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THREE RESULTS ON TRANSCENDENTAL MEROMORPHIC SOLUTIONS OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the transcendental meromorphic solutions for the nonlinear differential equations: $f^n + P(f) = R(z)e^{\alpha(z)}$ and $f^n + P_*(f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}$ in the complex plane, where $P(f)$ and $P_*(f)$ are differential polynomials in f of degree $n-1$ with coefficients being small functions and rational functions respectively, R is a non-vanishing small function of f, α is a nonconstant entire function, p_1, p_2 are non-vanishing rational functions, and α_1, α_2 are nonconstant polynomials. Particularly, we consider the solutions of the second equation when p_1, p_2 are nonzero constants, and $\deg \alpha_1 = \deg \alpha_2 = 1$. Our results are improvements and complements of Liao ([9]), and Rong-Xu ([11]), etc., which partially answer a question proposed by Li ([7]).

1. Introduction

Let $f(z)$ be a transcendental meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations and main results in Nevanlinna theory (see [4, 6, 12]). Throughout this paper, the term $S(r, f)$ always has the property that $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set E (which is not necessarily the same at each occurrence) of finite linear measure. A meromorphic function $a(z)$ is said to be a small function with respect to $f(z)$ if and only if $T(r, a) = S(r, f)$. In addition, $N_{1)}(r, 1/f)$ and $N_{(2}(r, 1/f)$ are used to denote the counting functions corresponding to simple and multiple zeros of f, respectively.

In the past few decades, many scholars, see [7–10] etc., focus on the solutions of the nonlinear differential equations of the form

$$f^n + P(f) = h, \tag{1}$$

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where $P(f)$ denotes a differential polynomial in f of degree at most $n-2$, and h is a given meromorphic function.

In 2015, Liao [9] investigated the forms of meromorphic solutions of the equation (1) for specific h, and obtained the following result.

Theorem A. *Let $n \geq 2$ and $P(f)$ be a differential polynomial in f of degree d with rational functions as its coefficients. Suppose that p is a non-zero rational function, α is a non-constant polynomial and $d \leq n-2$. If the following differential equation*

$$f^n + P(f) = p(z)e^{\alpha(z)} \tag{2}$$

admits a meromorphic function f with finitely many poles, then f has the following form $f(z) = q(z)e^{r(z)}$ and $P(f) \equiv 0$, where $q(z)$ is a rational function and $r(z)$ is a polynomial with $q^n = p, nr(z) = \alpha(z)$. In particular, if p is a polynomial, then q is a polynomial, too.

If the condition $d \leq n-2$ is omitted, then the conclusions in Theorem A can not hold. For example, $f_0(z) = e^z - 1$ is a solution of the equation $f^2 + f' + f = e^{2z}$, here $n = 2$ and $d = 1 = n-1$. So it is natural to ask what will happen to the solutions of the equation (2) when $d = n-1$? In this paper, we study this problem and obtain the following result, which is a complement of Theorem A.

Theorem 1.1. *Let $n \geq 2$ be an integer and $P(f)$ be a differential polynomial in f of degree $n-1$ with coefficients being small functions. Then for any entire function α and any small function R, if the equation*

$$f^n + P(f) = R(z)e^{\alpha(z)} \tag{3}$$

possesses a meromorphic solution f with $N(r, f) = S(r, f)$, then f has the following form:

$$f(z) = s(z)e^{\alpha(z)/n} + \gamma(z),$$

where s and γ are small functions of f with $s^n = R$.

The following Example 1 shows that the case in Theorem 1.1 occurs.

Example 1. $f_0 = e^z + 1$ is a solution of the following equation

$$f^3 - 2ff' - (f')^2 - f = e^{3z}.$$

Here, $P(f) = -2ff' - (f')^2 - f$, $n = 3$, and $\deg P(f) = 2 = n-1$.

In 2011, Li [7] considered to find all entire solutions of the equation (1) for $h = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}$, where α_1 and α_2 are distinct constants, and obtained the following result.

Theorem B. *Let $n \geq 2$ be an integer, $P(f)$ be a differential polynomial in f of degree at most $n-2$ and α_1, α_2, p_1, p_2 be nonzero constants satisfying $\alpha_1 \neq \alpha_2$. If f is a transcendental meromorphic solution of the following equation*

$$f^n(z) + P(f) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z} \tag{4}$$

satisfying $N(r, f) = S(r, f)$, then one of the following relations holds:

(1) $f = c_0 + c_1 e^{\frac{\alpha_1 z}{n}}$;
(2) $f = c_0 + c_2 e^{\frac{\alpha_2 z}{n}}$;
(3) $f = c_1 e^{\frac{\alpha_1 z}{n}} + c_2 e^{\frac{\alpha_2 z}{n}}$ *and* $\alpha_1 + \alpha_2 = 0$,

where $c_0(z)$ *is a small function of* f *and constants* c_1 *and* c_2 *satisfy* $c_1^n = p_1$ *and* $c_2^n = p_2$*, respectively.*

For further study, Li [7] proposed the following question:

Question 1. *How to find the solutions of the equation* (4) *under the condition* $\deg P(f) = n - 1$*?*

For the case $\alpha_2 = -\alpha_1$, Li [7] has already given the detailed forms of the entire solutions of the equation (4) when $\deg P(f) = n-1$; For the case $\alpha_2 = \alpha_1$, (4) can be reduced to $f^n + P(f) = (p_1 + p_2)e^{\alpha_1 z}$, then we can get the forms of entire solutions by using Theorem 1.1. So it's natural to ask: what will happen when $\alpha_2 \pm \alpha_1 \neq 0$.

Chen and Gao [2] studied the above question, and obtained the following result.

Theorem C. *Let* $a(z)$ *be a nonzero polynomial and* $p_1, p_2, \alpha_1, \alpha_2$ *be nonzero constants such that* $\alpha_1 \neq \alpha_2$*. Suppose that* $f(z)$ *is a transcendental entire solution of finite order of the differential equation*

$$f^2(z) + a(z)f'(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} \tag{5}$$

satisfying $N(r, 1/f) = S(r, f)$*, then* $a(z)$ *must be a constant and one of the following relations holds*:

(1) $f = c_1 e^{\frac{\alpha_1 z}{2}}$, $ac_1\alpha_1 = 2p_2$ *and* $\alpha_1 = 2\alpha_2$;
(2) $f = c_2 e^{\frac{\alpha_2 z}{2}}$, $ac_2\alpha_2 = 2p_1$ *and* $\alpha_2 = 2\alpha_1$,

where c_1 *and* c_2 *are constants satisfying* $c_1^2 = p_1$ *and* $c_2^2 = p_2$*, respectively.*

Later, Rong and Xu [11] improved Theorem C by removing the condition that $f(z)$ is a finite-order function. In [11], they also considered the general case in Question 1, and obtained the following result.

Theorem D. *Let* $n \geq 2$ *be an integer. Suppose that* $P(f)$ *is a differential polynomial in* $f(z)$ *of degree* $n - 1$ *and that* α_1, α_2, p_1 *and* p_2 *are nonzero constants such that* $\alpha_1 \neq \alpha_2$*. If* $f(z)$ *is a transcendental meromorphic solution of the differential equation* (4) *satisfying* $N(r, f) = S(r, f)$*, then* $\rho(f) = 1$ *and one of the following relations holds*:

(1) $f(z) = c_1 e^{\frac{\alpha_1 z}{n}}$ *and* $c_1^n = p_1$;
(2) $f(z) = c_2 e^{\frac{\alpha_2 z}{n}}$ *and* $c_2^n = p_2$*, where* c_1 *and* c_2 *are constants*;
(3) $T(r, f) \leq N_{1)}(r, 1/f) + T(r, \varphi) + S(r, f)$*, where* $\varphi\,(\not\equiv 0)$ *is equal to* $\alpha_1\alpha_2 f^2 - n(\alpha_1 + \alpha_2)ff' + n(n-1)(f')^2 + nff''$.

In this paper, we go on investigating Question 1 and obtain the following results, which are improvements of Theorems C and D.

Theorem 1.2. *Let $n \geq 2$ be an integer. Suppose that $P_*(f)$ is a differential polynomial in $f(z)$ of degree $n-1$ and with rational functions as its coefficients, α_1, α_2 be nonconstant polynomials, and p_1, p_2 be non-vanishing rational functions. If $f(z)$ is a transcendental meromorphic solution of the following nonlinear differential equation*

$$f^n(z) + P_*(f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \tag{6}$$

with $\lambda_f = \max\{\lambda(f), \lambda(1/f)\} < \sigma(f)$, then $\sigma(f) = \deg \alpha_1 = \deg \alpha_2$, and one of the following relations holds:

(I) $\alpha_2' = \alpha_1'$. *In this case, $f = s_1(z)\exp(\alpha_1(z)/n) = s_2(z)\exp(\alpha_2(z)/n)$, where s_1 and s_2 are rational functions satisfying $s_1^n = p_1 + p_2 c_2$ and $s_2^n = \frac{1}{c_2}p_1 + p_2$, $c_2 = e^{\alpha_2 - \alpha_1}$ is a non-zero constant*;

(II) $k_1\alpha_1' = n\alpha_2'$, *where k_1 is an integer satisfying $1 \leq k_1 \leq n-1$. In this case, $f(z) = s_3(z)e^{\frac{\alpha_1(z)}{n}}$, where s_3 is a rational function satisfying $s_3^n = p_1$*;

(III) $k_2\alpha_2' = n\alpha_1'$, *where k_2 is an integer satisfying $1 \leq k_2 \leq n-1$. In this case, $f(z) = s_4(z)e^{\frac{\alpha_2(z)}{n}}$, where s_4 is a rational function satisfying $s_4^n = p_2$.*

Theorem 1.3. *Let $n \geq 2$ be an integer. Suppose that $P_*(f)$ is a differential polynomial in $f(z)$ of degree $n-1$ with rational functions as its coefficients, $\alpha_1, \alpha_2, p_1, p_2$ be nonzero constants such that $\alpha_1 \pm \alpha_2 \neq 0$. If $f(z)$ is an transcendental meromorphic solution of the following nonlinear differential equation*

$$f^n(z) + P_*(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{7}$$

satisfying $N(r, f) = S(r, f)$, then $\sigma(f) = 1$ and there exist two cases:

(I) $N\left(r, \frac{1}{f}\right) = S(r, f)$, *then one of the following relations holds*: (a) $k_1\alpha_1 = n\alpha_2$ *and* $f = s_1\exp(\alpha_1 z/n)$; (b) $k_2\alpha_2 = n\alpha_1$ *and* $f = s_2\exp(\alpha_2 z/n)$, *where k_1, k_2 are integers satisfying $1 \leq k_1, k_2 \leq n-1$, s_1, s_2 are constants with $s_1^n = p_1$ and $s_2^n = p_2$*;

(II) $N\left(r, \frac{1}{f}\right) \neq S(r, f)$, *then* $T(r, f) \leq N_{1)}\left(r, \frac{1}{f}\right) + \frac{1}{2}T(r, \varphi) + \frac{1}{2}N\left(r, \frac{1}{\varphi}\right) + S(r, f)$, *where* $\varphi = \alpha_1\alpha_2 f^2 - n(\alpha_1 + \alpha_2)ff' + n(n-1)(f')^2 + nff'' \not\equiv 0$, *and* (1) *if φ is a nonzero constant, then $f(z) = c_1 e^{\frac{\alpha_1+\alpha_2}{2n-1}z} + c_2$, where c_1, c_2 are nonzero constants, and one of the following relations holds*: (a) $(n-1)\alpha_1 = n\alpha_2$ *and* $f(z) = c_1 e^{\alpha_1 z/n} - c_2$ $(c_1^n = p_1)$; (b) $(n-1)\alpha_2 = n\alpha_1$, *and* $f(z) = c_1 e^{\alpha_2 z/n} - c_2, (c_1^n = p_2)$; (2) *if φ is a nonconstant meromorphic function, then $T(r, \varphi) \neq S(r, f)$. Particularly, suppose $n = 2$ and $\varphi = P(z)e^{Q(z)}$, where P and Q are non-vanishing polynomials such that* $\deg Q \geq 1$. *Then we have* $\deg Q = 1$ *and $f^2 = d_1 e^{\alpha_1 z} + d_2 e^{\alpha_2 z} - R(z)e^{Q(z)}$, where d_1, d_2 are constants, and R is a non-vanishing polynomial with* $\deg R \leq \deg P + 2$.

The following Examples 2 and 3 are shown to illustrate the cases (II)(1) and (II)(2) of Theorem 1.3.

Example 2. $f_0 = e^z - 1$ is a solution of the equation

$$f^2 + 2f' + f = e^{2z} + e^z.$$

Here $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_1 = 2\alpha_2$ and $\varphi = 2$. It implies that case (II)(1)(a) occurs.

Example 3. $f_0 = e^{2z} + e^z$ is a solution of

$$f^2 + \frac{1}{2}f' - \frac{1}{2}f'' = e^{4z} + 2e^{3z}.$$

Here $\alpha_1 = 4$, $\alpha_2 = 3$, $n = 2$, $\varphi = 2e^{2z}$, and $f_0^2 = e^{4z} + 2e^{3z} + e^{2z}$. It implies that case (II)(2) occurs.

2. Preliminary lemmas

The following lemma plays an important role in uniqueness problems of meromorphic functions.

Lemma 2.1 ([12])**.** *Let $f_j(z)\,(j = 1, \dots, n)\,(n \geq 2)$ be meromorphic functions, and let $g_j(z)\,(j = 1, \dots, n)$ be entire functions satisfying*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(ii) *when $1 \leq j < k \leq n$, then $g_j(z) - g_k(z)$ is not a constant*;

(iii) *when $1 \leq j \leq n$, $1 \leq h < k \leq n$, then*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or logarithmic measure.

Then, $f_j(z) \equiv 0\,(j = 1, \dots, n)$.

Lemma 2.2 (the Clunie lemma [6])**.** *Let f be a transcendental meromorphic solution of the equation*:

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda \mid \lambda \in I\}$ such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is at most n, then $m(r, P(z, f)) = S(r, f)$.

Lemma 2.3 (the Hadamard factorization theorem [12, Theorem 2.7] or [3, Theorem 1.9])**.** *Let f be a meromorphic function of finite order $\sigma(f)$. Write*

$$f(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots \ (c_k \neq 0)$$

near $z = 0$ and let $\{a_1, a_2, \dots\}$ and $\{b_1, b_2, \dots\}$ be the zeros and poles of f in $\mathbb{C}\backslash\{0\}$, respectively. Then

$$f(z) = z^k e^{Q(z)} \frac{P_1(z)}{P_2(z)},$$

where $P_1(z)$ and $P_2(z)$ are the canonical products of f formed with the non-null zeros and poles of $f(z)$, respectively, and $Q(z)$ is a polynomial of degree $\le \sigma(f)$.

Remark 1. A well known fact about Lemma 2.3 asserts that $\lambda(f) = \lambda(z^k P_1) = \sigma(z^k P_1) \le \sigma(f)$, $\lambda(1/f) = \lambda(P_2) = \sigma(P_2) \le \sigma(f)$ if $k \ge 0$; and $\lambda(f) = \lambda(P_1) = \sigma(P_1) \le \sigma(f)$, $\lambda(1/f) = \lambda(z^{-k}P_2) = \sigma(z^{-k}P_2) \le \sigma(f)$ if $k < 0$. So we have $\sigma(f) = \sigma(e^Q)$ when $\lambda_f < \sigma(f)$.

The following lemma, which is a slight generalization of Tumura–Clunie type theorem, is referred to [5, Corollary], can also see [1, Theorem 4.3.1].

Lemma 2.4 ([1, 5])**.** *Suppose that $f(z)$ is meromorphic and not constant in the plane, that*

$$g(z) = f(z)^n + P_{n-1}(f),$$

where $P_{n-1}(f)$ is a differential polynomial of degree at most $n-1$ in f, and that

$$N(r, f) + N\left(r, \frac{1}{g}\right) = S(r, f).$$

Then $g(z) = (f + \gamma)^n$, where γ is meromorphic and $T(r, \gamma) = S(r, f)$.

Lemma 2.5 ([7])**.** *Suppose that f is a transcendental meromorphic function, a, b, c, d are small functions with respect to f and $acd \not\equiv 0$. If*

$$af^2 + bff' + c(f')^2 = d,$$

then

$$c(b^2 - 4ac)\frac{d'}{d} + b(b^2 - 4ac) - c(b^2 - 4ac)' + (b^2 - 4ac)c' = 0.$$

Lemma 2.6. *Let α_1, α_2 and a be nonzero constants, and $P_m(z)$ be a non-vanishing polynomial. Then the differential equation*

$$y'' - (\alpha_1 + \alpha_2)y' + \alpha_1\alpha_2 y = P_m(z)e^{az} \tag{8}$$

has a special solution $y^* = R(z)e^{az}$, where $R(z)$ is a nonzero polynomial with $\deg R \le \deg P_m + 2$.

Proof. Set

$$P_m(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0, \quad a_m \ne 0. \tag{9}$$

We guess

$$y^* = R(z)e^{az}, \quad \text{where } R(z) \text{ is a polynomial,}$$

maybe a special solution of (8). By substituting y^*, $(y^*)'$, $(y^*)''$ into the equation (8), and eliminating e^{az}, we get

$$R'' + (2a - \alpha_1 - \alpha_2)R' + \left(a^2 - a(\alpha_1 + \alpha_2) + \alpha_1\alpha_2\right) R = P_m(z). \tag{10}$$

We derive the polynomial solution $R(z)$ by using the method of undetermined coefficients.

Case I. $a \neq \alpha_1$ and $a \neq \alpha_2$. Then $a^2 - a(\alpha_1 + \alpha_2) + \alpha_1\alpha_2 \neq 0$. We choose $R(z)$ is a polynomial with degree m as follow:

$$R(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0. \tag{11}$$

By substituting (9) and (11) into (10), comparing the coefficients of the same power of z at both sides of the equation (10), we get the following system of linear equations,

$$\begin{cases} a_m = & \big(a^2 - a(\alpha_1 + \alpha_2) + \alpha_1\alpha_2\big)\, b_m, \\ a_{m-1} = & \big(a^2 - a(\alpha_1 + \alpha_2) + \alpha_1\alpha_2\big)\, b_{m-1} + (2a - \alpha_1 - \alpha_2) m b_m, \\ a_i = & \big(a^2 - a(\alpha_1 + \alpha_2) + \alpha_1\alpha_2\big)\, b_i + (2a - \alpha_1 - \alpha_2)(i+1) b_{i+1} \\ & + (i+2)(i+1) b_{i+2}, \quad i = m-2, \ldots, 1, 0. \end{cases}$$

Since $a^2 - a(\alpha_1 + \alpha_2) + \alpha_1\alpha_2 \neq 0$, we can solve $b_i\,(i = 0, 1, \ldots, m)$ by using Cramer's rule to the above system.

Case II. $\alpha_1 \neq \alpha_2$, and either $a = \alpha_1$ or $a = \alpha_2$. Then $2a - \alpha_1 - \alpha_2 \neq 0$, and (10) reduces to

$$R'' + (2a - \alpha_1 - \alpha_2) R' = P_m(z). \tag{12}$$

We choose $R(z)$ is a polynomial with degree $m+1$ as follow:

$$R(z) = c_{m+1} z^{m+1} + c_m z^m + \cdots + c_1 z. \tag{13}$$

By substituting (9) and (13) into (12), comparing the coefficients of the same power of z at both sides of the equation (12), we get the following system of linear equations,

$$\begin{cases} a_m = (2a - \alpha_1 - \alpha_2)(m+1) c_{m+1}, \\ a_i = (2a - \alpha_1 - \alpha_2)(i+1) c_{i+1} + (i+2)(i+1) c_{i+2},\ i = m-1, \ldots, 1, 0. \end{cases}$$

Since $2a - \alpha_1 - \alpha_2 \neq 0$, we can solve $c_i\,(i = 1, \ldots, m+1)$ by using Cramer's rule to the above system.

Case III. $a = \alpha_1 = \alpha_2$. Then $2a - \alpha_1 - \alpha_2 = 0$, $a^2 - a(\alpha_1 + \alpha_2) + \alpha_1\alpha_2 = 0$, and (10) reduces to

$$R'' = P_m(z). \tag{14}$$

We choose $R(z)$ is another polynomial with degree $m+2$ as follow:

$$R(z) = d_{m+2} z^{m+2} + d_{m+1} z^{m+1} + \cdots + d_2 z^2. \tag{15}$$

By substituting (9) and (15) into (14), comparing the coefficients of the same power of z at both sides of the equation (14), we get the following system of linear equations,

$$\begin{cases} a_m = (m+2)(m+1) d_{m+2}, \\ a_{m-1} = (m+1) m d_{m+1}, \\ \quad \cdots \\ a_0 = 2 d_2. \end{cases}$$

Obviously, we can solve d_i $(i = 2, \dots, m+2)$ directly from the above system. $\square$

By the proof of [13, Theorem 1.3] (or [6, Lemma 2.4.2.Clunie lemma]), we get the following lemma, see also [8].

Lemma 2.7 ([8]). *Let $P_d(f)$ be a differential polynomial in f of degree d with small functions of f as coefficients. Then we have*

$$m\left(r, P_d(f)\right) \leq dm(r, f) + S(r, f).$$

Lemma 2.8. *Let $n \geq 2$ be integers and $P_d(f)$ denote an algebraic differential polynomial in $f(z)$ of degree $d \leq n-1$ with small functions of f as coefficients. If $p_1(z)$, $p_2(z)$ are small functions of f, $\alpha_1(z)$, $\alpha_2(z)$ are nonconstant entire functions and if f is a transcendental meromorphic solution of the equation*

$$f^n + P_d(f) = p_1 e^{\alpha_1} + p_2 e^{\alpha_2} \tag{16}$$

with $N(r, f) = S(r, f)$, then we have

$T(r, f) = O\left(T\left(r, p_1 e^{\alpha_1} + p_2 e^{\alpha_2}\right)\right), T(r, p_1 e^{\alpha_1} + p_2 e^{\alpha_2}) = O(T(r, f))$, and $T(r, f^n + P_d(f)) \neq S(r, f)$.

Proof. By Lemma 2.7, we get that

$$m\left(r, P_d(f)\right) \leq dm(r, f) + S(r, f). \tag{17}$$

By combining (16), (17) with $N(r, f) = S(r, f)$, we get that

$$\begin{aligned} nT(r, f) = T(r, f^n) &\leq m\left(r, p_1 e^{\alpha_1} + p_2 e^{\alpha_2}\right) + m\left(r, P_d(f)\right) + S(r, f) \\ &\leq T\left(r, p_1 e^{\alpha_1} + p_2 e^{\alpha_2}\right) + dT(r, f) + S(r, f). \end{aligned}$$

This gives that

$$(n - d)T(r, f) \leq T\left(r, p_1 e^{\alpha_1} + p_2 e^{\alpha_2}\right) + S(r, f),$$

i.e.,

$$T(r, f) = O\left(T\left(r, p_1 e^{\alpha_1} + p_2 e^{\alpha_2}\right)\right). \tag{18}$$

From (17), $N(r, f) = S(r, f)$ and the equation (16), we can also get

$$T(r, p_1 e^{\alpha_1} + p_2 e^{\alpha_2}) = O(T(r, f)). \tag{19}$$

Therefore, combining with (16), (18) and (19) we get that $T(r, f^n + P_d(f)) = T(r, p_1 e^{\alpha_1} + p_2 e^{\alpha_2}) \neq S(r, f)$. $\square$

3. Proof of Theorem 1.1

Let f be a transcendental meromorphic solution of the equation (3) with $N(r, f) = S(r, f)$.

Since

$$N(r, f) + N\left(r, \frac{1}{R(z)e^{\alpha(z)}}\right) = S(r, f),$$

by Lemma 2.4 we get

$$(f-\gamma)^n = R(z)e^{\alpha(z)}, \quad T(r,\gamma) = S(r,f).$$

Thus we have

$$f = s(z)e^{\alpha(z)/n} + \gamma(z),$$

where s and γ are small functions of f with $s^n = R$.

4. Proof of Theorem 1.2.

Let f be a transcendental meromorphic solution of the equation (6) with $\lambda_f < \sigma(f)$. Then f is of regular growth, and we have

$$N(r,f) = S(r,f), \text{ and } N(r,1/f) = S(r,f). \tag{20}$$

By combining with Lemma 2.8, we have

$$T(r, f^n + P_*(f)) \neq S(r,f), \tag{21}$$

and

$$\sigma(f) = \sigma(p_1e^{\alpha_1} + p_2e^{\alpha_2}) = \max\{\deg\alpha_1, \deg\alpha_2\}. \tag{22}$$

Therefore, by Lemma 2.3 and Remark 1, we can factorize $f(z)$ as

$$f(z) = \frac{d_1(z)}{d_2(z)}e^{g(z)} = d(z)e^{g(z)}, \tag{23}$$

where g is a polynomial with $\deg g = \sigma(f) = \max\{\deg\alpha_1, \deg\alpha_2\} \geq 1$, d_1 and d_2 are the canonical products formed by zeros and poles of f with $\sigma(d_1) = \lambda(f) < \sigma(f)$ and $\sigma(d_2) = \lambda(1/f) < \sigma(f)$.

Next we assert that $\deg\alpha_1 = \deg\alpha_2$. Otherwise, we have $\deg\alpha_1 \neq \deg\alpha_2$.

Suppose that $\deg\alpha_1 < \deg\alpha_2$, then $T(r, e^{\alpha_1}) = S(r, e^{\alpha_2})$. From Lemma 2.8, we get

$$(1+o(1))T(r,e^{\alpha_2}) = T(r, p_1e^{\alpha_1} + p_2e^{\alpha_2}) \leq K_1T(r,f), \quad K_1 > 0,$$

which means that a small function of e^{α_2} is also a small function of f. So we have $T(r, e^{\alpha_1}) = S(r,f)$. We rewritten (6) as follow:

$$f^n(z) + P_*(f) - p_1e^{\alpha_1} = p_2e^{\alpha_2}. \tag{24}$$

Therefore, by using Theorem 1.1, we get that $f = s_0(z)\exp(\alpha_2(z)/n) + t_0(z)$, where s_0, t_0 are small functions of f with $s_0^n = p_2$. If $t_0 \not\equiv 0$, then combining (20) with Nevanlinna's Second Main Theorem, we have

$$T(r,f) \leq N\left(r, \frac{1}{f-t_0}\right) + N\left(r, \frac{1}{f}\right) + N(r,f) + S(r,f) = S(r,f),$$

a contradiction. So we have $t_0 \equiv 0$. Moreover, we also have that s_0 is a rational function because of the fact that p_2 is a rational function. Substituting $f = s_0(z)\exp(\alpha_2(z)/n)$ into (24), we get that

$$p_1e^{\alpha_1} = P_*(f) = R_{n-1}e^{\frac{n-1}{n}\alpha_2} + \cdots + R_1e^{\frac{1}{n}\alpha_2} + R_0,$$

where $R_0, R_1, \dots, R_{n-1}$ are rational functions. By using Lemma 2.1 and $\deg\alpha_2 > \deg\alpha_1 > 0$, we get that $p_1 \equiv 0$, a contradiction.

Suppose that $\deg\alpha_1 > \deg\alpha_2$, we can also get a contradiction as in the case $\deg\alpha_1 < \deg\alpha_2$.

Therefore, $\deg\alpha_1 = \deg\alpha_2$. By combining with (22) and (23), we have $\sigma(f) = \deg g = \deg\alpha_1 = \deg\alpha_2$, and $S(r,f) = S(r,e^{\alpha_1}) = S(r,e^{\alpha_2})$.

Case 1. $(\alpha_2-\alpha_1)' = 0$. Then $\alpha_2 - \alpha_1$ is a constant, by the equation (6), we get

$$f^n(z) + P_*(f) = (p_1 + p_2c_2)e^{\alpha_1} = \left(\frac{1}{c_2}p_1 + p_2\right)e^{\alpha_2},$$

where $c_2 = e^{\alpha_2-\alpha_1}$ is a non-zero constant. Obviously, from (21) we have that $p_1 + p_2c_2 \neq 0$ and $\frac{1}{c_2}p_1 + p_2 \neq 0$. Therefore, by using Theorem 1.1, we get that $f = s_1(z)\exp(\alpha_1(z)/n) + t_1(z) = s_2(z)\exp(\alpha_2(z)/n) + t_2(z)$, where s_1, t_1, s_2, t_2 are small functions of f with $s_1^n = p_1 + p_2c_2$ and $s_2^n = \frac{1}{c_2}p_1 + p_2$. Combining (20) with Nevanlinna's Second Main Theorem, we have $t_1 \equiv 0$ and $t_2 \equiv 0$. From p_1, p_2 are rational functions, we have s_1 and s_2 are rational functions. This belongs to Case I in Theorem 1.2.

Case 2. $(\alpha_2-\alpha_1)' \neq 0$. By differentiating both sides of (6), we have

$$nf^{n-1}f' + P_*'(f) = (p_1' + p_1\alpha_1')e^{\alpha_1} + (p_2' + p_2\alpha_2')e^{\alpha_2}. \tag{25}$$

Obviously, we have that $p_1' + p_1\alpha_1' \not\equiv 0$ and $p_2' + p_2\alpha_2' \not\equiv 0$. Otherwise, we will get that $p_1 = c_0e^{-\alpha_1}$ and $p_2 = c_1e^{-\alpha_2}$, where c_0, $c_1 \in \mathbb{C}\setminus\{0\}$, which contradict with the facts that α_1, α_2 are nonconstant polynomials, and p_1, p_2 are non-vanishing rational functions.

By eliminating e^{α_2} from equations (6) and (25), we have

$$(p_2' + p_2\alpha_2')f^n - np_2f^{n-1}f' + Q_1(f) = A_1e^{\alpha_1}, \tag{26}$$

where

$$A_1 = p_1\left(p_2' + p_2\alpha_2'\right) - p_2\left(p_1' + p_1\alpha_1'\right), \tag{27}$$

and

$$Q_1(f) = \left(p_2' + p_2\alpha_2'\right)P_* - p_2P_*'. \tag{28}$$

We assert that $A_1(z) \not\equiv 0$. Otherwise, if $A_1(z) \equiv 0$, then we have

$$\left(p_2' + p_2\alpha_2'\right)p_1 = p_2\left(p_1' + p_1\alpha_1'\right).$$

Therefore

$$p_2e^{\alpha_2} = c_3p_1e^{\alpha_1}, \quad c_3 \in \mathbb{C}\setminus\{0\}. \tag{29}$$

So we get $\alpha_2 - \alpha_1$ is a constant, a contradiction with the assumption $(\alpha_2-\alpha_1)' \neq 0$. Therefore, $A_1(z) \not\equiv 0$.

By differentiating (26), we have

$$\begin{aligned}&(p_2' + p_2\alpha_2')'f^n + np_2\alpha_2'f^{n-1}f' - np_2(n-1)f^{n-2}(f')^2\\ &- np_2f^{n-1}f'' + Q_1'(f) = (A_1' + A_1\alpha_1')e^{\alpha_1}.\end{aligned} \tag{30}$$

By eliminating e^{α_1} from equations (26) and (30), we obtain

$$f^{n-2}\varphi = Q(f), \tag{31}$$

where

$$\begin{aligned}\varphi = &\ ((A_1' + A_1\alpha_1')(p_2' + p_2\alpha_2') - A_1(p_2' + p_2\alpha_2')')\, f^2 + n(n-1)p_2A_1(f')^2 \\ &- np_2\,(A_1' + A_1(\alpha_1' + \alpha_2'))\, ff' + np_2A_1ff''\end{aligned} \tag{32}$$

and

$$Q(f) = A_1Q_1'(f) - (A_1' + A_1\alpha_1')Q_1(f). \tag{33}$$

Next we discuss two cases.

Subcase 2.1. $Q(f) \equiv 0$. Then by (31), we have $\varphi \equiv 0$, i.e.,

$$\begin{aligned}&((A_1' + A_1\alpha_1')(p_2' + p_2\alpha_2') - A_1(p_2' + p_2\alpha_2')')\, f^2 \\ &= np_2\,(A_1' + A_1(\alpha_1' + \alpha_2'))\, ff' - n(n-1)p_2A_1(f')^2 - np_2A_1ff''.\end{aligned} \tag{34}$$

Next we assert that f has at most finitely many zeros and poles. Otherwise, f has infinitely many zeros or poles.

Suppose that f has infinitely many zeros. Let z_0 be a zero of f with multiplicity k but neither a zero nor a pole of the coefficients in the equation (34), then $k \geq 2$ and $f(z) = a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \cdots$ $(a_k \neq 0)$ holds in some small neighborhood of z_0.

If $(A_1' + A_1\alpha_1')(p_2' + p_2\alpha_2') - A_1(p_2' + p_2\alpha_2')' \equiv 0$, then we have

$$\frac{A_1'}{A_1} + \alpha_1' = \frac{(p_2' + p_2\alpha_2')'}{p_2' + p_2\alpha_2'}.$$

This gives

$$A_1e^{\alpha_1} = c_4(p_2' + p_2\alpha_2'), \quad c_4 \in \mathbb{C} \setminus \{0\},$$

which yields a contradiction with $A_1(\not\equiv 0)$, $p_2' + p_2\alpha_2'(\not\equiv 0)$ are rational functions, and α_1 is a nonconstant polynomial. Therefore, $(A_1'+A_1\alpha_1')(p_2'+p_2\alpha_2') - A_1(p_2' + p_2\alpha_2')' \not\equiv 0$.

Obviously, z_0 is a zero with multiplicity $2k$ of the left side of (34). As to the right side, the coefficient of $(z-z_0)^{2k-2}$ is

$$-nkp_2A_1((n-1)k + (k-1))a_k^2,$$

which can not equal to zero when $n, k \geq 2$. Therefore, z_0 is a zero with multiplicity $2k-2$ of the right side of (34). This is a contradiction.

Suppose that f has infinitely many poles. Let z_1 be a pole of f with multiplicity m but neither a zero nor a pole of the coefficients in the equation (34), then $f(z) = \frac{a_{-m}}{(z-z_1)^m} + \frac{a_{-m+1}}{(z-z_1)^{m-1}} + \cdots$ $(a_{-m} \neq 0)$ holds in some small neighborhood of z_1. Obviously, z_1 is a pole with multiplicity $2m$ of the left side of (34). As to the right side, the coefficient of $(z-z_0)^{-2(m+1)}$ is

$$-nmp_2A_1((n-1)m + (m+1))a_{-m}^2,$$

which can not be equal to zero when $m \geq 1$ and $n \geq 2$. Therefore, z_1 is a pole with multiplicity $2(m+1)$ of the right side of (34). This is a contradiction.

Therefore, f has at most finitely many zeros and poles. So

$$f(z) = d(z)e^{g(z)}, \tag{35}$$

where g is a polynomial with $\deg g = \deg \alpha_1 = \deg \alpha_2 \geq 1$, and d is a rational function.

By substituting (35) into the equation (6), we get

$$d^n e^{ng} + \widetilde{R}_{n-1} e^{(n-1)g} + \cdots + \widetilde{R}_1 e^g + \widetilde{R}_0 = p_1 e^{\alpha_1} + p_2 e^{\alpha_2}, \tag{36}$$

where $\widetilde{R}_0, \widetilde{R}_1, \ldots, \widetilde{R}_{n-1}$ are rational functions.

If neither $ng(z) - \alpha_1(z)$ nor $ng(z) - \alpha_2(z)$ are constants, then by Lemma 2.1, we get that $d(z) \equiv 0$, which yields a contradiction.

If $ng(z) - \alpha_1(z)$ is a constant, then $ng(z) - \alpha_2(z)$ is not a constant, otherwise we have $\alpha_2(z) - \alpha_1(z)$ is a constant, which yields a contradiction. We set $ng(z) - \alpha_1(z) = c_5$, then (36) can be reduced to

$$(d^n - p_1 e^{-c_5}) e^{ng} + \widetilde{R}_{n-1} e^{(n-1)g} + \cdots + \widetilde{R}_1 e^g + \widetilde{R}_0 - p_2 e^{\alpha_2} = 0.$$

By Lemma 2.1, there must exist some integer k_1 $(1 \leq k_1 \leq n-1)$ such that

$$k_1 g' = \alpha_2' \ \text{ and } \ d^n - p_1 e^{-c_5} = 0.$$

Therefore, by combining with (35) we have

$$f(z) = s_3(z) e^{\frac{\alpha_1(z)}{n}},$$

where $s_3^n = p_1$, and $k_1 \alpha_1' = n\alpha_2'$.

If $ng(z) - \alpha_2(z)$ is a constant, then $ng(z) - \alpha_1(z)$ is not a constant, following the similar reason, we have

$$f(z) = s_4(z) e^{\frac{\alpha_2(z)}{n}},$$

where $s_4^n = p_2$, and $k_2 \alpha_2' = n\alpha_1'$ $(1 \leq k_2 \leq n-1)$.

Subcase 2.2. $Q(f) \not\equiv 0$. By combining Logarithmic Derivative Lemma with (32), we get

$$m\left(r, \frac{\varphi}{f^2}\right) = S(r, f). \tag{37}$$

We rewritten (31) as follow:

$$f^{n-1} \frac{\varphi}{f} = Q(f). \tag{38}$$

From (32), we have

$$\begin{aligned} \frac{\varphi}{f} = \ & ((A_1' + A_1\alpha_1')(p_2' + p_2\alpha_2') - A_1(p_2' + p_2\alpha_2')')\, f + n(n-1)p_2 A_1 \frac{f'}{f} \cdot f' \\ & - np_2\, (A_1' + A_1(\alpha_1' + \alpha_2'))\, f' + np_2 A_1 f'' \end{aligned} \tag{39}$$

is a polynomial in f, f' and f'' with meromorphic coefficients such that

$$m\left(r,(A_1'+A_1\alpha_1')(p_2'+p_2\alpha_2')-A_1(p_2'+p_2\alpha_2')'\right)=S(r,f),$$
$$m(r,p_2A_1)=S(r,f),$$
$$m\left(r,p_2A_1\frac{f'}{f}\right)=S(r,f),\text{ and } m\left(r,p_2\left(A_1'+A_1(\alpha_1'+\alpha_2')\right)\right)=S(r,f).$$

By combining with (38), (39), (33), and Lemma 2.2, we have that

$$m\left(r,\frac{\varphi}{f}\right)=S(r,f). \tag{40}$$

From (20), (32), (37) and (40), we get that

$$\begin{aligned}2T(r,f)+S(r,f)&=T\left(r,\frac{1}{f^2}\right)=m\left(r,\frac{1}{f^2}\right)+S(r,f)\\&\le m\left(r,\frac{\varphi}{f^2}\right)+m\left(r,\frac{1}{\varphi}\right)+S(r,f)\\&\le T(r,\varphi)+S(r,f)\\&\le m\left(r,\frac{\varphi}{f}\right)+m(r,f)+S(r,f)\\&\le T(r,f)+S(r,f),\end{aligned}$$

which yields a contradiction.

5. Proof of Theorem 1.3.

Let f be a transcendental meromorphic solution of the equation (7) with $N(r,f)=S(r,f)$. By Lemma 2.8, we have that f is of finite order and

$$\sigma(f)=\sigma(p_1e^{\alpha_1z}+p_2e^{\alpha_2z})=1. \tag{41}$$

If $N(r,1/f)=S(r,f)$, by the proof of Theorem 1.2, we can get the conclusion.

Next, we consider the case when $N(r,1/f)\neq S(r,f)$. By differentiating (7), we get

$$nf^{n-1}f'+P_*'(f)=p_1\alpha_1e^{\alpha_1z}+p_2\alpha_2e^{\alpha_2z} \tag{42}$$

By eliminating e^{α_2z} from (7) and (42), we have

$$\alpha_2f^n+\alpha_2P_*(f)-nf^{n-1}f'-P_*'(f)=p_1(\alpha_2-\alpha_1)e^{\alpha_1z}. \tag{43}$$

Differentiating (43) yields

$$\begin{aligned}&n\alpha_2f^{n-1}f'+\alpha_2P_*'-n(n-1)f^{n-2}(f')^2-nf^{n-1}f''-P_*''\\&=p_1\alpha_1(\alpha_2-\alpha_1)e^{\alpha_1z}.\end{aligned} \tag{44}$$

It follows from (43) and (44) that

$$f^{n-2}\varphi=-P_*''+(\alpha_1+\alpha_2)P_*'-\alpha_1\alpha_2P_*, \tag{45}$$

where

$$\varphi(z) = \alpha_1\alpha_2 f^2 - n(\alpha_1+\alpha_2)ff' + n(n-1)(f')^2 + nff''. \tag{46}$$

Next we assert that $\varphi(z) \not\equiv 0$. Otherwise, we have

$$\alpha_1\alpha_2 f^2 - n(\alpha_1+\alpha_2)ff' + n(n-1)(f')^2 + nff'' = 0. \tag{47}$$

Since $N(r,1/f) \neq S(r,f)$, let z_0 be a zero of f with multiplicity k. By (47) we have $k \geq 2$ and $f(z) = a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \cdots$ $(a_k \neq 0)$ holds in some small neighborhood of z_0. We rewrite (47) as follow,

$$\alpha_1\alpha_2 f^2 = n(\alpha_1+\alpha_2)ff' - n(n-1)(f')^2 - nff''. \tag{48}$$

Obviously, z_0 is a zero with multiplicity $2k$ of the left side of (48). As to the right side, the coefficient of $(z-z_0)^{2k-2}$ is

$$-nk((n-1)k+(k-1))a_k^2,$$

which can not equal to zero when $n, k \geq 2$. Therefore, z_0 is a zero with multiplicity $2k-2$ of the right side of (48). This is a contradiction. Therefore, $\varphi(z) \not\equiv 0$.

From (45) and (46), by using Lemma 2.2 and Logarithmic Derivative Lemma, we have

$$m\left(r, \frac{\varphi}{f}\right) = S(r,f), \text{ and } m\left(r, \frac{\varphi}{f^2}\right) = S(r,f). \tag{49}$$

From (49), we have

$$\begin{aligned} 2m\left(r,\frac{1}{f}\right) = m\left(r,\frac{1}{f^2}\right) &\leq m\left(r,\frac{\varphi}{f^2}\right) + m\left(r,\frac{1}{\varphi}\right) \\ &\leq m\left(r,\frac{1}{\varphi}\right) + S(r,f). \end{aligned} \tag{50}$$

By (46), we have

$$\begin{aligned} N\left(r,\frac{1}{f}\right) &= N_{1)}\left(r,\frac{1}{f}\right) + N_{(2}\left(r,\frac{1}{f}\right) \\ &\leq N_{1)}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\varphi}\right) + S(r,f). \end{aligned} \tag{51}$$

Combining with (50) and (51), we have

$$T(r,f) \leq N_{1)}\left(r,\frac{1}{f}\right) + \frac{1}{2}T(r,\varphi) + \frac{1}{2}N\left(r,\frac{1}{\varphi}\right) + S(r,f).$$

Case 1. $\varphi(z)$ is a nonzero constant. Since $N(r,1/f) \neq S(r,f)$, let z_1 be a zero of f with multiplicity m. By (46) we have $n(n-1)(f')^2(z_1) = \varphi \neq 0$. Thus, $m = 1$, i.e., z_1 is a simple zero of f. This gives that all zeros of f are simple zeros. So we have

$$N(r,1/f) = N_{1)}(r,1/f) + S(r,f). \tag{52}$$

By the assumption that φ is a nonzero constant, differentiating (46) yields

$$\frac{\varphi'}{n} = \frac{2\alpha_1\alpha_2}{n} ff' - (\alpha_1+\alpha_2)(f')^2 - (\alpha_1+\alpha_2)ff'' + (2n-1)f'f'' + ff''' = 0. \tag{53}$$

It follows from (53) and $f'(z_1) \neq 0$ that

$$(2n-1)f''(z_1) - (\alpha_1+\alpha_2)f'(z_1) = 0.$$

We set

$$h(z) = \frac{(2n-1)f''(z) - (\alpha_1+\alpha_2)f'(z)}{f(z)}. \tag{54}$$

Subcase 1.1. $h(z) \equiv 0$. Hence, by (54), we have $(2n-1)f''(z) - (\alpha_1 + \alpha_2)f'(z) \equiv 0$. Rewrite it as

$$\frac{f''}{f'} = \frac{\alpha_1+\alpha_2}{2n-1}.$$

By integrating the above equation, we have

$$f'(z) = \widetilde{c}e^{\frac{\alpha_1+\alpha_2}{2n-1}z}, \quad \widetilde{c} \in \mathbb{C}\backslash\{0\}.$$

Integrating the function f' yields

$$f(z) = c_1 e^{\frac{\alpha_1+\alpha_2}{2n-1}z} + c_2, \tag{55}$$

where $c_1(\neq 0)$, c_2 are two constants. Obviously, $c_2 \neq 0$. Otherwise, f has no zeros, which yields a contradiction. Substitute (55) into the equation (7) yields

$$c_1^n e^{\frac{n(\alpha_1+\alpha_2)}{2n-1}z} + \widetilde{P_*}(e^{\frac{\alpha_1+\alpha_2}{2n-1}z}) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

where $\widetilde{P_*}(e^{\frac{\alpha_1+\alpha_2}{2n-1}z})$ is a polynomial of $e^{\frac{\alpha_1+\alpha_2}{2n-1}z}$ with degree $\leq n-1$, and with rational functions as coefficients. By using Lemma 2.1, we have $\frac{n(\alpha_1+\alpha_2)}{2n-1} = \alpha_1$, i.e., $(n-1)\alpha_1 = n\alpha_2$, and $c_1^n = p_1$; or $\frac{n(\alpha_1+\alpha_2)}{2n-1} = \alpha_2$, i.e., $(n-1)\alpha_2 = n\alpha_1$, and $c_1^n = p_2$.

Subcase 1.2. $h(z) \not\equiv 0$. By (54) and Logarithmic Derivative Lemma, we get $m(r,h) = S(r,f)$. It follows from (54) that poles of h may occur at zeros and poles of f. But any simple zero of f is also a zero of $(2n-1)f'' - (\alpha_1+\alpha_2)f'$, so by combining with (52), (54) and $N(r,f) = S(r,f)$, we get $N(r,h) \leq N(r,f) + S(r,f) = S(r,f)$. Therefore, $T(r,h) = m(r,h) + N(r,h) = S(r,f)$, i.e., $h(z)$ is a small function of f. We rewrite (54) as follow,

$$f'' = H_1 f' + H_2 f, \tag{56}$$

where $H_1 = \frac{\alpha_1+\alpha_2}{2n-1}$, and $H_2 = \frac{h}{2n-1}$. Differentiating (56) yields

$$f''' = (H_1^2 + H_2)f' + (H_1H_2 + H_2')f. \tag{57}$$

Substituting (56) and (57) into (53), we get that

$$A_1 f + A_2 f' = 0, \tag{58}$$

where

$$A_1 = H_1H_2 + H_2' - (\alpha_1+\alpha_2)H_2$$

and

$$A_2 = \frac{2\alpha_1\alpha_2}{n} - (\alpha_1+\alpha_2)H_1 + 2nH_2 + H_1^2.$$

Suppose that $A_1 \not\equiv 0$, then by (52), (58) and $T(r,h) = S(r,f)$, we have

$$N\left(r,\frac{1}{f}\right) = N_{1)}\left(r,\frac{1}{f}\right) + S(r,f) \le N\left(r,\frac{1}{A_2}\right) + N(r,A_1) + S(r,f) = S(r,f),$$

a contradiction with the assumption that $N(r,1/f) \neq S(r,f)$. Therefore, combining with (58) we have $A_1 \equiv 0$, and $A_2 \equiv 0$. That is

$$\begin{cases} \dfrac{\alpha_1+\alpha_2}{2n-1}\dfrac{h}{2n-1} + \dfrac{h'}{2n-1} - (\alpha_1+\alpha_2)\dfrac{h}{2n-1} \equiv 0, \\ \dfrac{2\alpha_1\alpha_2}{n} - \dfrac{(\alpha_1+\alpha_2)^2}{2n-1} + \left(\dfrac{\alpha_1+\alpha_2}{2n-1}\right)^2 + \dfrac{2nh}{2n-1} \equiv 0, \end{cases}$$

which yields a contradiction since $n \ge 2$, $h \not\equiv 0$ and $\alpha_1+\alpha_2 \neq 0$.

Case 2. $\varphi(z)$ is a nonconstant small function of f. Differentiating (46) gives

$$\varphi' = 2\alpha_1\alpha_2 ff' - n(\alpha_1+\alpha_2)(f')^2 - n(\alpha_1+\alpha_2)ff'' + n(2n-1)f'f'' + nff'''. \tag{59}$$

It follows from (46) and (59) that

$$\begin{aligned} &\alpha_1\alpha_2\varphi' f^2 - [n(\alpha_1+\alpha_2)\varphi' + 2\alpha_1\alpha_2\varphi]\, ff' \\ &+ n\,[(n-1)\varphi' + (\alpha_1+\alpha_2)\varphi]\,(f')^2 \\ &+ n\,[(\alpha_1+\alpha_2)\varphi + \varphi']\, ff'' - n(2n-1)\varphi f'f'' - n\varphi ff''' = 0. \end{aligned} \tag{60}$$

Since $N(r,1/f) \neq S(r,f)$ and $T(r,\varphi) = S(r,f)$, let z_2 be a zero of f, which is neither a zero of φ nor a pole of the coefficients in (60), with multiplicity l, then by (46) we have $l = 1$, i.e., z_2 is a simple zero of f. And it follows from (60) that z_2 is also a zero of $[(n-1)\varphi' + (\alpha_1+\alpha_2)\varphi]\, f' - (2n-1)\varphi f''$.

We set

$$g = \frac{(2n-1)\varphi f'' - [(n-1)\varphi' + (\alpha_1+\alpha_2)\varphi]\, f'}{f}. \tag{61}$$

Subcase 2.1. $g(z) \not\equiv 0$. Then by combining (61) with Logarithmic Derivative Lemma, $N(r,f) = S(r,f)$, and $T(r,\varphi) = S(r,f)$, we have

$$T(r,g) = O\left(m(r,\varphi) + N\left(r,\frac{1}{\varphi}\right) + N(r,\varphi) + N(r,f)\right) + S(r,f) = S(r,f),$$

i.e., g is a small function of f. We rewrite (61) as follow,

$$f'' = t_1 f' + \frac{g}{(2n-1)\varphi}f, \quad \text{where } t_1 = \frac{1}{2n-1}\left((n-1)\frac{\varphi'}{\varphi} + \alpha_1 + \alpha_2\right). \tag{62}$$

Differentiating (62) gives that

$$f''' = \left(t_1^2 + t_1' + \frac{g}{(2n-1)\varphi}\right) f' + \frac{1}{2n-1}\left(t_1 \frac{g}{\varphi} + \left(\frac{g}{\varphi}\right)'\right) f. \tag{63}$$

By substituting (62) and (63) into (60), combining with $\varphi \not\equiv 0$, we get

$$B_1 f = B_2 f', \tag{64}$$

where

$$B_1 = \alpha_1\alpha_2 \frac{\varphi'}{\varphi} + n\left(\alpha_1 + \alpha_2 + \frac{\varphi'}{\varphi}\right) \frac{g}{(2n-1)\varphi} - \frac{n}{2n-1}\left(\left(\frac{g}{\varphi}\right)' + t_1 \frac{g}{\varphi}\right),$$

and

$$B_2 = n(\alpha_1+\alpha_2)\left(\frac{\varphi'}{\varphi} - t_1\right) + 2\alpha_1\alpha_2 - n\frac{\varphi'}{\varphi} t_1 + \frac{ng}{\varphi} + n\left(t_1' + \frac{g}{(2n-1)\varphi} + t_1^2\right).$$

If $B_2 \not\equiv 0$, then from (64) and f is transcendental, we have $B_1 \not\equiv 0$. Since $N(r, 1/f) \neq S(r,f)$, $T(r,\varphi) = S(r,f)$, and $T(r,g) = S(r,f)$, let z_3 be a zero of f with multiplicity q, which is neither a zero nor a pole of B_1 and B_2. Then z_3 is a zero with multiplicity q of the left side of (64), but a zero with multiplicity $q-1$ of the right side, which yields a contradiction. Therefore, we have $B_2 \equiv 0$ and $B_1 \equiv 0$, i.e.,

$$\left(\frac{g}{\varphi}\right)' = \left(\frac{2(n-1)}{2n-1}(\alpha_1+\alpha_2) + \frac{n}{2n-1}\gamma\right)\frac{g}{\varphi} + \frac{2n-1}{n}\alpha_1\alpha_2\gamma, \tag{65}$$

and

$$\begin{aligned} -\frac{2n}{2n-1}\frac{g}{\varphi} &= (\alpha_1+\alpha_2)\gamma + \frac{2}{n}\alpha_1\alpha_2 - \frac{1}{2n-1}(\alpha_1+\alpha_2+\gamma)(\alpha_1+\alpha_2+(n-1)\gamma) \\ &\quad + \frac{1}{(2n-1)^2}(\alpha_1+\alpha_2+(n-1)\gamma)^2 + \frac{n-1}{2n-1}\gamma', \end{aligned} \tag{66}$$

where $\gamma = \frac{\varphi'}{\varphi}$.

Substituting (62) into (46),

$$\varphi(z) = af^2 + bff' + n(n-1)(f')^2.$$

where

$$a = \alpha_1\alpha_2 + \frac{n}{2n-1}\frac{g}{\varphi}, \quad \text{and} \quad b = \frac{n(n-1)}{2n-1}\left(\gamma - 2(\alpha_1+\alpha_2)\right).$$

If $a \not\equiv 0$, then by Lemma 2.5, we have

$$\begin{aligned} &n(n-1)(b^2 - 4an(n-1))\frac{\varphi'}{\varphi} + b(b^2 - 4an(n-1)) \\ &- n(n-1)(b^2 - 4an(n-1))' = 0. \end{aligned} \tag{67}$$

Suppose that $b^2 - 4an(n-1) \not\equiv 0$. It follows from (67) that

$$2n\frac{\varphi'}{\varphi} = (2n-1)\frac{(b^2-4an(n-1))'}{b^2-4an(n-1)} + 2(\alpha_1+\alpha_2). \tag{68}$$

By integration, we see that there exists a $c_5 \in \mathbb{C} \setminus \{0\}$ such that

$$e^{2(\alpha_1+\alpha_2)z} = c_5\varphi^{2n}(b^2-4an(n-1))^{-(2n-1)},$$

which implies $e^{2(\alpha_1+\alpha_2)z} \in S(r,f)$, then $\alpha_2 = -\alpha_1$, a contradiction.

Suppose that $b^2 - 4an(n-1) \equiv 0$. Then we have

$$\frac{n(n-1)}{(2n-1)^2}\left(\gamma - 2(\alpha_1+\alpha_2)\right)^2 = 4\left(\alpha_1\alpha_2 + \frac{n}{2n-1}\frac{g}{\varphi}\right). \tag{69}$$

Differentiating (69) yields

$$\frac{n-1}{2n-1}\left(\gamma - 2(\alpha_1+\alpha_2)\right)\gamma' = 2\left(\frac{g}{\varphi}\right)'. \tag{70}$$

Differentiating (66) yields

$$2\left(\frac{g}{\varphi}\right)' = \frac{2(n-1)}{2n-1}\gamma\gamma' - \frac{(2n+1)(n-1)}{(2n-1)n}(\alpha_1+\alpha_2)\gamma' - \frac{n-1}{n}\gamma''. \tag{71}$$

Combining with (70) and (71), we obtain that

$$n\gamma\gamma' = (\alpha_1+\alpha_2)\gamma' + (2n-1)\gamma''. \tag{72}$$

We assert that $\gamma' \not\equiv 0$. Otherwise, by $\gamma' \equiv 0$ and φ is nonconstant we have

$$\frac{\varphi'}{\varphi} = c_6,\ c_6 \in \mathbb{C} \setminus \{0\}.$$

Then

$$\varphi = c_7 e^{c_6 z},\ c_7 \in \mathbb{C} \setminus \{0\},$$

which contradicts with the assumption that φ is a nonconstant small function of f.

Therefore, (72) gives that

$$\alpha_1 + \alpha_2 = n\gamma - (2n-1)\frac{\gamma''}{\gamma'}. \tag{73}$$

Thus

$$c_8 e^{(\alpha_1+\alpha_2)z} = \varphi^n\left(\left(\frac{\varphi'}{\varphi}\right)'\right)^{-(2n-1)},\ c_8 \in \mathbb{C} \setminus \{0\},$$

which implies that $e^{(\alpha_1+\alpha_2)z} \in S(r,f)$, then $\alpha_2 = -\alpha_1$, a contradiction.

If $a \equiv 0$, that is $\frac{g}{\varphi} = -\frac{2n-1}{n}\alpha_1\alpha_2$. By substituting it into (65), we get

$$\frac{\varphi'}{\varphi} = 2\left(\alpha_1+\alpha_2\right).$$

So we have

$$\varphi = c_9 e^{2(\alpha_1+\alpha_2)z},\ c_9 \in \mathbb{C} \setminus \{0\},$$

which implies that $e^{2(\alpha_1+\alpha_2)z} \in S(r,f)$, then $\alpha_2 = -\alpha_1$, a contradiction.

Subcase 2.2. $g(z) \equiv 0$. Hence, by (61), we have

$$(2n-1)\varphi f'' - [(n-1)\varphi' + (\alpha_1+\alpha_2)\varphi]\, f' \equiv 0.$$

Rewrite it as

$$f'' = t_1 f'. \tag{74}$$

Differentiating (74) yields

$$f''' = \left(t_1^2 + t_1'\right) f'. \tag{75}$$

By substituting (74) and (75) into (60), combining with $\varphi \not\equiv 0$, we get

$$\widetilde{B_1} f = \widetilde{B_2} f', \tag{76}$$

where

$$\widetilde{B_1} = \alpha_1\alpha_2 \frac{\varphi'}{\varphi},$$

and

$$\widetilde{B_2} = n(\alpha_1+\alpha_2)\left(\frac{\varphi'}{\varphi} - t_1\right) + 2\alpha_1\alpha_2 - n\frac{\varphi'}{\varphi}t_1 + n\left(t_1' + t_1^2\right).$$

By a similar method as in subcase 2.1, we have $\widetilde{B_1} \equiv 0$ and $\widetilde{B_2} \equiv 0$. Thus $\varphi' \equiv 0$, which yields that φ is a constant, a contradiction.

Case 3. $n = 2$ and $\varphi(z) = P(z)e^{Q(z)}$, where P, Q are nonvanishing polynomials and Q is non-constant. By (41) and (46), we get $\sigma(\varphi) \le \sigma(f) = 1$, combining with $\deg Q \ge 1$, we have $\deg Q = \sigma(\varphi) = 1$. Let $Q(z) = az + b$, where $a(\neq 0), b$ are constants, then $\varphi = e^b P e^{az}$. By (45) we get that

$$P_*'' - (\alpha_1+\alpha_2)P_*' + \alpha_1\alpha_2 P_* = -e^b P(z)e^{az}. \tag{77}$$

From Lemma 2.6 and the theory of ordinary differential equations, the general solutions of the equation (77) can be represented in the form

$$P_* = c_{10}e^{\alpha_1 z} + c_{11}e^{\alpha_2 z} + R(z)e^{Q(z)}, \tag{78}$$

where c_{10}, c_{11} are constants, and R is a polynomial with $\deg R \le \deg P + 2$.

By combining with (7), we get

$$f^2 = d_1 e^{\alpha_1 z} + d_2 e^{\alpha_2 z} - R(z)e^{Q(z)},$$

where $d_1 = p_1 - c_{10}$, and $d_2 = p_2 - c_{11}$.

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