on $\mathfrak{m}$. See [7] for more information on invariant Finsler metrics.

Definition. A homogeneous Finsler space $(G/H, F)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is said to be a (Finslerian) G-algebraic Ricci soliton if there exist $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{h} \subset \mathfrak{h}$ and

$$\mathrm{ric}(y) = cF^2(y) + g_y(y, D_{\mathfrak{m}} y), \quad \forall y \in \mathfrak{m}, \tag{3}$$

where $D_{\mathfrak{m}} := \mathrm{pr} \circ D|_{\mathfrak{m}}$ and $\mathrm{pr} : \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \longrightarrow \mathfrak{m}$ is the linear projection.

Moreover, it is called a shrinking, steady, or expanding algebraic Ricci soliton according to $c > 0$, $c = 0$, $c < 0$, respectively.

We should mention that (see Lemma 3.10 of [14]), if in addition that $K(\mathfrak{h}, \mathfrak{m}) = 0$, then $D\mathfrak{m} \subset \mathfrak{m}$, and thus $D_{\mathfrak{m}} = D|_{\mathfrak{m}}$, where K is the Killing form of $\mathfrak{g}$. When F is Riemannian, equation (3) is clearly equivalent to (1).

Definition. A homogeneous Ricci soliton Finsler space (M, F) is called algebraic if there exists a connected Lie subgroup $G \subset \mathrm{Iso}(M, F)$ acting transitively on M such that F is a G-algebraic Ricci soliton.

In the Riemannian case, any simply connected algebraic Ricci soliton is a Ricci soliton. Inspired by the proof of Proposition 3.3 of [14], we obtain:

Proposition 3.1. *Any simply connected Finslerian algebraic Ricci soliton is a Finslerian Ricci soliton.*

Proof. We can assume that G is simply connected and still have that G/H is almost effective. Notice that H is therefore connected as G/H is simply connected. Since $D \in \mathrm{Der}(\mathfrak{g})$ we have that $e^{tD} \in \mathrm{Aut}(\mathfrak{g})$ and thus there exists $\widetilde{\varphi_t} \in \mathrm{Aut}(G)$ such that $d\widetilde{\varphi_t}|_e = e^{tD}$ for all $t \in \mathbb{R}$. By using that H is connected and $D\mathfrak{h} \subset \mathfrak{h}$, it is easy to see that $\widetilde{\varphi_t}(H) = H$ for all t. This implies that $\widetilde{\varphi_t}$ defines a diffeomorphism of $M = G/H$ by $\varphi_t(gH) = \widetilde{\varphi_t}(g)H$ for any $g \in G$, which therefore satisfies at the origin that $d\varphi_t|_o = e^{tD_{\mathfrak{m}}}$. Set $c(t) = \sqrt{1 - 2ct}$ and $s(t) = \frac{1}{2c} \ln(1 - 2ct)$, we will show that $F_t = c(t)\varphi^*_{s(t)} F$ satisfies the Ricci flow equation (2). Obviously, F_t is G-invariant for all t, and hence it is sufficient to verify the equation (2) at the origin point.

Note that, for any $y \in \mathfrak{m}$, $(\varphi^*_t F)(y) = F(e^{tD_{\mathfrak{m}}} y)$, and then

$$\begin{aligned}
(\frac{\partial}{\partial t}\varphi^*_{s(t)} F)(y) &= \frac{\partial}{\partial t} F(e^{s(t)D_{\mathfrak{m}}} y) \\
&= \frac{1}{2F(e^{s(t)D_{\mathfrak{m}}} y)} \cdot \frac{\partial}{\partial t} g_{e^{s(t)D_{\mathfrak{m}}} y}(e^{s(t)D_{\mathfrak{m}}} y, e^{s(t)D_{\mathfrak{m}}} y) \\
&= \frac{1}{F(e^{s(t)D_{\mathfrak{m}}} y)} \cdot g_{e^{s(t)D_{\mathfrak{m}}} y}(s'(t) D_{\mathfrak{m}} e^{s(t)D_{\mathfrak{m}}} y, e^{s(t)D_{\mathfrak{m}}} y) \\
&= \frac{s'(t)}{F(e^{s(t)D_{\mathfrak{m}}} y)} \cdot [\mathrm{ric}(e^{s(t)D_{\mathfrak{m}}} y) - cF^2(e^{s(t)D_{\mathfrak{m}}} y)].
\end{aligned}$$

Notice that $e^{s(t)D_{\mathfrak{m}}} \in \mathrm{Aut}(G)$ for all t, we have

$$\mathrm{ric}_{F_t}(y) = \mathrm{ric}_{\varphi^*_{s(t)} F}(y) = \mathrm{ric}(d\varphi_{s(t)} y) = \mathrm{ric}(e^{s(t)D_{\mathfrak{m}}} y).$$

Therefore

$$\frac{\partial}{\partial t}\varphi^*_{s(t)}F = \frac{c(t)s'(t)}{F_t}\mathrm{ric}_{F_t} - \frac{cs'(t)}{c(t)}F_t.$$

Notice that $\frac{c'(t)}{c(t)} - cs'(t) = 0$ and $c^2(t)s'(t) = -1$, it follows that

$$\begin{aligned}\frac{\partial}{\partial t}\ln F_t &= \frac{1}{F_t}\cdot\frac{\partial}{\partial t}F_t\\ &= \frac{1}{F_t}\left[\frac{c'(t)}{c(t)}F_t + c(t)\frac{\partial}{\partial t}\varphi^*_{s(t)}F\right]\\ &= \frac{c'(t)}{c(t)} - cs'(t) + \frac{c^2(t)s'(t)}{F_t^2}\mathrm{ric}_{F_t}\\ &= -\frac{1}{F_t^2}\mathrm{ric}_{F_t}.\end{aligned}$$

This completes the proof of the proposition. □

4. Randers algebraic Ricci solitons

In this section, we study Randers algebraic Ricci solitons with vanishing S-curvature. We first recall some results on invariant Randers metrics. Randers metrics were introduced by G. Randers in 1941, in the context of general relativity. They are Finsler metrics of the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a smooth 1-form on M whose length with respect to α is everywhere less than 1. In the homogeneous case, both the Riemannian metric α and 1-form β are invariant under the action of G. Then the Randers metric F is uniquely determined by a pair $(\langle\cdot,\cdot\rangle, u)$, where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathfrak{m}$ induced by the Riemannian metric α, and u is an H-fixed vector in $\mathfrak{m}$ with length less than 1, satisfying $\beta(y) = \langle y, u\rangle$, $\forall y \in \mathfrak{m}$. In this case, $F(y) = \sqrt{\langle y,y\rangle} + \langle y,u\rangle$.

There is another presentation of a Randers metric, by the so-called navigation data

$$F(x,y) = \frac{\sqrt{h(y,W)^2 + \lambda h(y,y)}}{\lambda} - \frac{h(y,W)}{\lambda},$$

where h is a Riemannian metric, W is a vector field on M with $h(W,W) < 1$ and $\lambda = 1 - h(W,W)$. The pair (h,W) is called the navigation data of the Randers metric F. This version of a Randers metric is convenient when handling some problems concerning the flag curvature and Ricci curvature [4].

Lemma 4.1 (Theorem 1.4 of [11]). *Let $(G/H, F = \alpha + \beta)$ be a homogeneous Randers space with navigation data (h,W). Then F has vanishing S-curvature if and only if W is a Killing vector field with respect to h, if and only if*

$$\langle [u,x]_{\mathfrak{m}}, y\rangle + \langle x, [u,y]_{\mathfrak{m}}\rangle = 0, \quad \forall x,y \in \mathfrak{m}.$$

Theorem 4.2 (Theorem 3.4 of [9]). *Let G/H, F, α, $\mathfrak{m}$, u, be as above. Then the Ricci curvature of the homogeneous Randers metric is given by*

$$\begin{aligned}
\mathrm{ric}(y) = \mathrm{ric}_\alpha(y,y) &- |y|\langle [Z,y]_{\mathfrak{m}}, u\rangle + \frac{1}{2}|y|\sum_{k,l}\langle [u_k,u_l]_{\mathfrak{m}}, u\rangle\langle [u_k,u_l]_{\mathfrak{m}}, y\rangle \\
&+ \frac{1}{4}|y|^2\sum_{k,l}\langle [u_k,u_l]_{\mathfrak{m}}, u\rangle^2 - \frac{(n-1)|y|}{F(y)}\sum_l \langle U(u_l,y),u\rangle\langle [u_l,y]_{\mathfrak{m}}, u\rangle \\
&+ \frac{1}{2}\sum_l \langle [y,u_l]_{\mathfrak{m}}, u\rangle^2 - \frac{n-1}{F(y)}\langle U(U(y,y),y),u\rangle \\
&+ \frac{(n-1)|y|}{2F(y)}\langle U(u,u),U(y,y)\rangle + \frac{(n-1)|y|^2}{2F(y)}\langle [U(u,u),y]_{\mathfrak{m}}, u\rangle \\
&+ \frac{3(n-1)}{4F^2(y)}\Big(\langle U(y,y),u\rangle - |y|\langle U(u,u),y\rangle\Big)^2,
\end{aligned}$$

where y is a non-zero vector in $\mathfrak{m}$, $|y| = \sqrt{\langle y,y\rangle}$, $\{u_l\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\cdot,\cdot\rangle$, $U : \mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$ is a bilinear form defined by

$$2\langle U(x,y), z\rangle = \langle [z,x]_{\mathfrak{m}}, y\rangle + \langle x, [z,y]_{\mathfrak{m}}\rangle, \quad \forall z \in \mathfrak{m}.$$

Moreover, ric_α is the Ricci curvature of $(G/H,\alpha)$ given by

$$\begin{aligned}
\mathrm{ric}_\alpha(y,y) = &-\frac{1}{2}K(y,y) - \frac{1}{2}\sum_l \langle [y,u_l]_{\mathfrak{m}}, [y,u_l]_{\mathfrak{m}}\rangle \\
&+ \frac{1}{4}\sum_{k,l}\langle y, [u_k,u_l]_{\mathfrak{m}}\rangle^2 - \langle [Z,y]_{\mathfrak{m}}, y\rangle,
\end{aligned}$$

K is the Killing form of $\mathfrak{g}$, and Z is the unique vector in $\mathfrak{m}$ defined by $\langle Z,y\rangle =$ $\mathrm{tr}\,\mathrm{ad}\,y$, *$\forall y \in \mathfrak{m}$.*

Combining the above results, we have:

Lemma 4.3. *A Randers metric $F = \alpha+\beta$ on G/H with vanishing S-curvature is a G-algebraic Ricci soliton if and only if the following two equations hold*:

$$\begin{aligned}
\mathrm{ric}_\alpha(y,y) = {}& c(\langle y,y\rangle + \langle y,u\rangle^2) + \langle y, D_{\mathfrak{m}}y\rangle \\
&- \frac{1}{4}\langle y,y\rangle\sum_{k,l}\langle [u_k,u_l]_{\mathfrak{m}}, u\rangle^2 - \frac{1}{2}\sum_l \langle [y,u_l]_{\mathfrak{m}}, u\rangle^2,
\end{aligned} \tag{4}$$

$$\begin{aligned}
\langle D_{\mathfrak{m}}y, u\rangle = &-\langle [Z,y]_{\mathfrak{m}}, u\rangle + \frac{1}{2}\sum_{k,l}\langle [u_k,u_l]_{\mathfrak{m}}, u\rangle\langle [u_k,u_l]_{\mathfrak{m}}, y\rangle \\
&- 2c\langle y,u\rangle, \quad \forall y \in \mathfrak{m}.
\end{aligned} \tag{5}$$

Proof. Since F has vanishing S-curvature, by Lemma 4.1, one has $U(u,u) = 0$ and

$$\langle U(x,y), u\rangle = 0, \quad \forall x,y \in \mathfrak{m}.$$

Therefore the Ricci curvature of F can be written as

$$\operatorname{ric}(y)=\operatorname{ric}_\alpha(y,y)-|y|\langle [Z,y]_{\mathfrak m},u\rangle+\frac{1}{2}|y|\sum_{k,l}\langle [u_k,u_l]_{\mathfrak m},u\rangle\langle [u_k,u_l]_{\mathfrak m},y\rangle$$
$$+\frac{1}{4}|y|^2\sum_{k,l}\langle [u_k,u_l]_{\mathfrak m},u\rangle^2+\frac{1}{2}\sum_{l}\langle [y,u_l]_{\mathfrak m},u\rangle^2,\quad y\in\mathfrak m. \tag{6}$$

On the other hand, an easy computation shows that (see Lemma 6.5 of [21])

$$g_y(y,D_{\mathfrak m}y)=\langle y+|y|u,D_{\mathfrak m}y\rangle. \tag{7}$$

Plugging (7) into (3), we obtain

$$\operatorname{ric}(y)=c(|y|+\langle y,u\rangle)^2+\langle y+|y|u,D_{\mathfrak m}y\rangle. \tag{8}$$

Now it is easy to see that (6) and (8) is equivalent to (4) and (5). $\square$

Proposition 4.4. *Let $(G/H,F=\alpha+\beta)$ be a Randers G-algebraic Ricci soliton with navigation data (h,W). If F has vanishing S-curvature, then h is a G-algebraic Ricci soliton on G/H.*

Proof. Let $\overline{\alpha}$ be the invariant Riemannian metric on G/H defined by

$$\overline{\alpha}(X,Y)=\langle X,Y\rangle-\langle X,u\rangle\langle Y,u\rangle,\quad X,Y\in\mathfrak m.$$

It is easy to see that $h=(1-\langle u,u\rangle)\overline{\alpha}$. The following is to prove that $\overline{\alpha}$ is a G-algebraic Ricci soliton on G/H by a direct computation.

Putting $y=u$ into (4) and (5), we obtain

$$\operatorname{ric}_\alpha(u,u)=c(\langle u,u\rangle+\langle u,u\rangle^2)+\langle u,D_{\mathfrak m}u\rangle-\frac{1}{4}\langle u,u\rangle\sum_{k,l}\langle [u_k,u_l]_{\mathfrak m},u\rangle^2,$$

and

$$\langle D_{\mathfrak m}u,u\rangle=\frac{1}{2}\sum_{k,l}\langle [u_k,u_l]_{\mathfrak m},u\rangle^2-2c\langle u,u\rangle.$$

Therefore,

$$\operatorname{ric}_\alpha(u,u)=c\langle u,u\rangle(\langle u,u\rangle-1)+(\frac{1}{2}-\frac{1}{4}\langle u,u\rangle)\sum_{k,l}\langle [u_k,u_l]_{\mathfrak m},u\rangle^2.$$

On the other hand, $\operatorname{ric}_\alpha(u,u)=\frac{1}{4}\sum_{k,l}\langle [u_k,u_l]_{\mathfrak m},u\rangle^2$, then one has

$$(-\frac{1}{4}+\frac{1}{4}\langle u,u\rangle)\sum_{k,l}\langle [u_k,u_l]_{\mathfrak m},u\rangle^2=c\langle u,u\rangle(\langle u,u\rangle-1).$$

This implies that

$$\sum_{k,l}\langle [u_k,u_l]_{\mathfrak m},u\rangle^2=4c\langle u,u\rangle. \tag{9}$$

In particular, $\langle D_{\mathfrak m}u,u\rangle=0$.

Now let $\mathrm{ric}_{\overline{\alpha}}$ be the Ricci curvature of $(G/H,\overline{\alpha})$. Then for any $y\in\mathfrak{m}$,

$$\begin{aligned}
\mathrm{ric}_{\overline{\alpha}}(y,y) = & -\frac{1}{2}K(y,y)-\frac{1}{2}\sum_l \overline{\alpha}([y,u_l]_{\mathfrak{m}},[y,u_l]_{\mathfrak{m}})\\
&+\frac{1}{4}\sum_{k,l}\overline{\alpha}(y,[u_k,u_l]_{\mathfrak{m}})^2-\overline{\alpha}([Z,y]_{\mathfrak{m}},y)\\
= & -\frac{1}{2}K(y,y)-\frac{1}{2}\sum_l\langle[y,u_l]_{\mathfrak{m}},[y,u_l]_{\mathfrak{m}}\rangle+\frac{1}{2}\sum_l\langle[y,u_l]_{\mathfrak{m}},u\rangle^2\\
&+\frac{1}{4}\sum_{k,l}\Big(\langle y,[u_k,u_l]_{\mathfrak{m}}\rangle-\langle y,u\rangle\langle[u_k,u_l]_{\mathfrak{m}},u\rangle\Big)^2\\
&-\langle[Z,y]_{\mathfrak{m}},y\rangle+\langle[Z,y]_{\mathfrak{m}},u\rangle\langle y,u\rangle\\
=&\ \mathrm{ric}_\alpha(y,y)+\frac{1}{2}\sum_l\langle[y,u_l]_{\mathfrak{m}},u\rangle^2+\frac{1}{4}\langle y,u\rangle^2\sum_{k,l}\langle[u_k,u_l]_{\mathfrak{m}},u\rangle^2\\
&-\frac{1}{2}\langle y,u\rangle\sum_{k,l}\langle y,[u_k,u_l]_{\mathfrak{m}}\rangle\langle[u_k,u_l]_{\mathfrak{m}},u\rangle+\langle[Z,y]_{\mathfrak{m}},u\rangle\langle y,u\rangle.
\end{aligned}$$

Furthermore, according to (4), (5) and (9), we have

$$\mathrm{ric}_\alpha(y,y)+\frac{1}{2}\sum_l\langle[y,u_l]_{\mathfrak{m}},u\rangle^2=c(\langle y,y\rangle+\langle y,u\rangle^2)+\langle y,D_{\mathfrak{m}}y\rangle-c\langle y,y\rangle\langle u,u\rangle,$$

and

$$-\frac{1}{2}\sum_{k,l}\langle y,[u_k,u_l]_{\mathfrak{m}}\rangle\langle[u_k,u_l]_{\mathfrak{m}},u\rangle+\langle[Z,y]_{\mathfrak{m}},u\rangle=-\langle D_{\mathfrak{m}}y,u\rangle-2c\langle y,u\rangle.$$

Therefore

$$\begin{aligned}
\mathrm{ric}_{\overline{\alpha}}(y,y)&=c(\langle y,y\rangle+\langle y,u\rangle^2)+\langle y,D_{\mathfrak{m}}y\rangle-c\langle y,y\rangle\langle u,u\rangle\\
&\quad+c\langle y,u\rangle^2\langle u,u\rangle+\langle y,u\rangle\Big(-\langle D_{\mathfrak{m}}y,u\rangle-2c\langle y,u\rangle\Big)\\
&=c(1-\langle u,u\rangle)\langle y,y\rangle+c(\langle u,u\rangle-1)\langle y,u\rangle^2\\
&\quad+\langle y,D_{\mathfrak{m}}y\rangle-\langle y,u\rangle\langle D_{\mathfrak{m}}y,u\rangle\\
&=c(1-\langle u,u\rangle)\overline{\alpha}(y,y)+\overline{\alpha}(y,D_{\mathfrak{m}}y).
\end{aligned}\tag{10}$$

This asserts that $\overline{\alpha}$ is a G-semi-algebraic Ricci soliton on G/H. Now according to a deep result of [12], every G-semi-algebraic Ricci soliton is necessarily G-algebraic. □

Now we can state the main result in this paper.

Theorem 4.5. *Let (M,F) be a Randers algebraic Ricci soliton with vanishing S-curvature.*

(1) *If (M,F) is shrinking, then it is Einstein.*
(2) *If (M,F) is steady, then it is locally Minkowskian.*

(3) *If (M, F) is expanding, then it is Riemannian.*

Proof. Assume G is a Lie subgroup of $\mathrm{Iso}(M, F)$ acting transitively on M, H is the isotropy subgroup of G at a point $p \in M$, and $F = \alpha + \beta$ is a G-algebraic Ricci soliton satisfying (3).

If (M, F) is expanding, that is $c < 0$. Then it follows from (9) that $u = 0$, which implies that F is Riemnannian.

If (M, F) is shrinking or steady, then by (10), the homogeneous Riemannian Ricci soliton (M, h) is also shrinking or steady. It follows that (M, h) is in fact Einstein [17, 19]. That is $D_{\mathfrak{m}} = 0$, F is Einstein. In particular, if $c = 0$, (M, h) is Ricci flat, then it is flat [1]. By a result of [16], (M, F) is flat. Finally, according to a result of Akbar-Zadeh (see Theorem 12.4.1 of [3]), (M, F) is locally Minkowskian. □

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