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AN INTERPOLATING HARNACK INEQUALITY FOR NONLINEAR HEAT EQUATION ON A SURFACE

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Abstract. In this short note we prove new differential Harnack inequalities interpolating those for the static surface and for the Ricci flow. In particular, for $0 \leq \varepsilon \leq 1$, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \leq 1$ and u being a positive solution to

$$\frac{\partial u}{\partial t} = \Delta u - \alpha u \log u + \varepsilon R u + \beta u^{\gamma}$$

on closed surfaces under the flow $\frac{\partial}{\partial t} g_{ij} = -\varepsilon R g_{ij}$ with $R > 0$, we prove that

$$\frac{\partial}{\partial t} \log u - |\nabla \log u|^2 + \alpha \log u - \beta u^{\gamma-1} + \frac{1}{t} = \Delta \log u + \varepsilon R + \frac{1}{t} \geq 0.$$

1. Introduction and the main result

Geometric flow is one of central problems in geometric analysis. Curve shortening flow in the plane is the simplest flow, and recently we established interpolating inequalities in [8, 9]. Since there is no Riemannian curvature on curves, the simplest intrinsic flow is on surfaces. Hamilton studied the Ricci flow on surfaces in [10]. One of the useful tools to study geometric flows is Li-Yau-Hamilton Harnack inequality. Chow proved an interpolating Harnack estimate linking the Li-Yau estimate to the linear trace estimate in [3]. Wu and Wu-Zheng generalized the interpolating Harnack estimates on surfaces to nonlinear and constraint cases, see [11–13]. We have also studied Harnack inequalities in various settings in [2, 5–7].

On the other hand, assuming that M is a static complete Riemannian manifold Yang [14] proved gradient estimates for solutions to the following nonlinear parabolic equation:

$$\frac{\partial u}{\partial t} = \Delta u + a u \log u + b u. \tag{1.1}$$

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Very recently Wu [12] proved several new Harnack estimates, and one of them is a new interpolating Harnack inequality for the equation

$$\frac{\partial u}{\partial t} = \Delta u - u \log u + \varepsilon R u \tag{1.2}$$

on closed surfaces under the $\varepsilon-$Ricci flow:

$$\frac{\partial}{\partial t} g_{ij} = -2\varepsilon R_{ij} = -\varepsilon R g_{ij}. \tag{1.3}$$

Inspired by their work, in this note we consider

$$\frac{\partial u}{\partial t} = \Delta u - \alpha u \log u + \varepsilon R u + \beta u^{\gamma} \tag{1.4}$$

with $0 \leq \varepsilon \leq 1, \alpha \geq 0, \beta \geq 0, \gamma \leq 1$ on closed surfaces under the flow (1.3).

We prove:

Theorem 1.1. *Let* $(M^2, g(t))$ *be a solution to the* ε*-Ricci flow* (1.3) *on a closed surface with* $R > 0$*. Let* u *be a positive solution to the equation* (1.4)*. Then for all time* t *one has*

$$\frac{\partial}{\partial t} \log u - |\nabla \log u|^2 + \alpha \log u - \beta u^{\gamma - 1} + \frac{1}{t} = \Delta \log u + \varepsilon R + \frac{1}{t} \geq 0. \tag{1.5}$$

2. Proof of the main theorem

In this section, by routine calculations we prove Theorem 1.1.

Proof. We know in the ε-Ricci flow [4]

$$\frac{\partial R}{\partial t} = \varepsilon(\Delta R + R^2)$$

and

$$\frac{\partial}{\partial t}(\Delta) = \varepsilon R \Delta.$$

We can get

$$\frac{\partial}{\partial t} \log R - \varepsilon |\nabla \log R|^2 = \varepsilon(\Delta \log R + R).$$

As in [4] we define $P_{ij} = \nabla_i \nabla_j \log u + \frac{1}{2}\varepsilon R g_{ij}$ and $P = g^{ij} P_{ij} = \Delta \log u + \varepsilon R$. For $\eta \leq 0$ we have

$$\begin{aligned}
\Delta u^{\eta} &= \mathrm{Div}(\nabla u^{\eta}) \\
&= \mathrm{Div}(\eta u^{\eta - 1} \nabla u) \\
&= \eta(\eta - 1) u^{\eta - 2} |\nabla u|^2 + \eta u^{\eta - 1} \Delta u \\
&= \eta(\eta - 1) u^{\eta} |\nabla \log u|^2 + \eta u^{\eta} (\Delta \log u + |\nabla \log u|^2) \\
&= \eta^2 u^{\eta} |\nabla \log u|^2 + \eta u^{\eta} \Delta \log u \\
&= \eta^2 u^{\eta} |\nabla \log u|^2 + \eta u^{\eta} P - \eta u^{\eta} \varepsilon R \\
&\geq \eta u^{\eta} P.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}\Delta|\nabla \log u|^2 &= 2\nabla\Delta \log u \cdot \nabla \log u + 2\,\mathrm{Rc}(\nabla \log u, \nabla \log u) + 2|\nabla\nabla \log u|^2 \\ &= 2\nabla P \cdot \nabla \log u - 2\varepsilon \nabla R \cdot \nabla \log u + 2\,\mathrm{Rc}(\nabla \log u, \nabla \log u) \\ &\quad + 2|\nabla\nabla \log u|^2.\end{aligned}$$

Since on surfaces $\mathrm{Rc} = \frac{1}{2}Rg$ we have

$$\begin{aligned}\Delta|\nabla \log u|^2 &= 2\nabla P \cdot \nabla \log u + R|\nabla \log u - \varepsilon\nabla \log R|^2 - \varepsilon^2 R|\nabla \log R|^2 \\ &\quad + 2|\nabla\nabla \log u|^2.\end{aligned}$$

We know that

$$|P_{ij}|^2 = |\nabla\nabla \log u|^2 + \varepsilon R\Delta \log u + \frac{1}{2}\varepsilon^2 R^2$$

and furthermore

$$|P_{ij}|^2 = |\nabla\nabla \log u|^2 + \varepsilon RP - \frac{1}{2}\varepsilon^2 R^2.$$

Combining with the inequality $2|P_{ij}|^2 \geq P^2$ we get the following inequality

$$2|\nabla\nabla \log u|^2 \geq P^2 - 2\varepsilon RP + \varepsilon^2 R^2.$$

Then we arrive at

$$\begin{aligned}\Delta|\nabla \log u|^2 &\geq 2\nabla P \cdot \nabla \log u + R|\nabla \log u - \varepsilon\nabla \log R|^2 - \varepsilon^2 R|\nabla \log R|^2 \\ &\quad + P^2 - 2\varepsilon RP + \varepsilon^2 R^2 \\ &\geq 2\nabla P \cdot \nabla \log u - \varepsilon^2 R|\nabla \log R|^2 + P^2 - 2\varepsilon RP + \varepsilon^2 R^2.\end{aligned}$$

We compute that

$$\begin{aligned}\frac{\partial P}{\partial t} &= \frac{\partial}{\partial t}(\Delta \log u) + \varepsilon\frac{\partial R}{\partial t} \\ &= (\frac{\partial}{\partial t}\Delta) \log u + \Delta(\frac{\partial}{\partial t} \log u) + \varepsilon R\frac{\partial}{\partial t} \log R \\ &= \varepsilon R\Delta \log u + \Delta(P + |\nabla \log u|^2 - \alpha \log u + \beta u^{\gamma-1}) + \varepsilon R\frac{\partial}{\partial t} \log R \\ &\geq \varepsilon RP - \varepsilon^2 R^2 + \Delta P + 2\nabla P \cdot \nabla \log u - \varepsilon^2 R|\nabla \log R|^2 + P^2 - 2\varepsilon RP \\ &\quad + \varepsilon^2 R^2 - \alpha P + \alpha\varepsilon R + \beta(\gamma - 1)u^{\gamma-1}P + \varepsilon R\frac{\partial}{\partial t} \log R \\ &\geq \Delta P + 2\nabla P \cdot \nabla \log u - (\varepsilon R + \alpha + \beta(1 - \gamma)u^{\gamma-1})P + P^2 \\ &\quad + \varepsilon R(\frac{\partial}{\partial t} \log R - \varepsilon|\nabla \log R|^2) \\ &\geq \Delta P + 2\nabla P \cdot \nabla \log u - (\varepsilon R + \alpha + \beta(1 - \gamma)u^{\gamma-1})P + P^2 \\ &\quad + \varepsilon R(\varepsilon(\Delta \log R + R)).\end{aligned}$$

Recall that the trace Harnack inequality for the ε-Ricci flow on closed surfaces [4]

$$\frac{\partial}{\partial t}\log R - \varepsilon|\nabla \log R|^2 = \varepsilon(\Delta \log R + R) \geq -\frac{1}{t}.$$

Hence

$$\begin{aligned}\frac{\partial}{\partial t}(P+\frac{1}{t}) \geq{}& \Delta(P+\frac{1}{t}) + 2\nabla(P+\frac{1}{t})\cdot\nabla \log u\\ &- (\varepsilon R + \alpha + \beta(1-\gamma)u^{\gamma-1})(P+\frac{1}{t})\\ &+ (P+\frac{1}{t})(P-\frac{1}{t}) + \varepsilon R(\varepsilon(\Delta \log R + R) + \frac{1}{t}).\end{aligned}$$

It's very clear to see that

$$P + \frac{1}{t} > 0$$

for very small positive t. Then applying the maximum principle, we conclude that

$$P + \frac{1}{t} > 0$$

for all positive time t. □

In particular, when $\gamma = 0$ and $\varepsilon = 1$ we can extend the parameter α to all real numbers, which has been proved in [1].

It is standard to get:

Corollary 2.1 (Comparing the solution at different points and times)**.** *For any $x_1, x_2 \in M^2$, we pick a space-time path $\Gamma(x,t)$ joining (x_1,t_1) and (x_2,t_2) with $0 < t_1 < t_2$. Along Γ we have*

$$\exp(\alpha t_1)\log u(x_1,t_1) \leq \exp(\alpha t_2)\log u(x_2,t_2) + \int_{t_1}^{t_2} \exp(\alpha t)\left(\frac{1}{4}\left|\frac{d\Gamma}{dt}\right|^2 + \frac{1}{t}\right)dt.$$

Proof. Indeed for such a path Γ, we have

$$\begin{aligned}\frac{d}{dt}\log u(x,t) &= \frac{\partial}{\partial t}\log u + \nabla \log u \cdot \frac{d\Gamma}{dt}\\ &\geq |\nabla \log u|^2 - \alpha \log u + \beta u^{\gamma-1} - \frac{1}{t} + \nabla \log u \cdot \frac{d\Gamma}{dt}\\ &\geq -\frac{1}{4}\left|\frac{d\Gamma}{dt}\right|^2 - \alpha \log u + \beta u^{\gamma-1} - \frac{1}{t}\\ &\geq -\frac{1}{4}\left|\frac{d\Gamma}{dt}\right|^2 - \alpha \log u - \frac{1}{t}.\end{aligned}$$

Hence,

$$\frac{d}{dt}(\exp(\alpha t)\log u(x,t)) \geq -\exp(\alpha t)\left(\frac{1}{4}\left|\frac{d\Gamma}{dt}\right|^2 + \frac{1}{t}\right).$$

Integrating this inequality from time t_1 to t_2 yields

$$\begin{aligned}&\exp(\alpha t_1)\log u(x_1,t_1)-\exp(\alpha t_2)\log u(x_2,t_2)\\ \leq &\int_{t_1}^{t_2}\exp(\alpha t)\left(\frac{1}{4}\left|\frac{d\Gamma}{dt}\right|^2+\frac{1}{t}\right)dt.\end{aligned}$$

□

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