

MINIMAL TRANSLATION SURFACES WITH RESPECT TO SEMI-SYMMETRIC CONNECTIONS IN \mathbb{R}^3 AND \mathbb{R}_1^3

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ABSTRACT. In this paper, we define and classify minimal translation surfaces with respect to a kind of semi-symmetric metric connections and a kind of semi-symmetric non-metric connections in \mathbb{R}^3 and \mathbb{R}_1^3 .

1. Introduction

Minimal surfaces are among the most natural objects in differential geometry, and have been studied during the last two and half centuries since J. L. Lagrange. In particular, minimal surfaces have encountered striking applications in other fields, like mathematical physics, conformal geometry, computer aided design, among others. In order to search for more minimal surfaces, some natural geometric assumptions arise. Translation surfaces were studied in the Euclidean 3-dimensional space and they are represented as graphs $z = f(x) + g(y)$, where f and g are smooth functions. Scherk proved in 1835 that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right|,$$

where a is a non-zero constant. Since then, minimal translation surfaces were generalized in several directions. For example, the Euclidean space \mathbb{R}^3 was replaced with other spaces of dimension 3—usually being 3-dimensional Lie groups and the notion of translation was often replaced by using the group operation. See for example [9, 11, 12, 15, 18, 20]. Another generalizations of Scherk surfaces are: affine translation surfaces in Euclidean 3-space [10], affine translation surfaces in affine 3-dimensional space [16] and translation surfaces in Galilean 3-space [19]. On the other hand, Scherk surfaces were generalized to minimal translation surfaces in Euclidean spaces of arbitrary dimensions. See for example [5, 13].

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H. A. Hayden introduced the notion of a semi-symmetric metric connection on a Riemannian manifold [6]. K. Yano studied a Riemannian manifold endowed with a semi-symmetric metric connection [17]. Some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection were studied by T. Imai [7, 8]. Z. Nakao [14] studied submanifolds of a Riemannian manifold with semi-symmetric metric connections. In [3], Gozutok and Esin studied the tangent bundle of a hypersurface with semi-symmetric metric connections. In [4], Demirbag investigated the properties of a weakly Ricci symmetric manifold admitting a semi-symmetric metric connection. N. S. Agashe and M. R. Chafle introduced the notion of a semisymmetric non-metric connection and studied some of its properties and submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection [1, 2].

In this paper, we classify minimal translation surfaces with respect to a kind of semi-symmetric metric connections and a kind of semi-symmetric non-metric connections in the 3-dimensional Euclidean space \mathbb{R}^3 and Minkowski space \mathbb{R}_1^3 . In Section 2, we classify minimal translation surfaces with respect to a kind of semi-symmetric metric connections and a kind of semi-symmetric non-metric connections in \mathbb{R}^3 . In Section 3, we classify minimal translation surfaces with respect to a kind of semi-symmetric metric connections and a kind of semi-symmetric non-metric connections in \mathbb{R}_1^3 .

2. Minimal translation surfaces with respect to semi-symmetric connections in \mathbb{R}^3

Let \mathbb{R}^3 be the 3-dimensional Euclidean space with the canonical Euclidean metric \tilde{g} . Let $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial y}$, $X_3 = \frac{\partial}{\partial z}$. Let $\nabla_{X_i}^L X_j = 0$ for $1 \leq i, j \leq 3$ be the Levi-Civita connection on \mathbb{R}^3 . We recall the general semi-symmetric metric connection on \mathbb{R}^3 defined by $\nabla_X^L Y + \tilde{g}(Y, P)X - \tilde{g}(X, Y)P$ where $P = h_1 X_1 + h_2 X_2 + h_3 X_3$ and $h_1, h_2, h_3 \in C^\infty(\mathbb{R}^3)$. In order to simplify computations in the following, we choose $P = X_3$. We define a special semi-symmetric metric connection by

$$(1) \quad \nabla_X Y = \nabla_X^L Y + \tilde{g}(Y, X_3)X - \tilde{g}(X, Y)X_3.$$

The connection ∇ of \mathbb{R}^3 is given by

$$(2) \quad \begin{aligned} \nabla_{X_1} X_1 &= -X_3, \quad \nabla_{X_1} X_2 = 0, \quad \nabla_{X_1} X_3 = X_1, \\ \nabla_{X_2} X_1 &= 0, \quad \nabla_{X_2} X_2 = -X_3, \quad \nabla_{X_2} X_3 = X_2, \\ \nabla_{X_3} X_1 &= 0, \quad \nabla_{X_3} X_2 = 0, \quad \nabla_{X_3} X_3 = 0. \end{aligned}$$

Definition 2.1. A surface M in \mathbb{R}^3 is a translation surface if it is given by an isometric immersion $\mathcal{F} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ of the form

$$(3) \quad \mathcal{F}(u, v) = (u, v, f(u) + g(v)), \quad (\text{Type I})$$

or

$$(4) \quad \mathcal{F}(u, v) = (u, f(u) + g(v), v), \text{ (Type II)}$$

or

$$(5) \quad \mathcal{F}(u, v) = (f(u) + g(v), u, v), \text{ (Type III)}$$

where f and g are smooth functions on open sets of \mathbb{R} .

Let $E_1 = \mathcal{F}_u, E_2 = \mathcal{F}_v$ and $\{E_1, E_2\}$ be the basis of TM and N be the unit normal vector field of TM in \mathbb{R}^3 . Let $p : T\mathbb{R}^3|_M \rightarrow TM$ be the projection. The we have the Gauss formula with respect to ∇ as follows:

$$(6) \quad \nabla_X Y = p\nabla_X Y + \sigma(X, Y)N,$$

where $X, Y \in TM$ and $\sigma(X, Y)$ is the second fundamental form with respect to ∇ . In general, $\sigma(X, Y) \neq \sigma(Y, X)$. Let $\{e_1, e_2\}$ be an orthonormal basis of TM . We define the mean curvature of M with respect to ∇ as follows: $H = \frac{1}{2}[\sigma(e_1, e_1) + \sigma(e_2, e_2)]$. Let

$$(7) \quad E = \tilde{g}(\mathcal{F}_u, \mathcal{F}_u), \quad F = \tilde{g}(\mathcal{F}_u, \mathcal{F}_v), \quad G = \tilde{g}(\mathcal{F}_v, \mathcal{F}_v).$$

Then

$$(8) \quad H = \frac{G\tilde{g}(\nabla_{E_1} E_1, N) - F\tilde{g}(\nabla_{E_1} E_2, N) - F\tilde{g}(\nabla_{E_2} E_1, N) + E\tilde{g}(\nabla_{E_2} E_2, N)}{2(EG - F^2)}.$$

We called that M is minimal with respect to ∇ if $H = 0$. So by (8), M is minimal with respect to ∇ if and only if

$$(9) \quad G\tilde{g}(\nabla_{E_1} E_1, N) - F\tilde{g}(\nabla_{E_1} E_2, N) - F\tilde{g}(\nabla_{E_2} E_1, N) + E\tilde{g}(\nabla_{E_2} E_2, N) = 0.$$

Let us consider a translation surface M of Type I in \mathbb{R}^3 parametrized by $\mathcal{F}(u, v) = (u, v, f(u) + g(v))$. The tangent plane of M is spanned by

$$(10) \quad \mathcal{F}_u = X_1 + f'(u)X_3, \text{ and } \mathcal{F}_v = X_2 + g'(v)X_3,$$

while the unit normal N (up to orientation) is given by

$$(11) \quad N = \frac{1}{\alpha}[-f'(u)X_1 - g'(v)X_2 + X_3],$$

where $\alpha^2 = f'(u)^2 + g'(v)^2 + 1$.

We obtain the coefficients of the first fundamental form of \mathcal{F} as

$$(12) \quad E = 1 + f'(u)^2, \quad F = f'(u)g'(v), \quad G = 1 + g'(v)^2.$$

Then, the semi-symmetric metric connection (1) on the surface is given by

$$(13) \quad \begin{cases} \nabla_{\mathcal{F}_u} \mathcal{F}_u = f'(u)X_1 + [f''(u) - 1]X_3, \\ \nabla_{\mathcal{F}_u} \mathcal{F}_v = g'(v)X_1, \\ \nabla_{\mathcal{F}_v} \mathcal{F}_u = f'(u)X_2, \\ \nabla_{\mathcal{F}_v} \mathcal{F}_v = g'(v)X_2 + [g''(v) - 1]X_3. \end{cases}$$

Consequently, the minimality condition (9) may be expressed as follows:

$$(14) \quad f''(u)g'(v)^2 - 2f'(u)^2 - 2g'(v)^2 + f'(u)^2g''(v) + f''(u) + g''(v) - 2 = 0.$$

We will solve (14). Let us assume first that f', g', f'' and g'' are different from zero at every point. Taking successive derivatives with respect to u and v , we obtain

$$(15) \quad \frac{f'''}{f'f''} = -\frac{g'''}{g'g''}.$$

In the following, c_j denotes a constant where j is a positive integer. Remark that the left-hand side of equation (15) is a function of u , while the right-hand side is a function of v . Therefore, there exist three constants c_0, c_1, c_2 such that

$$(16) \quad f'' = \frac{c_0}{2}f'^2 + c_1, \quad g'' = -\frac{c_0}{2}g'^2 + c_2.$$

Now, plugging (16) into (14) we obtain

$$(17) \quad (c_2 + \frac{c_0}{2} - 2)f'^2 + c_1 + c_2 - 2 = (-c_1 + 2 + \frac{c_0}{2})g'^2.$$

Remark that the left-hand side of equation (17) is a function of u , while the right-hand side is a function of v . Therefore, $(-c_1 + 2 + \frac{c_0}{2})g'^2 = \bar{c}_0$. Then $g'^2 = \tilde{c}_0$ or $-c_1 + 2 + \frac{c_0}{2} = 0$. But we assume that g' and g'' are different from zero at every point, so we get

$$(18) \quad -c_1 + 2 + \frac{c_0}{2} = 0.$$

By (17), we get

$$(19) \quad (c_2 + \frac{c_0}{2} - 2)f'^2 + c_1 + c_2 - 2 = 0.$$

Since we assume that f' and f'' are different from zero at every point, so we get

$$(20) \quad c_2 + \frac{c_0}{2} - 2 = 0, \quad c_1 + c_2 - 2 = 0.$$

By (18) and (20), we have a contradiction. So in this case, we have no solutions.

Case 1) There exists u_0 such that $f''(u_0) \neq 0$ and there exists v_0 such that $g''(v_0) \neq 0$. Then there is an open interval U of u_0 such that $f''|_U \neq 0$. Then there exists $u_1 \in U$ such that $f'(u_1) \neq 0$ and there is an open interval $U_1 \subset U$ of u_1 such that $f'|_{U_1} \neq 0$ and $f''|_{U_1} \neq 0$. Similarly, there is an open interval V_1 such that $g'|_{V_1} \neq 0$ and $g''|_{V_1} \neq 0$. By the above discussions, we know that we have no solutions in this case.

Case 2) There exists u_0 such that $f''(u_0) \neq 0$ and $g''(v) = 0$. So $g'(v) = c_3$ and $g(v) = c_3v + c_4$. By (14), we have

$$(21) \quad f''(u) - \frac{2}{c_3^2 + 1}f'(u)^2 - 2 = 0.$$

The general solution of this ODE is found as

$$(22) \quad f(u) = -\frac{c_3^2 + 1}{2} \ln \left| \cos \left(\frac{2}{\sqrt{c_3^2 + 1}}u - a \right) \right| + b,$$

where a, b are constant. So

$$(23) \quad \mathcal{F}(u, v) = (u, v, -\frac{c_3^2 + 1}{2} \ln \left| \cos \left(\frac{2}{\sqrt{c_3^2 + 1}} u - a \right) \right| + c_3 v + c_5).$$

Case 3) $f''(u) = 0$ and there exists v_0 such that $g''(v_0) \neq 0$. So $f'(u) = \bar{c}_3$ and $f(u) = \bar{c}_3 u + \bar{c}_4$. Similar to Case 2), we have

$$(24) \quad \mathcal{F}(u, v) = (u, v, -\frac{\bar{c}_3^2 + 1}{2} \ln \left| \cos \left(\frac{2}{\sqrt{\bar{c}_3^2 + 1}} v - a_1 \right) \right| + \bar{c}_3 u + c_6).$$

where \bar{c}_3, a_1 are constant.

Case 4) $f''(u) = 0$ and $g''(v) = 0$. By (14), we have $-2f'(u)^2 - 2g'(v)^2 - 2 = 0$. This is a contradiction, so we have no solutions in this case.

So we have the following theorem:

Theorem 2.2. *Type I minimal translation surfaces with respect to ∇ in \mathbb{R}^3 are of the forms (23) and (24).*

In the following, we obtain all Type II minimal translation surfaces with respect to ∇ in \mathbb{R}^3 . Let M be a translation surface of Type II parametrized by $\mathcal{F}(u, v) = (u, f(u) + g(v), v)$. The tangent plane of M is spanned by

$$(25) \quad \mathcal{F}_u = X_1 + f'(u)X_2, \text{ and } \mathcal{F}_v = g'(v)X_2 + X_3,$$

while the unit normal N (up to orientation) is given by

$$(26) \quad N = \frac{1}{\beta} [f'(u)X_1 - X_2 + g'(v)X_3],$$

where $\beta^2 = f'(u)^2 + g'(v)^2 + 1$.

We obtain the coefficients of the first fundamental form of \mathcal{F} as

$$(27) \quad E = 1 + f'(u)^2, \quad F = f'(u)g'(v), \quad G = 1 + g'(v)^2.$$

Then, the semi-symmetric metric connection (1) on the surface is given by

$$(28) \quad \begin{cases} \nabla_{\mathcal{F}_u} \mathcal{F}_u = f''(u)X_2 - [f'(u)^2 + 1]X_3, \\ \nabla_{\mathcal{F}_u} \mathcal{F}_v = X_1 + f'(u)X_2 - f'(u)g'(v)X_3, \\ \nabla_{\mathcal{F}_v} \mathcal{F}_u = -f'(u)g'(v)X_3, \\ \nabla_{\mathcal{F}_v} \mathcal{F}_v = [g'(v) + g''(v)]X_2 - g'(v)^2 X_3. \end{cases}$$

Consequently, the minimality condition (9) may be expressed as follows:

$$(29) \quad 2g'^3 + 2f'^2g' + g'^2f'' + f'^2g'' + f'' + g'' + 2g' = 0.$$

We will solve (29). Let us assume first that f', g', f'' and g'' are different from zero at every point. Taking successive derivatives with respect to u and v , we obtain

$$(30) \quad \frac{f'''}{f'f''} = -\frac{2g'' + g'''}{g'g''}.$$

Remark that the left-hand side of equation (30) is a function of u , while the right-hand side is a function of v . Therefore, there exist two constants c_0, c_1 such that

$$(31) \quad f'' = \frac{c_0}{2} f'^2 + c_1, \quad \frac{g'''}{g''} = -c_0 g' - 2.$$

For (29), taking derivative with respect to v , then we have

$$(32) \quad 6g'^2 g'' + 2f'^2 g'' + 2g' g'' f'' + f'^2 g''' + g''' + 2g'' = 0.$$

Plugging (31) into (32) and g', g'' are different from zero, we obtain $g' = -\frac{2c_1 - c_0}{6}$. Then $g'' = 0$. This is a contradiction, so we have no solutions in this case.

Case 1) There exists u_0 such that $f''(u_0) \neq 0$ and there exists v_0 such that $g''(v_0) \neq 0$. By the above discussions, we know that we have no solutions in this case.

Case 2) There exists u_0 such that $f''(u_0) \neq 0$ and $g''(v) = 0$. So $g'(v) = \tilde{c}_0$ and $g(v) = \tilde{c}_0 v + \tilde{c}_1$. By (29), we have

$$(33) \quad f''(u) + \frac{2\tilde{c}_0}{\tilde{c}_0^2 + 1} f'(u)^2 + 2\tilde{c}_0 = 0.$$

Since $f'' \neq 0$, so $\tilde{c}_0 \neq 0$. The general solution of this ODE is found as

$$(34) \quad f(u) = \frac{\tilde{c}_0^2 + 1}{2\tilde{c}_0} \ln \left| \cos \left(\frac{2\tilde{c}_0}{\sqrt{\tilde{c}_0^2 + 1}} u - \tilde{a} \right) \right| + \tilde{b},$$

where \tilde{a}, \tilde{b} are constant. So

$$(35) \quad \mathcal{F}(u, v) = \left(u, \frac{\tilde{c}_0^2 + 1}{2\tilde{c}_0} \ln \left| \cos \left(\frac{2\tilde{c}_0}{\sqrt{\tilde{c}_0^2 + 1}} u - \tilde{a} \right) \right| + \tilde{c}_0 v + \tilde{b}, v \right).$$

Case 3) $f''(u) = 0$ and there exists v_0 such that $g''(v_0) \neq 0$. So $f'(u) = \hat{c}_0$ and $f(u) = \hat{c}_0 u + \hat{c}_1$. By (29), we have

$$(36) \quad g'' + \frac{2}{\hat{c}_0 + 1} g'^3 + 2g' = 0.$$

Let $g' = h$. Then we get $h' = -\frac{2}{\hat{c}_0 + 1} h^3 - 2h$. By $g''(v_0) \neq 0$, we may set $h \neq 0$ on an open interval. Let $W = h^{-2}$. Then we have

$$(37) \quad W' = \frac{4}{\hat{c}_0^2 + 1} + 4W.$$

Then its solution is $W = \hat{a}e^{4v} - \frac{1}{\hat{c}_0^2 + 1}$, where $\hat{a} > 0$. So we get

$$(38) \quad g(v) = \int_0^v \frac{\pm 1}{\sqrt{\hat{a}e^{4x} - \frac{1}{\hat{c}_0^2 + 1}}} dx + \hat{b},$$

where \widehat{a}, \widehat{b} are constant. In this case, the surface is parametrized by

$$(39) \quad \mathcal{F}(u, v) = \left(u, \int_0^v \frac{\pm 1}{\sqrt{\widehat{a}e^{4x} - \frac{1}{\widehat{c}_0^2 + 1}}} dx + \widehat{c}_0 u + b_0, v \right).$$

Case 4) $f''(u) = 0$ and $g''(v) = 0$. Then $f'(u) = c'_0$ and $g'(v) = c'_1$. By (29), we get $c'_1 = 0$. Then

$$(40) \quad \mathcal{F}(u, v) = (u, c'_0 u + b', v),$$

where c'_0, b' are constant.

So we have the following theorem:

Theorem 2.3. *Type II minimal translation surfaces with respect to ∇ in \mathbb{R}^3 are of the forms (35), (39) and (40).*

In the following, we consider Type III minimal translation surfaces with respect to ∇ in \mathbb{R}^3 . Let M be a translation surface of Type III parametrized by $\mathcal{F}(u, v) = (f(u) + g(v), u, v)$. The tangent plane of M is spanned by

$$(41) \quad \mathcal{F}_u = f'(u)X_1 + X_2, \text{ and } \mathcal{F}_v = g'(v)X_1 + X_3,$$

while the unit normal N (up to orientation) is given by

$$(42) \quad N = \frac{1}{\gamma} [X_1 - f'(u)X_2 - g'(v)X_3],$$

where $\gamma^2 = f'(u)^2 + g'(v)^2 + 1$.

We obtain the coefficients of the first fundamental form of \mathcal{F} as

$$(43) \quad E = 1 + f'(u)^2, \quad F = f'(u)g'(v), \quad G = 1 + g'(v)^2.$$

Then, the semi-symmetric metric connection (1) on the surface is given by

$$(44) \quad \begin{cases} \nabla_{\mathcal{F}_u} \mathcal{F}_u = f''(u)X_1 - [f'(u)^2 + 1]X_3, \\ \nabla_{\mathcal{F}_u} \mathcal{F}_v = f'(u)X_1 + X_2 - f'(u)g'(v)X_3, \\ \nabla_{\mathcal{F}_v} \mathcal{F}_u = -f'(u)g'(v)X_3, \\ \nabla_{\mathcal{F}_v} \mathcal{F}_v = [g'(v) + g''(v)]X_1 - g'(v)^2 X_3. \end{cases}$$

Consequently, the minimality condition (9) may be expressed as follows:

$$(45) \quad 2g'^3 + 2f'^2g' + g'^2f'' + f'^2g'' + f'' + g'' + 2g' = 0.$$

(45) is the same as (29), so similar to Theorem 2.3, we can get Type III minimal translation surfaces with respect to ∇ in \mathbb{R}^3 .

We define a special semi-symmetric non-metric connection by

$$(46) \quad \overline{\nabla}_X Y = \nabla_X^L Y + \widetilde{g}(Y, X_3)X.$$

The connection $\overline{\nabla}$ of \mathbb{R}^3 is given by

$$(47) \quad \begin{aligned} \overline{\nabla}_{X_1} X_1 &= 0, \quad \overline{\nabla}_{X_1} X_2 = 0, \quad \overline{\nabla}_{X_1} X_3 = X_1, \\ \overline{\nabla}_{X_2} X_1 &= 0, \quad \overline{\nabla}_{X_2} X_2 = 0, \quad \overline{\nabla}_{X_2} X_3 = X_2, \\ \overline{\nabla}_{X_3} X_1 &= 0, \quad \overline{\nabla}_{X_3} X_2 = 0, \quad \overline{\nabla}_{X_3} X_3 = X_3. \end{aligned}$$

In the following, we consider Type I minimal translation surfaces with respect to $\bar{\nabla}$ in \mathbb{R}^3 . For $\mathcal{F}(u, v) = (u, v, f(u) + g(v))$, then $\mathcal{F}_u, \mathcal{F}_v, N, E, F, G$ are computed by (10)-(12).

Then, the semi-symmetric non-metric connection (46) on the surface is given by

$$(48) \quad \begin{cases} \bar{\nabla}_{\mathcal{F}_u} \mathcal{F}_u = f'(u)X_1 + [f''(u) + f'(u)^2]X_3, \\ \bar{\nabla}_{\mathcal{F}_u} \mathcal{F}_v = g'(v)X_1 + f'(u)g'(v)X_3, \\ \bar{\nabla}_{\mathcal{F}_v} \mathcal{F}_u = f'(u)X_2 + f'(u)g'(v)X_3, \\ \bar{\nabla}_{\mathcal{F}_v} \mathcal{F}_v = g'(v)X_2 + [g'(v)^2 + g''(v)]X_3. \end{cases}$$

Similar to (9), we have the minimality condition with respect to $\bar{\nabla}$. Consequently, the minimality condition may be expressed as follows:

$$(49) \quad \frac{f''(u)}{1 + f'(u)^2} = -\frac{g''(v)}{1 + g'(v)^2}.$$

Solving (49), we obtain:

Theorem 2.4. *Type I minimal translation surfaces with respect to $\bar{\nabla}$ in \mathbb{R}^3 are of the following forms*

$$(50) \quad \mathcal{F}(u, v) = (u, v, c_0u + c_1v + c_2),$$

$$(51) \quad \mathcal{F}(u, v) = (u, v, \frac{1}{c} \ln \left| \frac{\cos(cu - c_3)}{\cos(cv - c_4)} \right| + c_5),$$

where $c \neq 0$.

For Type II and III minimal translation surfaces with respect to $\bar{\nabla}$ in \mathbb{R}^3 , we also get the equation (49) and we have Theorems similar to Theorem 2.4.

3. Minimal translation surfaces with respect to semi-symmetric connections in \mathbb{R}_1^3

Let \mathbb{R}_1^3 be the 3-dimensional Minkowski space with the canonical Minkowski metric $\tilde{g}_1 = dx^2 + dy^2 - dz^2$. Let $\nabla_{X_i}^L X_j = 0$ for $1 \leq i, j \leq 3$ be the Levi-Civita connection on \mathbb{R}_1^3 . We define a special semi-symmetric metric connection by

$$(52) \quad \nabla_X Y = \nabla_X^L Y + \tilde{g}_1(Y, X_3)X - \tilde{g}_1(X, Y)X_3.$$

The connection ∇ of \mathbb{R}_1^3 is given by

$$(53) \quad \begin{aligned} \nabla_{X_1} X_1 &= -X_3, \quad \nabla_{X_1} X_2 = 0, \quad \nabla_{X_1} X_3 = -X_1, \\ \nabla_{X_2} X_1 &= 0, \quad \nabla_{X_2} X_2 = -X_3, \quad \nabla_{X_2} X_3 = -X_2, \\ \nabla_{X_3} X_1 &= 0, \quad \nabla_{X_3} X_2 = 0, \quad \nabla_{X_3} X_3 = 0. \end{aligned}$$

In the following, we assume that M is a spacelike surface of \mathbb{R}_1^3 , that is the induced metric on M is Riemannian metric. When M is a timelike surface of \mathbb{R}_1^3 , we have similar discussions. Let us consider a translation surface M of

Type I in \mathbb{R}_1^3 parametrized by $\mathcal{F}(u, v) = (u, v, f(u) + g(v))$. \mathcal{F}_u and \mathcal{F}_v are given by (10). While the unit normal N (up to orientation) is given by

$$(54) \quad N = \frac{1}{|\alpha_1|}[-f'(u)X_1 - g'(v)X_2 - X_3],$$

where $\alpha_1^2 = 1 - f'(u)^2 - g'(v)^2$.

We obtain the coefficients of the first fundamental form of \mathcal{F} as

$$(55) \quad E = 1 - f'(u)^2, \quad F = -f'(u)g'(v), \quad G = 1 - g'(v)^2.$$

Then, the semi-symmetric metric connection (52) on the surface is given by

$$(56) \quad \begin{cases} \nabla_{\mathcal{F}_u}\mathcal{F}_u = -f'(u)X_1 + [f''(u) - 1]X_3, \\ \nabla_{\mathcal{F}_u}\mathcal{F}_v = -g'(v)X_1, \\ \nabla_{\mathcal{F}_v}\mathcal{F}_u = -f'(u)X_2, \\ \nabla_{\mathcal{F}_v}\mathcal{F}_v = -g'(v)X_2 + [g''(v) - 1]X_3. \end{cases}$$

In this case, we have the same minimality condition with (9). So the minimality condition may be expressed as follows:

$$(57) \quad f''(u)g'(v)^2 - 2f'(u)^2 - 2g'(v)^2 + f'(u)^2g''(v) - f''(u) - g''(v) + 2 = 0.$$

Similar to the discussions of the case 1) in the page 962, we get

Case 1) There exists u_0 such that $f''(u_0) \neq 0$ and there exists v_0 such that $g''(v_0) \neq 0$. We know that we have no solutions in this case.

Case 2) There exists u_0 such that $f''(u_0) \neq 0$ and $g''(v) = 0$. So $g'(v) = c$ and $g(v) = cv + \bar{c}$. By (57), we have

$$(58) \quad (c^2 - 1)f''(u) - 2f'(u)^2 - 2(c^2 - 1) = 0.$$

If $c^2 = 1$, then $f' = 0$ and $f'' = 0$. This is a contradiction. So $c^2 \neq 1$ and we get

$$(59) \quad f''(u) - \frac{2}{c^2 - 1}f'(u)^2 - 2 = 0.$$

If $c^2 > 1$, then the general solution of this ODE (59) is found as

$$(60) \quad f(u) = -\frac{c^2 - 1}{2} \ln \left| \cos \left(\frac{2}{\sqrt{c^2 - 1}}u - a \right) \right| + b,$$

where a, b are constant. So

$$(61) \quad \mathcal{F}(u, v) = (u, v, -\frac{c^2 - 1}{2} \ln \left| \cos \left(\frac{2}{\sqrt{c^2 - 1}}u - a \right) \right| + cv + \bar{b}).$$

If $c^2 < 1$, then the general solution of this ODE (59) is found as

$$(62) \quad f(u) = \int_0^u \frac{\sqrt{1 - c^2} + \tilde{c}e^{-\frac{4}{\sqrt{1 - c^2}}x}}{1 - \tilde{c}e^{-\frac{4}{\sqrt{1 - c^2}}x}} dx,$$

where \tilde{c} is a nonzero constant. So

$$(63) \quad \mathcal{F}(u, v) = (u, v, \int_0^u \frac{\sqrt{1-c^2} + \tilde{c}e^{-\frac{4}{\sqrt{1-c^2}}x}}{1 - \tilde{c}e^{-\frac{4}{\sqrt{1-c^2}}x}} dx + cv + \tilde{b}).$$

Case 3) $f''(u) = 0$ and there exists v_0 such that $g''(v_0) \neq 0$. So $f'(u) = \hat{c}$. Similar to case 2), we have if $\hat{c}^2 > 1$, then

$$(64) \quad \mathcal{F}(u, v) = (u, v, -\frac{\hat{c}^2 - 1}{2} \ln \left| \cos \left(\frac{2}{\sqrt{\hat{c}^2 - 1}} v - a_1 \right) \right| + \hat{c}u + \bar{b}_1).$$

If $\hat{c}^2 < 1$, then

$$(65) \quad \mathcal{F}(u, v) = (u, v, \int_0^v \frac{\sqrt{1-\hat{c}^2} + \tilde{c}_1 e^{-\frac{4}{\sqrt{1-\hat{c}^2}}x}}{1 - \tilde{c}_1 e^{-\frac{4}{\sqrt{1-\hat{c}^2}}x}} dx + \hat{c}u + \tilde{b}).$$

Case 4) $f''(u) = 0$ and $g''(v) = 0$. By (57), we have $1 - f'(u)^2 - g'(v)^2 = 0$. By N is timelike, then $1 - f'(u)^2 - g'(v)^2 > 0$. This is a contradiction, so we have no solutions in this case.

So we have the following theorem:

Theorem 3.1. *Type I minimal translation surfaces with respect to ∇ in \mathbb{R}_1^3 are of the forms (61), (63), (64) and (65).*

In the following, we obtain all Type II minimal translation surfaces with respect to ∇ in \mathbb{R}_1^3 . Let M be a translation surface of Type II parametrized by $\mathcal{F}(u, v) = (u, f(u) + g(v), v)$. \mathcal{F}_u and \mathcal{F}_v are given by (25). While the unit normal N (up to orientation) is given by

$$(66) \quad N = \frac{1}{|\beta_1|} [f'(u)X_1 - X_2 - g'(v)X_3],$$

where $\beta_1^2 = -1 - f'(u)^2 + g'(v)^2$.

We obtain the coefficients of the first fundamental form of \mathcal{F} as

$$(67) \quad E = 1 + f'(u)^2, \quad F = f'(u)g'(v), \quad G = g'(v)^2 - 1.$$

Then, the semi-symmetric metric connection (52) on the surface is given by

$$(68) \quad \begin{cases} \nabla_{\mathcal{F}_u} \mathcal{F}_u = f''(u)X_2 - [f'(u)^2 + 1]X_3, \\ \nabla_{\mathcal{F}_u} \mathcal{F}_v = -X_1 - f'(u)X_2 - f'(u)g'(v)X_3, \\ \nabla_{\mathcal{F}_v} \mathcal{F}_u = -f'(u)g'(v)X_3, \\ \nabla_{\mathcal{F}_v} \mathcal{F}_v = [-g'(v) + g''(v)]X_2 - g'(v)^2X_3. \end{cases}$$

Consequently, the minimality condition may be expressed as follows:

$$(69) \quad 2g'^3 - 2f'^2g' + g'^2f'' + f'^2g'' - f'' + g'' - 2g' = 0.$$

We will solve (69). Let us assume first that f', g', f'' and g'' are different from zero at every point. Taking successive derivatives with respect to u and v , we obtain

$$(70) \quad \frac{f'''}{f'f''} = \frac{-2g'' + g'''}{g'g''}.$$

Therefore, there exist two constants c_0, c_1 such that

$$(71) \quad f'' = \frac{c_0}{2} f'^2 + c_1, \quad \frac{g'''}{g''} = c_0 g' + 2.$$

For (69), taking derivative with respect to v , then we have

$$(72) \quad 6g'^2 g'' - 2f'^2 g'' + 2g' g'' f'' + f'^2 g''' + g''' - 2g'' = 0.$$

Plugging (71) into (72) and g', g'' are different from zero, we obtain $6g' = -2c_0 f'^2 - 2c_1 - c_0$. Then $g'' = 0$. This is a contradiction, so we have no solutions in this case.

Case 1) There exists u_0 such that $f''(u_0) \neq 0$ and there exists v_0 such that $g''(v_0) \neq 0$. By the above discussions, we know that we have no solutions in this case.

Case 2) There exists u_0 such that $f''(u_0) \neq 0$ and $g''(v) = 0$. So $g'(v) = \tilde{c}_0$ and $g(v) = \tilde{c}_0 v + \tilde{c}_1$. By (69), we have

$$(73) \quad (\tilde{c}_0^2 - 1)f''(u) - 2\tilde{c}_0 f'(u)^2 + 2\tilde{c}_0(\tilde{c}_0^2 - 1) = 0.$$

Since $f'' \neq 0$, so $\tilde{c}_0 \neq 0$ and $\tilde{c}_0^2 \neq 1$. So

$$(74) \quad f''(u) - \frac{2\tilde{c}_0}{\tilde{c}_0^2 - 1} f'(u)^2 + 2\tilde{c}_0 = 0.$$

The general solution of this ODE (74) is found as if $\tilde{c}_0^2 < 1$

$$(75) \quad f(u) = \frac{1 - \tilde{c}_0^2}{2\tilde{c}_0} \ln \left| \cos \left(\frac{2\tilde{c}_0}{\sqrt{1 - \tilde{c}_0^2}} u - \tilde{a} \right) \right| + \tilde{b},$$

where \tilde{a}, \tilde{b} are constant. So

$$(76) \quad \mathcal{F}(u, v) = \left(u, \frac{1 - \tilde{c}_0^2}{2\tilde{c}_0} \ln \left| \cos \left(\frac{2\tilde{c}_0}{\sqrt{1 - \tilde{c}_0^2}} u - \tilde{a} \right) \right| + \tilde{c}_0 v + \tilde{b}, v \right).$$

If $\tilde{c}_0^2 > 1$, then the general solution of this ODE (74) is found as

$$(77) \quad f(u) = \int_0^u \frac{\sqrt{\tilde{c}_0^2 - 1} + c_1 e^{\frac{4\tilde{c}_0}{\sqrt{\tilde{c}_0^2 - 1}} x}}{1 - c_1 e^{\frac{4\tilde{c}_0}{\sqrt{\tilde{c}_0^2 - 1}} x}} dx,$$

where c_1 is a nonzero constant. So

$$(78) \quad \mathcal{F}(u, v) = \left(u, \int_0^u \frac{\sqrt{\tilde{c}_0^2 - 1} + c_1 e^{\frac{4\tilde{c}_0}{\sqrt{\tilde{c}_0^2 - 1}} x}}{1 - c_1 e^{\frac{4\tilde{c}_0}{\sqrt{\tilde{c}_0^2 - 1}} x}} dx + \tilde{c}_0 v + \tilde{b}_1, v \right).$$

Case 3) $f''(u) = 0$ and there exists v_0 such that $g''(v_0) \neq 0$. So $f'(u) = \hat{c}_0$ and $f(u) = \hat{c}_0 u + \hat{c}_1$. By (3.18), we have

$$(79) \quad g'' + \frac{2}{\hat{c}_0^2 + 1} g'^3 - 2g' = 0.$$

We get

$$(80) \quad g(v) = \int_0^v \frac{\pm 1}{\sqrt{\widehat{a}e^{-4x} + \frac{1}{\widehat{c}_0^2 + 1}}} dx + \widehat{b},$$

where $\widehat{a} \neq 0, \widehat{b}$ are constant. So

$$(81) \quad \mathcal{F}(u, v) = (u, \int_0^v \frac{\pm 1}{\sqrt{\widehat{a}e^{-4x} + \frac{1}{\widehat{c}_0^2 + 1}}} dx + \widehat{c}_0 u + b_0, v).$$

Case 4) $f''(u) = 0$ and $g''(v) = 0$. Then $f'(u) = c'_0$ and $g''(v) = c'_1$. By (69), we get $c'_1(c_1'^2 - c_0'^2 - 2) = 0$. Then

$$(82) \quad \mathcal{F}(u, v) = (u, c'_0 u + c'_1 v + b', v),$$

where $c'_1 = 0$ or $(c_1'^2 - c_0'^2 - 2) = 0$.

So we have the following theorem:

Theorem 3.2. *Type II minimal translation surfaces with respect to ∇ in \mathbb{R}_1^3 are of the forms (76), (78), (81) and (82).*

For Type III minimal translation surfaces with respect to ∇ in \mathbb{R}_1^3 , we also get (69) and we have Theorem similar to Theorem 3.2.

We define a special semi-symmetric non-metric connection in \mathbb{R}_1^3 by

$$(83) \quad \overline{\nabla}_X Y = \nabla_X^L Y + \widetilde{g}_1(Y, X_3)X.$$

The connection $\overline{\nabla}$ of \mathbb{R}_1^3 is given by

$$(84) \quad \begin{aligned} \overline{\nabla}_{X_1} X_1 &= 0, \quad \overline{\nabla}_{X_1} X_2 = 0, \quad \overline{\nabla}_{X_1} X_3 = -X_1, \\ \overline{\nabla}_{X_2} X_1 &= 0, \quad \overline{\nabla}_{X_2} X_2 = 0, \quad \overline{\nabla}_{X_2} X_3 = -X_2, \\ \overline{\nabla}_{X_3} X_1 &= 0, \quad \overline{\nabla}_{X_3} X_2 = 0, \quad \overline{\nabla}_{X_3} X_3 = -X_3. \end{aligned}$$

In the following, we consider Type I minimal translation surfaces with respect to $\overline{\nabla}$ in \mathbb{R}_1^3 . For $\mathcal{F}(u, v) = (u, v, f(u) + g(v))$, then $\mathcal{F}_u, \mathcal{F}_v, N, E, F, G$ is computed by (54) and (55). Then, the semi-symmetric non-metric connection (83) on the surface is given by

$$(85) \quad \begin{cases} \overline{\nabla}_{\mathcal{F}_u} \mathcal{F}_u = -f'(u)X_1 + [f''(u) - f'(u)^2]X_3, \\ \overline{\nabla}_{\mathcal{F}_u} \mathcal{F}_v = -g'(v)X_1 - f'(u)g'(v)X_3, \\ \overline{\nabla}_{\mathcal{F}_v} \mathcal{F}_u = -f'(u)X_2 - f'(u)g'(v)X_3, \\ \overline{\nabla}_{\mathcal{F}_v} \mathcal{F}_v = -g'(v)X_2 + [-g'(v)^2 + g''(v)]X_3. \end{cases}$$

Similar to (9), we have the minimality condition with respect to $\overline{\nabla}$ in \mathbb{R}_1^3 . N is timelike, so $1 - f'(u)^2 - g'(v)^2 > 0$ and $1 - f'(u)^2 > 0$ and $1 - g'(v)^2 > 0$. Consequently, the minimality condition may be expressed as follows:

$$(86) \quad \frac{f''(u)}{1 - f'(u)^2} = -\frac{g''(v)}{1 - g'(v)^2} = c_0.$$

When $c_0 = 0$, we get

$$(87) \quad \mathcal{F}(u, v) = (u, v, c_1u + c_2v + c_3).$$

When $c_0 \neq 0$, we get

$$(88) \quad f''(u) + c_0f'(u)^2 - c_0 = 0, \quad g''(u) - c_0g'(u)^2 + c_0 = 0.$$

Solving (88), we obtain:

Theorem 3.3. *Type I minimal translation surfaces with respect to $\bar{\nabla}$ in \mathbb{R}_1^3 are (87) and the following form*

$$(89) \quad \mathcal{F}(u, v) = (u, v, \frac{1}{c_0} \ln \frac{|e^{c_0u} - \hat{c}e^{-c_0u}|}{|e^{-c_0v} - \hat{c}_1e^{c_0v}|} + a),$$

where $c_0, \hat{c}, \hat{c}_1 \neq 0$.

For Type II minimal translation surfaces with respect to $\bar{\nabla}$ in \mathbb{R}_1^3 , and $\mathcal{F}(u, v) = (u, f(u) + g(v), v)$, then $\mathcal{F}_u, \mathcal{F}_v, N, E, F, G$ is computed by (66) and (67). Then, the semi-symmetric non-metric connection (83) on the surface is given by

$$(90) \quad \begin{cases} \bar{\nabla}_{\mathcal{F}_u} \mathcal{F}_u = f''(u)X_2, \\ \bar{\nabla}_{\mathcal{F}_u} \mathcal{F}_v = -X_1 - f'(u)X_2, \\ \bar{\nabla}_{\mathcal{F}_v} \mathcal{F}_u = 0, \\ \bar{\nabla}_{\mathcal{F}_v} \mathcal{F}_v = [-g'(v) + g''(v)]X_2 - X_3. \end{cases}$$

Consequently, the minimality condition may be expressed as follows:

$$(91) \quad \frac{f''(u)}{1 + f'(u)^2} = \frac{g''(v)}{1 - g'(v)^2} = \bar{c}_0.$$

When $\bar{c}_0 = 0$, we get

$$(92) \quad \mathcal{F}(u, v) = (u, c_1u + c_2v + c_3, v).$$

When $\bar{c}_0 \neq 0$, we get

$$(93) \quad f''(u) - \bar{c}_0f'(u)^2 - \bar{c}_0 = 0, \quad g''(u) + \bar{c}_0g'(u)^2 - \bar{c}_0 = 0.$$

Solving (93), we obtain:

Theorem 3.4. *Type II minimal translation surfaces with respect to $\bar{\nabla}$ in \mathbb{R}_1^3 are (92) and the following form*

$$(94) \quad \mathcal{F}(u, v) = (u, \frac{1}{\bar{c}_0} \ln \frac{|e^{\bar{c}_0v} - c_3e^{-\bar{c}_0v}|}{|\cos(\bar{c}_0u + c_4)|} + b, v),$$

where $\bar{c}_0, c_3 \neq 0$.

For Type III minimal translation surfaces with respect to $\bar{\nabla}$ in \mathbb{R}_1^3 , we also get (91) and we have Theorem similar to Theorem 3.4.

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