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ON PSEUDO 2-PRIME IDEALS AND ALMOST VALUATION DOMAINS

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Abstract. In this paper, we introduce the notion of pseudo 2-prime ideals in commutative rings. Let R be a commutative ring with a nonzero identity. A proper ideal P of R is said to be a pseudo 2-prime ideal if whenever $xy \in P$ for some $x, y \in R$, then $x^{2n} \in P^n$ or $y^{2n} \in P^n$ for some $n \in \mathbb{N}$. Various examples and properties of pseudo 2-prime ideals are given. We also characterize pseudo 2-prime ideals of PID's and von Neumann regular rings. Finally, we use pseudo 2-prime ideals to characterize almost valuation domains (AV-domains).

1. Introduction

Throughout the paper, we focus only on commutative rings with a nonzero identity. Let R will always denote such a ring. Assume that P is an ideal of R. Then we say that P is proper if $P \neq R$. For any proper ideal P of R, the radical $\sqrt{P}$ is defined as $\sqrt{P} := \{a \in R : a^n \in P \text{ for some } n \in \mathbb{N}\}$. Also for each nonempty subset K of R and each ideal P of R, the residual of P by K is denoted by $(P : K) = \{a \in R : aK \subseteq P\}$.

The notion of prime ideals and its generalizations play a central role in multiplicative ideal theory since they are used in classifying certain classes of rings such as Dedekind domains, valuation domains, divided domains and etc. The set of all maximal ideals and prime ideals of R will be designated by $Max(R)$ and $Spec(R)$, respectively. Recall from [19] that an integral domain R with quotient field K is said to be a *valuation domain* if for each $x \in K$, then either $x \in R$ or $x^{-1} \in R$. Note that an integral domain R is a valuation domain if and only if the lattice of all ideals $L(R)$ of R is totally ordered by inclusion if and only if for each $x, y \in R - \{0\}$, either x divides y or y divides x. In 2016, Beddani and Messirdi defined the concept of 2-prime ideals and they characterized valuation domains in terms of this concept. A proper ideal P of R is said to be a *2-prime ideal* if whenever $xy \in P$ for some $x, y \in R$, then $x^2 \in P$ or $y^2 \in P$ [9]. Note that every prime ideal is 2-prime but the

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converse is not true. For example, $P = (X^2, XY, Y^2)$ is not a prime ideal of $R = k[X, Y]$, where k is a field, while it is a 2-prime ideal of R. Afterwards, Koç et al. in [18] gave a generalization of 2-prime ideals and they used it to characterize divided domains. Recall from [12] that a prime ideal P of R is said to be a *divided prime ideal* if for each $x \in R - P$, we have $P \subset Rx$. In particular, an integral domain R is said to be a *divided domain* if its each prime ideal is divided prime. It is well known that an integral domain R is a divided domain if and only if for every elements $x, y \in R - \{0\}$, either y divides x or x divides y^n for some $n \in \mathbb{N}$. For more information on divided domains, we refer [5], [6], [8] and [22] to the reader. A proper ideal P of R is said to be a *strongly quasi primary ideal* if whenever $xy \in P$ for some $x, y \in R$, then $x^2 \in P$ or $y^n \in P$ ($x^n \in P$ or $y^2 \in P$) for some $n \in \mathbb{N}$. The authors in [18, Theorem 2.2] showed that an integral domain R is a divided domain if and only if its each proper ideal is strongly quasi primary.

Another generalization of valuation domain is that almost valuation domain firstly defined by Anderson and Zafrullah in [3]. An integral domain R is said to be an *almost valuation domain* (briefly, AV-domain) if for each $x, y \in R - \{0\}$, then there exists $n \in \mathbb{N}$ such that either x^n divides y^n or y^n divides x^n. Equivalently, an integral domain R with quotient field K is an AV-domain if and only if for each $x \in K$, there exists $n \in \mathbb{N}$, either $x^n \in R$ or $x^{-n} \in R$. Recently, valuation domains and AV-domains have been center of interest and studied by many authors. See, for example, [13], [16] and [20]. The purpose of the paper is to introduce pseudo 2-prime ideals which is a generalization of prime ideals in commutative rings and to use them to characterize almost valuation domains. A proper ideal P of R is said to be a *pseudo 2-prime ideal* if whenever $xy \in P$ for some $x, y \in R$, then either $x^{2n} \in P^n$ or $y^{2n} \in P^n$ for some $n \in \mathbb{N}$. Among many results in this paper, we investigate the relations between pseudo 2-prime ideal and other classical ideals such as prime ideal, 2-prime ideal, quasi primary ideal (i.e., an ideal whose radical is prime [14]), 2-absorbing ideal, 2-absorbing primary ideal and irreducible ideal (See, Theorem 2.1, Example 2.2, Example 2.3 and Proposition 2.5). Also, we investigate the stability of pseudo 2-prime ideals under homomorphism, in factor ring, in cartesian product of rings, under localization of rings and in trivial extension $R \ltimes M$ of a unital R-module M (See, Theorem 2.6, Corollary 2.7, Proposition 2.9, Theorem 2.13, Theorem 2.14, Theorem 2.16). Also, we determine pseudo 2-prime ideals in certain commutative rings such as von Neumann regular rings and Principal ideal domains (PID's) (See, Theorem 2.11 and Proposition 2.12). Furthermore, we prove pseudo 2-prime avodiance theorem (See, Theorem 2.17 and Theorem 2.18). Finally, we characterize AV-domains in terms of pseudo 2-prime ideals (See, Theorem 2.15).

2. Characterizations of pseudo 2-prime ideals

Definition. Let R be a ring and P be a proper ideal of R. P is said to be pseudo 2-prime ideal of R if whenever $xy \in P$ for some $x, y \in R$, then either $x^{2n} \in P^n$ or $y^{2n} \in P^n$ for some $n \in \mathbb{N}$.

In 2007, Badawi in his celebrated paper [7], defined the concept of 2-absorbing ideals and used them to characterize Dedekind domains. Recall from [7] that a nonzero proper ideal P of R is said to be a *2-absorbing ideal* if whenever $xyz \in P$ for some $x, y, z \in R$, then either $xy \in P$ or $xz \in P$ or $yz \in P$. Also, a proper ideal P of R is called a *2-absorbing primary ideal* if for some $x, y, z \in R$ with $xyz \in P$, then either $xy \in P$ or $xz \in \sqrt{P}$ or $yz \in \sqrt{P}$ [10]. Note that every 2-absorbing ideal is also a 2-absorbing primary ideal but the converse is not true in general. For instance, $P = (12)$ is a 2-absorbing primary ideal of the ring $\mathbb{Z}$ of integers which is not 2-absorbing since $2.2.3 \in P$ but $2.2 \notin P$ and $2.3 \notin P$.

Theorem 2.1. *Let R be a ring and P a proper ideal of R. The following statements are satisfied.*

(i) *Every pseudo 2-prime ideal P of R is a quasi primary ideal of R, that is, $\sqrt{P}$ is a prime ideal of R.*

(ii) *Every 2-prime ideal P of R is a pseudo 2-prime ideal of R. In particular, every prime ideal is pseudo 2-prime.*

(iii) *Every pseudo 2-prime ideal P of R is a 2-absorbing primary ideal of R.*

Proof. (i) Let P be a pseudo 2-prime ideal of R. Take $x, y \in R$ such that $xy \in \sqrt{P}$. Then there exists $k \in \mathbb{N}$ such that $(xy)^k = x^k y^k \in P$. Since P is a pseudo 2-prime ideal of R, we conclude that $x^{2kn} \in P^n$ or $y^{2kn} \in P^n$, which implies that $x \in \sqrt{P^n} = \sqrt{P}$ or $y \in \sqrt{P}$.

(ii) Suppose that P is a 2-prime ideal of R. Choose $x, y \in R$ such that $xy \in P$. As P is a 2-prime ideal of R, we conclude that $x^2 \in P$ or $y^2 \in P$, which implies that $x^{2n} \in P^n$ or $y^{2n} \in P^n$. Therefore, P is a pseudo 2-prime ideal of R.

(iii) Note that by (i), $\sqrt{P}$ is a prime ideal of R. The rest is clear by [10, Theorem 2.8]. $\square$

The converses of Theorem 2.1(i) and (ii) need not be true in general. See the following examples.

Example 2.2 (A pseudo 2-prime ideal which is not 2-prime ideal)**.** Consider the ring $R = k[X, Y]/P$, where k is a field and $P = (X^6, XY, Y^6)$. Let $Q = (X^3, XY, Y^3)/P$. Then note that $\sqrt{Q} = (X, Y)/P$ is a prime ideal of R and $\sqrt{Q}^6 = (\overline{0})$, where $\overline{0} = 0 + P$. Note that $\overline{xy} \in Q$ and $\overline{x}^2, \overline{y}^2 \notin Q$, where $\overline{x} = X + P$ and $\overline{y} = Y + P$. Thus, Q is not a 2-prime ideal of R. Now, we will show that it is a pseudo 2-prime ideal. To see this, take $ab \in Q \subseteq \sqrt{Q}$ for some $a, b \in R$. Since $\sqrt{Q}$ is a prime ideal, we have $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$.

As $\sqrt{Q}^6 = (\overline{0})$, we have $a^{12} \in Q^6 = (\overline{0})$ or $b^{12} \in Q^6 = (\overline{0})$. Therefore, Q is a pseudo 2-prime ideal of R.

Example 2.3 (A quasi-primary ideal that is not pseudo 2-prime ideal)**.** Let $R = k[X, Y]$, where k is a field and consider the ideal $P = (X^3, XY, Y^3)$ of R. Then $\sqrt{P} = (X, Y)$ is a maximal ideal so that P is quasi primary. Since $XY \in P$ but $X^{2n} \notin P^n$ and $Y^{2n} \notin P^n$, we have P is not a pseudo 2-prime ideal of R. Also, note that P is a 2-absorbing primary ideal of R by [10, Theorem 2.8], and hence the converse of Theorem 2.1(iii) is not true.

Recall from [4] that a proper ideal P of R is said to be a *primary ideal* if whenever $ab \in P$ for some $a, b \in R$, then either $a \in P$ or $b \in \sqrt{P}$. It is clear that if P is a primary ideal, then it is a quasi primary ideal of R. Note that the concepts of primary ideals and pseudo 2-prime ideals are different. See, the following example.

Example 2.4. (i) Take the ring R and the ideal P as in Example 2.3. Then $\sqrt{P} = (X, Y)$ is a maximal ideal of R, so P is primary. But P is not a pseudo 2-prime ideal of R.

(ii) Take the ring R as in Example 2.3. Let $P = (X^2, XY)$. Then note that $\sqrt{P} = (X)$ is a prime ideal of R with $\sqrt{P}^2 = (X^2) \subseteq P$. Then one can easily see that P is a pseudo 2-prime ideal of R. Since $XY \in P$, $X \notin P$ and $Y \notin \sqrt{P}$, it follows that P is not primary.

By above theorem and examples, the following diagram clarifies the place of pseudo 2-prime ideals in the lattice of all ideals $L(R)$ of R.

$$\begin{tikzcd}
& \boxed{\text{prime ideal}} \arrow[dl] \arrow[d] \arrow[dr] & \\
\boxed{\text{2-absorbing ideal}} \arrow[d] & \boxed{\text{primary ideal}} \arrow[d] & \boxed{\text{2-prime ideal}} \arrow[d] \\
\boxed{\text{2-absorbing primary ideal}} & \boxed{\text{quasi primary ideal}} \arrow[l] & \boxed{\text{pseudo 2-prime ideal}} \arrow[l]
\end{tikzcd}$$

FIGURE 1. Pseudo 2-prime ideals vs other classical ideals

Now, we investigate the conditions under which pseudo 2-prime ideals, 2-prime ideals and quasi primary ideals are coincide.

Proposition 2.5. (i) *Let P be a proper ideal of R such that $\sqrt{P}^2 \subseteq P$. Then, P is a 2-prime ideal $\Leftrightarrow$ P is a pseudo 2-prime ideal $\Leftrightarrow$ P is a quasi primary ideal.*

(ii) *Let P be a 2-absorbing ideal of R. Then P is a 2-prime ideal of R if and only if P is a pseudo 2-prime ideal of R.*

(iii) *Let R be a ring. Then the zero ideal is a quasi primary ideal of R if and only if it is a pseudo 2-prime ideal of R.*

(iv) *Let R be a ring and P a proper ideal of R. If P^n is irreducible and $(P^n : x^{2n}) = (P^n : x^{2n-1})$ for some $n \in \mathbb{N}$ and each $x \in R - P$, then P is a pseudo 2-prime ideal of R.*

Proof. (i) Let P be a proper ideal of R with $\sqrt{P}^2 \subseteq P$. The implications "P is a 2-prime ideal $\Rightarrow$ P is a pseudo 2-prime ideal $\Rightarrow$ P is a quasi primary ideal" follows from Theorem 2.1. Now, we will show that the other directions. Let P be a quasi primary ideal of R. It is sufficient to show that P is a 2-prime ideal of R. To see this, let $xy \in P \subseteq \sqrt{P}$ for some $x, y \in R$. Since $\sqrt{P}$ is a prime ideal, we have $x \in \sqrt{P}$ or $y \in \sqrt{P}$. As $\sqrt{P}^2 \subseteq P$, we conclude that $x^2 \in \sqrt{P}^2 \subseteq P$ or $y^2 \in \sqrt{P}^2 \subseteq P$. Therefore, P is a 2-prime ideal of R.

(ii) Since P is a 2-absorbing ideal of R, by [7, Theorem 2.4], $\sqrt{P}^2 \subseteq P$. The rest follows from (i).

(iii) If the zero ideal is a pseudo 2-prime ideal, then by Theorem 2.1, so is quasi-primary. For the converse, assume that the zero ideal is a quasi primary ideal. Let $x, y \in R$ such that $xy = 0$. As (0) is quasi-primary, we conclude that $x \in \sqrt{0}$ or $y \in \sqrt{0}$ implying that $x^{2k} \in (0)^k = (0)$ or $y^{2k} \in (0)^k$. Therefore, the zero ideal is a pseudo 2-prime ideal of R.

(iv) Suppose that P^n is an irreducible ideal and $(P^n : x^{2n}) = (P^n : x^{2n-1})$ for some $n \in \mathbb{N}$ and each $x \in R - P$. Let $ab \in P$ for some $a, b \in R$. Now, we will show that $a^{2n} \in P^n$ or $b^{2n} \in P^n$. Suppose to the contrary. Then we have $a^{2n} \notin P^n$ and $b^{2n} \notin P^n$. Now, take $z \in (P^n + Ra^{2n}) \cap (P^n + Rb^{2n})$. Then we can write $z = c + ra^{2n} = d + sb^{2n}$ for some $c, d \in P^n$ and $r, s \in R$. Then we have $za^{2n} = ca^{2n} + ra^{4n} = da^{2n} + s(ab)^{2n} \in P^n$, which implies $ra^{4n} \in P^n$. Since $a \in R - P$, by assumption, we get $ra^{2n} \in (P^n : a^{2n}) = (P^n : a^{2n-1})$, which yields that $ra^{4n-1} \in P^n$. If we continue in this manner, we can get $ra \in (P^n : a^{2n}) = (P^n : a^{2n-1})$ and so we have $ra^{2n} \in P^n$. Then we have $z = c + ra^{2n} \in P^n$ implying that $(P^n + Ra^{2n}) \cap (P^n + Rb^{2n}) = P^n$ which is a contradiction. Therefore, $a^{2n} \in P^n$ or $b^{2n} \in P^n$, namely, P is a pseudo 2-prime ideal of R. $\square$

Theorem 2.6. *Let $f : R \to S$ be a ring epimorphism and P a proper ideal of R. The following statements are satisfied.*

(i) *If P is a pseudo 2-prime ideal of R containing $Ker(f)$, then $f(P)$ is a pseudo 2-prime ideal of S.*

(ii) *If $f(P)$ is a pseudo 2-prime ideal of S such that $Ker(f) \subseteq P^n$ for each $n \in \mathbb{N}$, then P is a pseudo 2-prime ideal of R.*

Proof. (i) Let $yz \in f(P)$ for some $y, z \in S$. Since f is surjective, there exist $a, b \in R$ such that $f(a) = y$ and $f(b) = z$. Then we have $yz = f(ab) \in f(P)$. As $Ker(f) \subseteq P$, we have $ab \in P$. Since P is a pseudo 2-prime ideal of R, there exists $n \in \mathbb{N}$ such that either $a^{2n} \in P^n$ or $b^{2n} \in P^n$. This yields that $y^{2n} = f(a^{2n}) \in f(P)^n$ or $z^{2n} = f(b^{2n}) \in f(P)^n$. Hence, $f(P)$ is a pseudo 2-prime ideal of S.

(ii) Let $ab \in P$ for some $a, b \in R$. Then we have $f(a)f(b) = f(ab) \in f(P)$. As $f(P)$ is a pseudo 2-prime ideal of R, there exists $n \in \mathbb{N}$ such that $f(a)^{2n} = f(a^{2n}) \in f(P^n)$ or $f(b)^{2n} = f(b^{2n}) \in f(P^n)$. Since $P^n \supseteq Ker(f)$, we conclude that $a^{2n} \in P^n$ or $b^{2n} \in P^n$. Therefore, P is a pseudo 2-prime ideal of R. □

Corollary 2.7. (i) *Let P be a pseudo 2-prime ideal of R and $I \subseteq P$ be an ideal of R. Then P/I is a pseudo 2-prime ideal of R/I.*

(ii) *Let P/I be a pseudo 2-prime ideal of R/I, where I is an ideal of R such that $I \subseteq P^n$ for each $n \in \mathbb{N}$. Then P is a pseudo 2-prime ideal of R.*

Proof. (i) Consider the surjective homomorphism $\pi : R \to R/I$ defined by $\pi(a) = a + I$ for each $a \in R$. Then note that $Ker(\pi) = I \subseteq P$. If P is a pseudo 2-prime ideal, then by Theorem 2.6, $\pi(P) = P/I$ is a pseudo 2-prime ideal R/I.

(ii) By applying Theorem 2.6, one can prove the claim. □

The conditions "$I \subseteq P^n$" and "$Ker(f) \subseteq P^n$" in Corollary 2.7(ii) and Theorem 2.6(ii) are necessary. See the following example.

Example 2.8. Consider the ring $R = k[X, Y]$ and the ideal $P = (X^3, XY, Y^3) = I$ as in Example 2.3. Then note that $I \nsubseteq P^n$ for each $n \geq 2$. By Example 2.3, we know that P is not a pseudo 2-prime ideal of R. Also note that $\sqrt{P/I} = (X, Y)/I$ is a prime ideal of R/I, so by Proposition 2.5, the zero ideal P/I is a pseudo 2-prime ideal of R/I.

Let R be a ring and P a proper ideal of R. Then we denote the set $\{x \in R : xy \in P \text{ for some } y \in R - P\}$ by $Z_R(P)$.

Proposition 2.9. *Let R be a ring and S be a multiplicatively closed set of R.*

(i) *If P is a pseudo 2-prime ideal of R with $P \cap S = \emptyset$, then $S^{-1}P$ is a pseudo 2-prime ideal of $S^{-1}R$.*

(ii) *If $S^{-1}P$ is a pseudo 2-prime ideal of $S^{-1}R$ with $S \cap Z_R(P^n) = \emptyset$ for each $n \in \mathbb{N}$, then P is a pseudo 2-prime ideal of R.*

Proof. (i) Let $\frac{a}{s}\frac{b}{t} \in S^{-1}P$ for some $a, b \in R$; $s, t \in S$. Then we have $u(ab) = (ua)b \in P$ for some $u \in S$. As P is a pseudo 2-prime ideal, then there exists $n \in \mathbb{N}$ such that $(ua)^{2n} = u^{2n}a^{2n} \in P^n$ or $b^{2n} \in P^n$. This yields that $(\frac{a}{s})^{2n} = \frac{a^{2n}}{s^{2n}} = \frac{u^{2n}a^{2n}}{u^{2n}s^{2n}} \in S^{-1}(P^n) = (S^{-1}P)^n$ or $(\frac{b}{t})^{2n} = \frac{b^{2n}}{t^{2n}} \in (S^{-1}P)^n$. Thus $S^{-1}P$ is a pseudo 2-prime ideal of $S^{-1}R$.

(ii) Let $ab \in P$ for some $a, b \in R$. Then we have $\frac{ab}{1} = \frac{a}{1}\frac{b}{1} \in S^{-1}P$. As $S^{-1}P$ is a pseudo 2-prime ideal, there exists $n \in \mathbb{N}$ such that either $(\frac{a}{1})^{2n} = \frac{a^{2n}}{1} \in$

$(S^{-1}P)^n = S^{-1}(P^n)$ or $\frac{b^{2n}}{1} \in S^{-1}(P^n)$. Then we get either $sa^{2n} \in P^n$ or $s'b^{2n} \in P^n$ for some $s, s' \in S$. Without loss of generality we may assume that $sa^{2n} \in P^n$. If $a^{2n} \notin P^n$, then we have $s \in Z_R(P^n) \cap S$ which is a contradiction. So that we have $a^{2n} \in P^n$. Therefore, P is a pseudo 2-prime ideal of R. □

The following example shows that we can not drop the condition "$S \cap Z_R(P^n) = \emptyset$" in Proposition 2.9.

Example 2.10. Consider the ring $R = \mathbb{Z}$ of integers and the ideal $P = (6)$ of R. Choose the multiplicatively closed set $S = R - M$, where $M = (3)$. Then it is easy to see that $P_M = M_M$ is a prime ideal of R_M, so is a pseudo 2-prime ideal by Theorem 2.1. Also note that $2^n \in Z_R(P^n) \cap S$ since $2^n \cdot 3^n \in P^n$ and $3^n \notin P^n$ for each $n \in \mathbb{N}$. However, P is not a pseudo 2-prime ideal since $2 \cdot 3 \in P$, $2^{2n} \notin P^n$ and $3^{2n} \notin P^n$ for each $n \in \mathbb{N}$.

Now, we determine all pseudo 2-prime ideals in principal ideal domains.

Theorem 2.11. *Let R be a principal ideal domain and P a nonzero proper ideal of R. The following statements are equivalent.*
(i) *P is a pseudo 2-prime ideal of R.*
(ii) *$P = (p^n)$ for some irreducible element $p \in R$ and $n \in \mathbb{N}$.*
(iii) *P is a primary ideal of R.*
(iv) *P is a quasi primary ideal of R.*
(v) *P is a 2-prime ideal of R.*

Proof. (i)⇒(ii): Let P be a pseudo 2-prime ideal of a principal ideal domain R. Suppose that P is nonzero. Then we can write $P = (x)$, where $x = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$ and p_i's are irreducible elements of R. If $m = 1$, then we are done. So assume that $m > 1$ and all p_i's are not associates. Now, put $a = p_1^{n_1}$ and $b = p_2^{n_2} p_3^{n_3} \cdots p_m^{n_m}$. Since $ab \in P$ and P is a pseudo 2-prime ideal, there exists $k \in \mathbb{N}$ such that $a^{2k} \in P^k = (x^k)$ or $b^{2k} \in P^k = (x^k)$. In the former case, we have $p_1^{2kn_1} = c p_1^{kn_1} p_2^{kn_2} \cdots p_m^{kn_m}$ for some $c \in R$, which implies that $p_1^{kn_1} = c p_2^{kn_2} \cdots p_m^{kn_m}$. Since $p_1 \mid p_1^{kn_1} = c p_2^{kn_2} \cdots p_m^{kn_m}$ and p_1 is a prime element, we conclude that p_1 divides p_j for some $j \neq 1$, and thus we have p_1, p_j are associates, a contradiction. In the later case, we have $p_2^{2kn_2} p_3^{2kn_3} \cdots p_m^{2kn_m} = c p_1^{kn_1} p_2^{kn_2} \cdots p_m^{kn_m}$ for some $c \in R$, which implies that $p_2^{kn_2} \cdots p_m^{kn_m} = c p_1^{kn_1}$. Similar argument shows that p_1, p_j are associates, again a contradiction. Thus we have $m = 1$ and so $P = (p^n)$ for some irreducible element $p \in R$ and $n \in \mathbb{N}$.

(ii)⇒(iii): Since $\sqrt{P} = (p)$ is a maximal ideal, P is primary.

(iii)⇒(iv)⇒(v): Follows from [21, Theorem 2.3].

(v)⇒(i): Follows from Theorem 2.1. □

Recall from [23] that a ring R is said to be a *von Neumann regular ring* if for each $x \in R$, there exists $y \in R$ such that $x = x^2 y$. In that case the principal ideal (x) is generated by an idempotent element $e \in R$. It is well known that a ring R is a von Neumann regular ring if and only if for each ideal I of R and

$n \in \mathbb{N}$, $I = I^n$ if and only $I = \sqrt{I}$ for each ideal I of R if and only if R is reduced and every prime ideal is maximal. The notion of von Neumann regular rings and its generalizations have an important place in commutative algebra and have been widely studied by many authors. See, for example, [1], [11] and [17].

Proposition 2.12. *Let R be a von Neumann regular ring and P a proper ideal of R. The following statements are equivalent.*
(i) *P is a maximal ideal.*
(ii) *P is a prime ideal.*
(iii) *P is a primary ideal.*
(iv) *P is a 2-prime ideal.*
(v) *P is a pseudo 2-prime ideal.*
(vi) *P is a quasi primary ideal.*

Proof. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(vi): The proofs follow from the fact that $I = \sqrt{I}$ for each ideal I of R and every prime ideal is maximal in von Neumann regular ring.

(iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi): The proofs follow from the fact that $\sqrt{P}^2 = P^2 \subseteq P$ in von Neumman regular ring and Proposition 2.5. $\square$

Theorem 2.13. *Let R_1, R_2 be two commutative rings and P_1, P_2 be ideals of R_1 and R_2, respectively. Suppose that $R = R_1 \times R_2$ and $P = P_1 \times P_2$. The following statements are equivalent.*
(i) *P is a pseudo 2-prime ideal of R.*
(ii) *$P_1 = R_1$ and P_2 is a pseudo 2-prime ideal of R_2 or $P_2 = R_2$ and P_1 is a pseudo 2-prime ideal of R_1.*

Proof. (i)$\Rightarrow$(ii): Let P be a pseudo 2-prime ideal of R. Since $(1,0)(0,1) = (0,0) \in P$, there exists an $n \in \mathbb{N}$ such that either $(1,0)^{2n} = (1,0) \in P^n \subseteq P$ or $(0,1)^{2n} = (0,1) \in P^n \subseteq P$. This implies either $P_1 = R_1$ or $P_2 = R_2$. Without loss of generality, we may assume that $P_1 = R_1$. Now, we will show that P_2 is a pseudo 2-prime ideal of R_2. To see this, take $ab \in P_2$ for some $a, b \in R_2$. Then we have $(0, ab) = (0, a)(0, b) \in P$. As P is a pseudo 2-prime ideal, there exists $m \in \mathbb{N}$ such that $(0, a)^{2m} = (0, a^{2m}) \in P^m$ or $(0, b)^{2m} = (0, b^{2m}) \in P^m$. Then we get either $a^{2m} \in P_2^m$ or $b^{2m} \in P_2^m$. Thus P_2 is a pseudo 2-prime ideal of R_2.

(ii)$\Rightarrow$(i): Without loss of generality, we may assume that $P_1 = R_1$ and P_2 is a pseudo 2-prime ideal of R_2. Let $(x, y)(z, t) = (xz, yt) \in P$ for some $x, z \in R_1$; $y, t \in R_2$. This implies that $yt \in P_2$. As P_2 is a pseudo 2-prime ideal, there exists $m \in \mathbb{N}$ such that $y^{2m} \in P_2^m$ or $t^{2m} \in P_2^m$. This gives $(x, y)^{2m} = (x^{2m}, y^{2m}) \in P^m$ or $(z, t)^{2m} = (z^{2m}, t^{2m}) \in P^m$. Hence, P is a pseudo 2-prime ideal of R. $\square$

Theorem 2.14. *Let $R_1, R_2, \dots, R_n$ be commutative rings and P_i be an ideal of R_i for each $i = 1, 2, \dots, n$. Suppose that $R = R_1 \times R_2 \times \cdots \times R_n$ and $P = P_1 \times P_2 \times \cdots \times P_n$. The following statements are equivalent.*

(i) *P is a pseudo 2-prime ideal of R.*

(ii) *P_i is a pseudo 2-prime ideal of R_i for some $i \in \{1, 2, \dots, n\}$ and $P_j = R_j$ for each $i \neq j$.*

Proof. We use induction on n. If $n = 1$, the claim is obvious. If $n = 2$, the claim follows from Theorem 2.13. So assume that the claim is true for all $k < n$. Now, let $R' = R_1 \times R_2 \times \cdots \times R_{n-1}$ and $P' = P_1 \times P_2 \times \cdots \times P_{n-1}$. Then by Theorem 2.13, P is a pseudo 2-prime ideal of R if and only if P' is a pseudo 2-prime ideal of R' and $P_n = R_n$ or $P' = R'$ and P_n is a pseudo 2-prime ideal of R_n. The rest follows from induction hypothesis. □

Now, we characterize AV-domains in terms of pseudo 2-prime ideals.

Theorem 2.15. *Let R be an integral domain. The following statements are equivalent.*

(i) *R is an AV-domain.*

(ii) *Every proper ideal is a pseudo 2-prime ideal.*

(iii) *Every proper principal ideal is a pseudo 2-prime ideal.*

Proof. (i)⇒(ii): Let R be an AV-domain. Suppose that P is a proper ideal of R and $xy \in P$ for some $x, y \in R - \{0\}$. Since R is an AV-domain, there exists $n \in \mathbb{N}$ such that either $x^n \mid y^n$ or $y^n \mid x^n$. If $x^n \mid y^n$, then we can write $y^n = rx^n$. As $xy \in P$, then we have $y^{2n} = rx^n y^n = r(xy)^n \in P^n$. In other case, we have $x^{2n} \in P^n$. Thus P is a pseudo 2-prime ideal of R.

(ii)⇒(iii): It is clear.

(iii)⇒(i): Suppose that $x, y \in R - \{0\}$. If x or y is unit, then we are done. So assume that x and y are not units of R. Thus (xy) is a proper principal ideal, so by assumption, (xy) is a pseudo 2-prime ideal. Since $xy \in (xy)$, there exists $n \in \mathbb{N}$ such that $x^{2n} \in (xy)^n = (x^n y^n)$ or $y^{2n} \in (x^n y^n)$. If $x^{2n} \in (x^n y^n)$, then there exists $a \in R$ such that $x^{2n} = ax^n y^n$. As R is an integral domain, we have $x^n = ay^n$, namely, $y^n \mid x^n$. For the other case, one can conclude that $x^n | y^n$. Thus R is an AV-domain. □

Let R be a ring and M be a unital R-module. Trivial extension $R \ltimes M = R \oplus M$ of an R-module M is a commutative ring with componentwise addition and multiplication defined by $(a, m)(b, m') = (ab, am' + bm)$ for each $a, b \in R$; $m, m' \in M$ [2]. If I is an ideal of R and N is a submodule of M. Then $I \ltimes N$ is an ideal of $R \ltimes M$ if and only if $IM \subseteq N$ [2, 15]. Now, we investigate pseudo 2-prime ideals of trivial extension $R \ltimes M$ of M.

Theorem 2.16. *Let R be a ring and M be a unital R-module. Suppose that P is an ideal of R and N is a submodule of M such that $PM \subseteq N$. The following statements are satisfied.*

(i) *If $P \ltimes N$ is a pseudo 2-prime ideal of $R \ltimes M$, then P is a pseudo 2-prime ideal of R.*

(ii) *If P is a pseudo 2-prime ideal of R and $(P^n : x^{2n}) = (P^n : x^{2n-1})$ for each $x \in R - P$ and $n \in \mathbb{N}$, then $P \ltimes N$ is a pseudo 2-prime ideal of $R \ltimes M$.*

(iii) *Assume that $(P^n : x^{2n}) = (P^n : x^{2n-1})$ for each $x \in R - P$ and $n \in \mathbb{N}$. Then $P \ltimes N$ is a pseudo 2-prime ideal of $R \ltimes M$ if and only if P is a pseudo 2-prime ideal of R.*

Proof. (i) Let $xy \in P$ for some $x, y \in R$. Then we have $(x, 0)(y, 0) \in P \ltimes N$. As $P \ltimes N$ is a pseudo 2-prime ideal of $R \ltimes M$, we conclude either $(x, 0)^{2n} = (x^{2n}, 0) \in (P \ltimes N)^n$ or $(y, 0)^{2n} = (y^{2n}, 0) \in (P \ltimes N)^n$ for some $n \in \mathbb{N}$. Also note that $(P \ltimes N)^n \subseteq P^n \ltimes P^{n-1}N$. Then we get $x^{2n} \in P^n$ or $y^{2n} \in P^n$. Therefore, P is a pseudo 2-prime ideal of R.

(ii) First note that $(P^n \ltimes P^n M) = (P \ltimes PM)^n \subseteq (P \ltimes N)^n$. Let

$$(x, m)(y, m') = (xy, xm' + ym) \in P \ltimes N$$

for some $x, y \in R$; $m, m' \in M$. This implies that $xy \in P$. Since P is a pseudo 2-prime ideal of R, we get either $x^{2n} \in P^n$ or $y^{2n} \in P^n$ for some $n \in \mathbb{N}$. Without loss of generality, we may assume that $x^{2n} \in P^n$. If $x \in P$, then we have $(x, m)^{2n} = (x^{2n}, (2n)x^{2n-1}m) \in P^{2n} \ltimes P^{2n-1}M \subseteq P^n \ltimes P^n M \subseteq (P \ltimes N)^n$ which completes the proof. So assume that $x \notin P$. Since $x^{2n} \in P^n$, by assumption, we have $x^{2n-1} \in P^n$, which implies that $(x, m)^{2n} = (x^{2n}, (2n)x^{2n-1}m) \in P^n \ltimes P^n M \subseteq (P \ltimes N)^n$. Therefore, $P \ltimes N$ is a pseudo 2-prime ideal of $R \ltimes M$.

(iii) It follows from (i) and (ii). $\square$

Theorem 2.17. *Let $I \subseteq \bigcup\limits_{i=1}^{n} P_i$ be an efficient covering of ideals of R. Suppose that $I \cap \sqrt{P_k} \nsubseteq I \cap \sqrt{P_m}$ for each $k \neq m$. Then no P_i is a pseudo 2-prime ideal for each $i \in \{1, 2, \ldots, n\}$.*

Proof. Suppose that $I \subseteq \bigcup\limits_{i=1}^{n} P_i$ is an efficient covering of ideals of R and $I \cap \sqrt{P_k} \nsubseteq I \cap \sqrt{P_m}$ for each $k \neq m$. Assume that P_1 is a pseudo 2-prime ideal of R. Then by Theorem 2.1, $\sqrt{P_1}$ is a prime ideal of R. Since the covering is efficient, we have $I \cap \bigcap\limits_{i=2}^{n} P_i \subseteq I \cap P_1$. Also, by assumption, there exists $x_j \in I \cap \sqrt{P_j} - \sqrt{P_1}$ for each $j = 2, 3, \ldots, n$. Then there exists $t_j \in \mathbb{N}$ such that $x_j^{t_j} \in I \cap P_j$. Now, put $m = \max\{t_2, t_3, \ldots, t_n\}$. This implies that $x_j^m \in I \cap P_j$. Let $a = x_2$ and $b = x_3 x_4 \cdots x_n$. Since $\sqrt{P_1}$ is a prime ideal and $x_3, x_4, \ldots, x_n \notin \sqrt{P_1}$, we have $b^{2nm} \notin P_1^n$ for all $n \in \mathbb{N}$. Otherwise, we would have $b = x_3 x_4 \cdots x_n \in \sqrt{P_1^n} = \sqrt{P_1}$, which implies that $x_i \in \sqrt{P_1}$ for some $i \geq 3$, a contradiction. On the other hand, since P_1 is a pseudo 2-prime ideal and $a^m b^m \in P_1$, we have $a^{2nm} = x_2^{2nm} \in P_1^n$ for some $n \in \mathbb{N}$, which implies that $x_2 \in \sqrt{P_1^n} = \sqrt{P_1}$, again a contradiction. Therefore, no P_i is a pseudo 2-prime ideal of R. □

Now, we prove that pseudo 2-prime avoidance theorem for commutative rings.

Theorem 2.18 (Pseudo 2-Prime Avoidance Theorem)**.** *Let* $I \subseteq \bigcup_{i=1}^{n} P_i$ *for some ideals* $I, P_1, P_2, \ldots, P_n$ *of* R*, where at most two of* P_i*'s are not pseudo 2-prime ideals of* R*. Suppose that* $I \cap \sqrt{P_k} \nsubseteq I \cap \sqrt{P_m}$ *for each* $k \neq m$*. Then* $I \subseteq P_i$ *for some* $i \in \{1, 2, \ldots, n\}$*.*

Proof. We can reduce the covering to efficient one. So suppose that the covering $I \subseteq \bigcup_{i=1}^{n} P_i$ is an efficient, where at least $n-2$ of P_i's are pseudo 2-prime ideals of R. If $n \geq 3$, then we get a contradiction by using Theorem 2.17. Thus we have $n \leq 2$. The rest is clear. □

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References

[1] D. D. Anderson, S. Chun, and J. R. Juett, *Module-theoretic generalization of commutative von Neumann regular rings*, Comm. Algebra **47** (2019), no. 11, 4713–4728. https://doi.org/10.1080/00927872.2019.1593427

[2] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra **1** (2009), no. 1, 3–56. https://doi.org/10.1216/JCA-2009-1-1-3

[3] D. D. Anderson and M. Zafrullah, *Almost Bézout domains*, J. Algebra **142** (1991), no. 2, 285–309. https://doi.org/10.1016/0021-8693(91)90309-V

[4] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, MA, 1969.

[5] A. Ayache and D. E. Dobbs, *Strongly divided pairs of integral domains*, in Advances in commutative algebra, 63–92, Trends Math, Birkhäuser/Springer, Singapore, 2019. https://doi.org/10.1007/978-981-13-7028-1_4

[6] A. Badawi, *On divided commutative rings*, Comm. Algebra **27** (1999), no. 3, 1465–1474. https://doi.org/10.1080/00927879908826507

[7] ———, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. **75** (2007), no. 3, 417–429. https://doi.org/10.1017/S0004972700039344

[8] A. Badawi and D. E. Dobbs, *On locally divided rings and going-down rings*, Comm. Algebra **29** (2001), no. 7, 2805–2825. https://doi.org/10.1081/AGB-100104988

[9] C. Beddani and W. Messirdi, *2-prime ideals and their applications*, J. Algebra Appl. **15** (2016), no. 3, 1650051, 11 pp. https://doi.org/10.1142/S0219498816500511

[10] A. Badawi, U. Tekir, and E. Yetkin, *On 2-absorbing primary ideals in commutative rings*, Bull. Korean Math. Soc. **51** (2014), no. 4, 1163–1173. https://doi.org/10.4134/BKMS.2014.51.4.1163

[11] S. Çeken Gezen, *On M-coidempotent elements and fully coidempotent modules*, Comm. Algebra **48** (2020), no. 11, 4638–4646. https://doi.org/10.1080/00927872.2020.1768266

[12] D. E. Dobbs, *Divided rings and going-down*, Pacific J. Math. **67** (1976), no. 2, 353–363. http://projecteuclid.org/euclid.pjm/1102817497

[13] D. E. Dobbs, A. El Khalfi, and N. Mahdou, *Trivial extensions satisfying certain valuation-like properties*, Comm. Algebra **47** (2019), no. 5, 2060–2077. https://doi.org/10.1080/00927872.2018.1527926

[14] L. Fuchs, *On quasi-primary ideals*, Acta Univ. Szeged. Sect. Sci. Math. **11** (1947), 174–183.

[15] J. A. Huckaba, *Commutative rings with zero divisors*, Monographs and Textbooks in Pure and Applied Mathematics, 117, Marcel Dekker, Inc., New York, 1988.

[16] R. Jahani-Nezhad and F. Khoshayand, *Almost valuation rings*, Bull. Iranian Math. Soc. **43** (2017), no. 3, 807–816.

[17] C. Jayaram and Tekir, *von Neumann regular modules*, Comm. Algebra **46** (2018), no. 5, 2205–2217. https://doi.org/10.1080/00927872.2017.1372460

[18] S. Koc, U. Tekir, and G. Ulucak, *On strongly quasi primary ideals*, Bull. Korean Math. Soc. **56** (2019), no. 3, 729–743. https://doi.org/10.4134/BKMS.b180522

[19] Max. D. Larsen and P. J. McCarthy, *Multiplicative Theory of Ideals*, Academic Press, New York, 1971.

[20] N. Mahdou, A. Mimouni, and M. A. S. Moutui, *On almost valuation and almost Bézout rings*, Comm. Algebra **43** (2015), no. 1, 297–308. https://doi.org/10.1080/00927872.2014.897586

[21] R. Nikandish, M. J. Nikmehr, and A. Yassine, *More on the 2-prime ideals of commutative rings*, Bull. Korean Math. Soc. **57** (2020), no. 1, 17–126.

[22] Ü. Tekir, G. Ulucak, and S. Koç, *On divided modules*, Iran. J. Sci. Technol. Trans. A Sci. **44** (2020), no. 1, 265–272. https://doi.org/10.1007/s40995-020-00827-1

[23] J. von Neumann, *On regular rings*, Proceedings of the National Academy of Sci. **22** (1936), no. 12, 707–713.

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