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An Elementary Proof of Catalan Theorem

Koh Sung-Eun 고성은

An elementary proof of the Catalan Theorem is given.

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Catalan Theorem states that a non-flat ruled minimal surface in $\mathbb{R}^3$ is a part of a helicoid. One can find several proofs with different geometric flavors; for example, the one in [1] solves directly the minimal surface equation, the one in [2] uses a fact that a Bertrand curve with more than one Bertrand mate is a circular helix and the one in [3] makes use of the Schwarz reflection principle. Whereas all of these proofs seem to require some extra knowledges beyond the usual undergraduate differential geometry courses, the one in [4] requires only elementary stuffs along with a standard parametrization for ruled surfaces. However, as we think the proof is a little complicated, we give some changes in the course of the proof of [4] to get simpler one. We think this new proof is constructive in the sense that it features how helicoids are made up with straight lines.

Let $X(t, u) = \alpha(t) + u\mathbf{w}(t)$ be a parametrization of a ruled surface in $\mathbb{R}^3$, where $\alpha(t)$ is a differentiable curve and $\mathbf{w}(t)$ is a differentiable vector field of unit length, $\mathbf{w}(t) \cdot \mathbf{w}(t) = 1$. As there are many choices for the directrix curve α giving the same ruled surface, some kinds of normalization seem to be of help. Hence we will choose a special parameter t and a special directrix curve. Firstly let us assume that the parameter t is so chosen that $\mathbf{w}'(t) \cdot \mathbf{w}'(t) = 1$. The geometric meaning of this parameter can be found in [5]. For the choice of the special directrix curve, let us consider the u-function

$$\left\| \frac{dX(t,u)}{dt} \right\|^2 = \|\alpha'(t) + u\mathbf{w}'(t)\|^2,$$

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Koh Sung-Eun: Dept. of Math., Konkuk Univ. E-mail: skoh@konkuk.ac.kr

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which has a minimum at $u = -\alpha'(t) \cdot \mathbf{w}'(t)$. (In fact, as the vector field $u \mapsto \frac{dX(t,u)}{dt}$ is a Jacobi field along the geodesic $u \mapsto X(t,u)$, this point u is the one where the size of the Jacobi field minimizes.) Now set

$$\beta(t) = \alpha(t) - (\alpha'(t) \cdot \mathbf{w}'(t))\mathbf{w}(t),$$

then the parametrization

$$Y(t,u) = \beta(t) + u\mathbf{w}(t)$$

is another parametrization of the given ruled surface $X(t,u)$. This curve β is called as a striction curve, solid geometrical meaning of which can also be found in [5]. Then one has

$$\beta'(t) \cdot \mathbf{w}'(t) = 0.$$

Note that, on points of this directrix curve β, the tangent plane is spanned by $\beta'(t)$, $\mathbf{w}(t)$ and the vector $\mathbf{w}'(t)$ is the unit normal vector to this tangent plane. Now the first fundamental forms E, F, G and the second fundamental forms e, f, g are computed as follows:

$$\begin{aligned}
E &= (\beta' + u\mathbf{w}') \cdot (\beta' + u\mathbf{w}') = \beta' \cdot \beta' + u^2, \\
F &= \beta' \cdot \mathbf{w}, \\
G &= 1, \\
De &= \beta'' \cdot \beta' \times \mathbf{w} + u(\beta'' \cdot \mathbf{w}' \times \mathbf{w} + \mathbf{w}'' \cdot \beta' \times \mathbf{w}) + u^2(\mathbf{w}'' \cdot \mathbf{w}' \times \mathbf{w}), \\
Df &= \mathbf{w}' \cdot \beta' \times \mathbf{w}, \\
Dg &= 0
\end{aligned}$$

where $D = \|Y_t \times Y_u\|$.

Now, let us suppose the ruled surface is minimal. Then the minimal surface equation $H = \frac{1}{2}(Eg - 2Ff + Ge) = 0$ gives

$$\beta'' \cdot \beta' \times \mathbf{w} - 2(\beta' \cdot \mathbf{w})(\mathbf{w}' \cdot \beta' \times \mathbf{w}) + u(\beta'' \cdot \mathbf{w}' \times \mathbf{w} + \mathbf{w}'' \cdot \beta' \times \mathbf{w}) + u^2(\mathbf{w}'' \cdot \mathbf{w}' \times \mathbf{w}) = 0.$$

Since this equation holds for every u, one has three equations:

$$\beta'' \cdot \beta' \times \mathbf{w} - 2(\beta' \cdot \mathbf{w})(\mathbf{w}' \cdot \beta' \times \mathbf{w}) = 0, \tag{1}$$

$$\beta'' \cdot \mathbf{w}' \times \mathbf{w} + \mathbf{w}'' \cdot \beta' \times \mathbf{w} = 0, \tag{2}$$

$$\mathbf{w}'' \cdot \mathbf{w}' \times \mathbf{w} = 0. \tag{3}$$

Since β' and $\mathbf{w}$ span the tangent space, the identity (1) implies the normal component of the vector β'' is $2(\beta' \cdot \mathbf{w})\mathbf{w}'$ and then one can set

$$\beta'' = a_1\beta' + a_2\mathbf{w} + 2(\beta' \cdot \mathbf{w})\mathbf{w}' \tag{4}$$

for scalar functions a_1, a_2. Since $\mathbf{w}' \cdot \mathbf{w}' = 1$, one has $\mathbf{w}'' \cdot \mathbf{w}' = 0$, which implies that $\mathbf{w}''$ is a tangent vector. Hence one can set $\mathbf{w}'' = a_3\beta' + a_4\mathbf{w}$ for scalar functions a_3, a_4. However, since $\beta' \cdot \mathbf{w}' \times \mathbf{w} \neq 0$, (3) gives $a_3 = 0$ and $\mathbf{w}'' = a_4\mathbf{w}$. Moreover, since $\mathbf{w}' \cdot \mathbf{w} = 0$, differentiating this, one has

$$0 = \mathbf{w}'' \cdot \mathbf{w} + \mathbf{w}' \cdot \mathbf{w}' = \mathbf{w}'' \cdot \mathbf{w} + 1$$

and hence one has

$$\mathbf{w}'' = -\mathbf{w}. \tag{5}$$

Inserting (4) and (5) into (2), one has $a_1 = 0$ and $\beta'' = a_2\mathbf{w} + 2(\beta' \cdot \mathbf{w})\mathbf{w}'$. Differentiating $\beta' \cdot \mathbf{w}' = 0$, one has

$$0 = (\beta' \cdot \mathbf{w}')' = \beta'' \cdot \mathbf{w}' + \beta' \cdot \mathbf{w}'' = 2(\beta' \cdot \mathbf{w}) - (\beta' \cdot \mathbf{w}) = (\beta' \cdot \mathbf{w})$$

and hence one has $\beta'' = a_2\mathbf{w}$. Moreover, differentiating $\beta' \cdot \mathbf{w} = 0$, one has

$$0 = (\beta' \cdot \mathbf{w})' = \beta'' \cdot \mathbf{w} + \beta' \cdot \mathbf{w}' = a_2$$

which gives finally $\beta'' = 0$.

Now summarizing the result of computation one has

(i) The directrix curve is a straight line since $\beta'' = 0$: hence one can set $\beta(t) = (0, 0, at + b)$ for constants a, b.

(ii) The ruling line is orthogonal to the directrix line since $\beta' \cdot \mathbf{w} = 0$: hence one can set $\mathbf{w}(t) = (\cos\phi(t), \sin\phi(t), 0)$ for a smooth function ϕ. Note here that $\mathbf{w} \cdot \mathbf{w} = 1$.

(iii) The ruling line rotates with respect to the directrix line with constant speed since $\mathbf{w}' \cdot \mathbf{w}' = 1$: hence one can set $\phi(t) = \pm t + c$ for a constant c.

Hence one has

$$Y(t, u) = \beta(t) + u\mathbf{w}(t) = (\cos(\pm t + c), \sin(\pm t + c), at + b)$$

where $a \neq 0$ since the surface is assumed non-flat, which represents a helicoid.

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