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Riesz and Tight Wavelet Frame Sets in Locally Compact Abelian Groups

Arvind Kumar Sinha* and Radhakrushna Sahoo
Department of Mathematics, National Institute of Technology Raipur, G.E. Road Raipur, Chhatisgarh-492010, India
e-mail : aksinha.maths@nitrr.ac.in *and* radhakrushnasahoo92@gmail.com

Abstract. In this paper, we attempt to obtain sufficient conditions for the existence of tight wavelet frame sets in locally compact abelian groups. The condition is generated by modulating a collection of characteristic functions that correspond to a generalized shift-invariant system via the Fourier transform. We present two approaches (for stationary and non-stationary wavelets) to construct the scaling function for $L^2(G)$ and, using the scaling function, we construct an orthonormal wavelet basis for $L^2(G)$. We propose an open problem related to the extension principle for Riesz wavelets in locally compact abelian groups.

1. Introduction and Motivation

Duffin and Schaeffer [10] introduced the notion of frames, and Daubechies et al. [9] the notion of wavelet frames. Among frames, tight wavelet frames, due to their numerical stability, play an essential role in the series representation of a function and signal transmission. Other types of wavelets with simple structure have been established on a Lebesgue measurable set in R^n with a non-negative and finite measure. The most fundamental one is the Haar wavelet $\Psi(x) = 1_{[0,1/2]} - 1_{[1/2,1]}$, introduced by Alfred Haar in 1910. It is a one-dimensional prototypical illustration of a wavelet. The Shannon wavelet, whose Fourier transform is $1_{[-1,-1/2)\cup(1/2,1]}$, is another typical example. The Shannon wavelet is an essential tool for analyzing and reconstructing functions. We concentrate on Shannon-type wavelets whose Fourier transform is supported in the finite measure domain. We study Shannon

* Corresponding Author.
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wavelet's Fourier transforms in the Shannon set $\Theta = [-2\pi, -\pi] \cup [\pi, 2\pi)$. We take the orthonormal wavelet $\Psi = 2\sin h(2x-1) - \sin h(x)$ with $\widehat{\Psi} = 1_\Theta$, where $\widehat{\Psi}$ is a Fourier transform of Ψ and 1_Θ is an indicator of set Θ. For more examples and constructions of wavelet sets in R^n see [2, 3, 7, 23, 24]. Wang [29] addressed the existence of wavelet sets with the documentation of spectral sets. The characterization of wavelets has been generalized to n-dimensional Euclidean spaces. The dilation set is defined on any countable subset of non-singular matrices, whereas the translation set is defined on any countable subset in R^n.

Grepstad and Lev [14, 15] introduced the concept of multi-tiling and Riesz bases on a bounded Riemann measurable set in R^n. Shah et al. [25] provided complete characterizations of orthogonal families, tight frames, and orthonormal bases of Gabor systems on local fields of positive characteristic. More results in this direction can be found in [5, 6, 26, 28]. Iosevich et al. [19] gave the concept of tight wavelet frame sets in finite vector spaces. This paper continues the line of research for locally compact abelian (LCA) groups. Mayeli [21] gave the concept of Riesz wavelets, tiling, and spectral sets in LCA groups. Further motivated by works of [1, 4, 12, 20, 27], we introduce the sufficient condition for the existence of tight wavelet frame sets in LCA groups. We also construct an orthonormal wavelet basis for $L^2(G)$.

Some LCA groups, such as the rational $p-$adic additive group Qp, have no discrete subgroup in the field of $p-$adic rational numbers. Consequently, such groups do not apply to the classical concepts of wavelet sets. Thus the finite abelian group does not possess any wavelet set in the conventional sense. Iosevich et al. [19] defined tilings of tight wavelet frame sets in finite vector space. We introduce the concept of tilings of tight wavelet frame sets in infinite LCA groups. We assume that G is an infinite group and admits a discrete subgroup. For the notion of wavelet sets in Q_p and a non-commutative setting we refer [3] and [8], respectively. For Riesz wavelets' characterizations generated by multiresolution analysis in the finite cases, see [4, 16, 22].

The main objective of this work is to investigate the characterization of tight wavelet frame sets for infinite LCA groups equipped with a Haar measure. We present two approaches to construct (for stationary and non-stationary wavelets) the scaling function for $L^2(G)$. We construct an orthonormal wavelet basis for $L^2(G)$. We propose an open problem related to the extension principle for Riesz wavelets in locally compact abelian groups.

2. Preliminaries

Now we give some well-known definitions and results.

Definition 2.1.([19]) A subset $\{x_i\}_{i\in Z}$ of a Hilbert space H is said to be a *frame* if there exists a sequence of vectors $\{x_i : i \in Z\}$ in H and two positive real numbers $P \leq Q$ such that for every $x \in H$

$$P\|x\|_H^2 \leq \sum_{i\in Z} |\langle x, x_i\rangle_H|^2 \leq Q\|x\|_H^2. \tag{2.1}$$

P and Q are the lower and upper bounds of the frame. If $P = Q$, then the frame is called a *tight wavelet frame* and if $P = Q = 1$, then the frame is called a Parseval frame.

Definition 2.2.([21]) A collection of Haar-measurable wavelet sets $\{\Theta_l\}_{l=1}^r$ in R^n is called a *Riesz wavelet set* if there exists a countable set $S \subset GL(n, R)$ and $\Psi_l \in R^n$ such that for $\widehat{\Psi}_l = 1_{\Theta_l}$, where $GL(n, R)$ is a general linear group of order n. The collection of elements

$$\omega = \bigcup_{l=1}^{r} \{\triangle\ (s)^{1/2}\Psi_l(s(x) - \alpha) : \alpha \in R, s \in S\}, \tag{2.2}$$

is called a *Riesz wavelet basis* for $L^2(R^n)$, where $\triangle$ is a unitary operator over R and r is a fixed positive integer.

The main feature of frames is their redundancy; for example, they play an essential role in robustness. In Hilbert's space, the frame behaves as a simplified, visualized characterization of sets. Engineers and applied mathematicians have used frames for frequency-domain analysis of discrete-time systems. Frames are more attractive in the space $L^2(G)$ as it is closed to the mother wavelet.

Let $\alpha, \beta, \gamma \in R$ be given. We define the linear operators: modulation M_α, dilation D_β and translation T_γ, for a function $f \in L^2(G)$ as follows:

$$M_\alpha f(x) = e^{2\pi i\alpha x}, \qquad D_\beta f(x) = |\beta|^{1/2} f(\beta x), \qquad T_\gamma f(x) = f(x - \gamma), \tag{2.3}$$

If $\{M_\alpha, D_\beta, T_\gamma\}$ is a frame in $L^2(G)$, then (α, β, γ) generates a wavelet frame in $L^2(G)$. The Fourier transform of a function $f \in L^2(G)$ is defined by $\widehat{f}(\omega) = \int_G f(x)e^{-2\pi i\omega x}dx$. The Fourier transform of modulation M_α, dilation D_β and translation T_γ, is defined by $\widehat{M_\alpha f} = M_{-\alpha}\widehat{f}$, $\widehat{D_\beta f} = D_{1/\beta}\widehat{f}$ and $\widehat{T_\gamma f} = T_\gamma \widehat{f}$ respectively.

Definition 2.3.([21]) Let $\Psi \in L^2(R^n)$, $n \geq 1$, then Ψ is called a *wavelet* for $L^2(R^n)$ if there exist $D \subset GL(n, R)$ and countable sets $T \subset R^n$, such that the family

$$\{|det(M)|^{1/2}\Psi(Mx - t) : M \in D, t \in T\} \tag{2.4}$$

is an orthonormal basis for $L^2(R^n)$. The sets D and T are called *dilations* and *translation sets*, respectively.

The size of D and T of orthonormal wavelet frames for $L^2(R^n)$ have been studied by many authors, see [17, 18, 29].

Definition 2.4.([21]) If G is an LCA group equipped with a Haar measure and $\widehat{G}$ its dual. A collection of measurable subsets Θ_k, $1 \leq k \leq n$, of $\widehat{G}$ is called a

(Riesz) wavelet collection of sets for G if there is a countable subset A of $Aut(G)$ and countable subsets $T_k \subset G$ $(1 \leq k \leq n)$ such that

$$\bigcup_{k=1}^{n} \{\Psi_{k,j,r} : 1 \leq k \leq n, j \in T_k, r \in A\} \tag{2.5}$$

is a (Riesz) orthogonal basis for $L^2(G)$, $\Psi_{k,j,r}(x) = \triangle(r)^{1/2}\Psi_k(r(x) - \lambda)$, $\widehat{\Psi}_k = 1_{\Theta_k}$ When $n = 1$, we may say $\Theta = \Theta_1$ is a (Riesz) wavelet set.

Wavelet sets and minimally supported frequency wavelets were introduced by Fang and Wang [11] and studied exclusively by Hernandez et al. [17, 18]. A measurable set $\Theta \subset R^n$ with non-zero finite measure is called a wavelet set, if for the function $\Psi \in L^2(R^n)$ with $\widehat{\Psi} = 1_\Theta$ and the set ω of (2.2) becomes an orthogonal basis for $L^2(R^n)$. A simple way to establish a wavelet frame in R^n is to choose a function whose Fourier transform is the indicator of a measurable set. The classical interpretation of wavelet frame sets obtains dilations and translations of the family (2.4). A set $\Theta \subset R^n$ is called a wavelet frame set with respect to dilations and translations if for the function $\Psi \in L^2(R^n)$ with $\widehat{\Psi} = 1_\Theta$, the family (2.4) becomes a frame in $L^2(R^n)$.

Wang [29] studied the characterization of wavelet sets. Shah and Debnath [26] developed tight wavelet frames using the unitary extension principle in local fields and gave sufficient conditions for the formation of tight wavelet frame sets with a limited number of refinable functions. Wavelet sets were discussed by many authors [5, 13]. Mayeli [21] gave the concept of tilings of Riesz wavelets in LCA groups. Let G be an infinite LCA group equipped with Haar measure and C a set of complex numbers. A linear function $\Psi : G \to C$ is said to be a wavelet frame for $L^2(G)$ if there exists a subset $A \subset Aut(G)$ and a countable subset $T \subset G$ such that the family

$$\{\Psi(\alpha x - \lambda) : \alpha \in A, \lambda \in T\} \tag{2.6}$$

is an orthonormal basis with respect to wavelet frames for $L^2(G)$. Note that in continuous cases, for a non-singular matrix M, the dilation factor $c = |\det M|^{n/2}$ makes the dilation map $f \to |det(M)|^{1/2} f(Mx)$ an isometry. If $c = 1$ then the dilation of function in R^n belongs to a discrete group. To overcome this problem in the discrete case, we take both stationary and non-stationary wavelet settings of LCA groups to define the class of automorphism in G. In this way, the dilation of a function in R^n can be obtained in a co-compact group as well as a discrete group. In [2, 19, 26] the relationship between spectral sets and translational tilings were discussed. A measurable set Θ with positive measure is called a *spectral set* if there exists a countable set T such that the collection of exponentials $\{e^{2\pi i\delta x} : \delta \in T\}$ forms an orthonormal basis for $L^2(\Theta)$. In this case, the pair (Θ, T) is said to be a *spectral pair.*

Proposition 2.5.([29]) *Let $D \in GL(n, R)$, $T \subset R^n$ and $\Theta \subset R^n$ be a positive finite Lebesgue measure. If $\{M^t(\Theta) : M \in D\}$ is a tiling of R^n and (Θ, T) is a spectral* *pair, then* $\Psi = \check{1}_\Theta$ *is a wavelet with respect to the dilation set* D *and the translation set* T*. Conversely, if* $\Psi = \check{1}_\Theta$ *is a wavelet with respect to the dilation set* D *and the translation set* T *and* $0 \in T$*, then* $\{M^t(\Theta) : M \in D\}$ *is a tiling of* R^n *and* (Θ, T) *is a spectral pair.*

Definition 2.6.([21]) Let Θ be a subset of R^n, then Θ is said to be a *multiplicative tiling set* for R^n corresponding to a collection of $n \times n$ non-singular matrices M if $\{\alpha(\Theta) : \alpha \in M\}$ is a set partition for R^n i.e.,

$$\sum_{\alpha \in M} 1_{\alpha(\Theta)}(x) = 1, \quad x \in R^n.$$

Equivalently, $R^n = \bigcup_{\alpha \in M} \alpha(\Theta)$ where for any $\alpha \neq \alpha^{'}$ the two sets $\alpha(\Theta)$ and $\alpha^{'}(\Theta)$ are disjoint in measure.

Definition 2.7.([21]) Let Θ be a subset of R^n, then Θ is said to be a *translational tiling set* for R^n corresponding a countable set $L \subset R^n$ if

$$\sum_{l \in L} 1_\Theta(x - l) = 1, \quad x \in R^n.$$

Equivalently, $R^n = \bigcup_{l \in L} \Theta + l$, where for each pair of independent elements l and $l^{'}$, the sets $\Theta + l$ and $\Theta + l^{'}$ disjoint in measure.

3. Sufficient Conditions for Tight Wavelet Frame Sets in LCA Groups

Let f be a function on an infinite LCA group G; then the Fourier coefficient is defined by

$$\widehat{f}(\xi) = \sum_{n \in G} a_n \overline{\chi_n(\xi)}, \quad \xi \in G,$$

where $\chi_n(\xi) = e^{2\pi i \frac{n.\xi}{\alpha}}$ is the characteristic function, α is in G, $n.\xi$ is an inner product and we have $a_n = f(n)$. Here n is a variable. The function $\widehat{f}$ is the Fourier transform of f. The Fourier transform of the function $f \in G$ is defined as

$$f(x) = \sum_{n \in G} b_n \chi_n(x),$$

where $b_n = \widehat{f}(n)$. The Fourier transform $F : L^2(G) \to L^2(G)$ given by $f \to \widehat{f}$ is a unitary map. For any $h \in L^2(G)$, we shall denote by $\check{h}$ the inverse Fourier transform of h.

Let $Aut(G)$ be the set of all automorphism of G. A linear function $\Psi : L^2(G) \to C$ is a wavelet frame if there exists a set of automorphism $A \subset Aut(G)$ and a subset $T \subset G$ such that the linear map

$$\{\Psi(\alpha x - \beta) : \alpha \in A, \beta \in T\}$$

is an orthonormal basis concerning a wavelet frame for $L^2(G)$.

When $\alpha \in Aut(G)$ and $\beta \in G$ is associated with dilation and translation operators σ_α and τ_β defined by

$$\sigma_\alpha \Psi(x) = \Psi(\alpha x) \quad \text{and} \quad \tau_\beta \Psi(x) = \Psi(x - \beta)$$

respectively.

Definition 3.1. A measurable subset $\Theta \subset R^n$ with finite and positive measure is called a *wavelet set* if there is a function $\Psi \in L^2(R^n)$ with $\widehat{\Psi} = 1_\Theta$, such that the family (2.4) is an orthonormal basis for $L^2(G)$.

Definition 3.2. Let J be a subset of an infinite LCA group G, then J is called a *multiplicative tiling* for G if there exists a set of group automorphism A in $Aut(G)$ such that J tiles G multiplicatively by A, i.e.,

$$G = \bigcup_{\alpha \in A} \alpha(J)$$

for $\alpha \neq \alpha^{'}$ the union of two sets $\alpha(J)$ and $\alpha^{'}(J)$ are disjoint.

Definition 3.3. Let J be a subset of an infinite LCA group G, then J is called a *translational tiling set* for G if there exists a subset $T \subseteq G$ such that

$$G = \bigcup_{\beta \in T} (J + \beta),$$

and for $\beta \neq \beta^{'}$ the union of the two sets $(J + \beta)$ and $(J + \beta^{'})$ is disjoint.

As a result, if J is a multiplicative tiling set for an infinite LCA group G and $0 \in G$, then $\alpha(0) = 0$ is a natural requirement for any group automorphism α. We say a set J has spectrum K if the characteristic $\{\chi_k\}_{k \in K}$ is an orthonormal basis for $L^2(J)$. In this case, we say J is a spectral and (J, K) is a spectral pair.

Proposition 3.4. *Let G be an infinite LCA group, C a set of complex numbers and $\Psi : L^2(G) \to C$, then for a certain group automorphism $\alpha \in Aut(G)$ and $\beta \in G$ we have*

$$\widehat{\sigma_\alpha \tau_\beta \Psi}(n) = \overline{\chi_{\alpha^{-1}\beta}(n)}\widehat{\Psi}(\alpha^* n),$$

where $\alpha^* = (\alpha^t)^{-1} = (\alpha^{-1})^t$ the inverse transpose of α.
If $\widehat{\Psi} = 1_J$, where J is a spectral set, then

$$\widehat{\sigma_\alpha \tau_\beta \Psi}(n) = \overline{\chi_{\alpha^{-1}\beta}(n)} 1_{\alpha^t(J)}(n).$$

Proof. Let $\alpha \in Aut(G)$ and $\beta \in G$. By applying a Fourier transform and modifying the variable using the definition of Fourier transforms, for all $n \in G$ we have,

$$\widehat{\tau_\beta \Psi}(n) = \sum_{m \in G} \Psi(m-\beta)\overline{\chi_n(m)} = \sum_{m \in G} \Psi(m)\overline{\chi_n(m+\beta)} = \overline{\chi_n(\beta)}\left(\sum_{m \in G} \Psi(m)\overline{\chi_n(m)}\right) = \overline{\chi_n(\beta)}\widehat{\Psi}(n) = \overline{\chi_\beta(n)}\widehat{\Psi}(n) \tag{3.1}$$

and

$$\widehat{\sigma_\alpha \Psi}(n) = \sum_{m \in G} \Psi(\alpha m)\overline{\chi_n(m)} = \sum_{m \in G} \Psi(m)\overline{\chi_n(\alpha^{-1}m)} = \sum_{m \in G} \Psi(m)\overline{\chi_{\alpha^* n}(m)} = \widehat{\Psi}(\alpha^* n). \tag{3.2}$$

Now a combination of both equations (3.1) and (3.2) gives the assertion of the first part of Proposition 3.4.:

$$\widehat{\sigma_\alpha \tau_\beta \Psi}(n) = \widehat{\tau_\beta \Psi}(\alpha^* n) = \overline{\chi_{\alpha^* n}(\beta)}\widehat{\Psi}(\alpha^* n) = \overline{\chi_{\alpha^{-1}\beta}(n)}\widehat{\Psi}(\alpha^* n). \tag{3.3}$$

For the proof of the second part of proposition 3.4., we take the equality $1_J(\alpha^* n) = 1_{\alpha^t(J)}(n)$, then the proposition also holds. □

Proposition 3.5. *Let $A \subseteq Aut(G)$ and $T \subseteq G$. The family*

$$\{\sigma_\alpha \tau_\beta \Psi : \alpha \in A, \beta \in T\}$$

is an orthonormal basis for $L^2(G)$ if and only if the family

$$\{\overline{\chi_{\alpha^{-1}\beta}(n)}\widehat{\Psi}(\alpha^* n) : \alpha \in A, \beta \in T\}$$

is an orthonormal basis for $L^2(G)$. Here n is a variable.

The proof of Proposition 3.5. is straightforward using a Fourier transform and Proposition 3.4.

Propostion 3.6. *There exists no non empty subset $J \subsetneq G$ such that $\Psi = \check{1}_J$, the inverse Fourier transform of the indicator function 1_J, is a generator of Parseval* *wavelet frame for* $L^2(G)$.

Proof. Now we have a contradictory argument to prove this proposition. Suppose that there exists a non-empty set J and a subset $A \subset Aut(G)$ and a set $T \subset G$ such that the set

$$\{\sigma_\alpha \tau_\beta \check{1}_J : \alpha \in A, \beta \in T\}$$

is a Parseval frame for $L^2(G)$. By Proposition 3.5., we get that

$$\{\overline{\chi_{\alpha^* n}(\beta)} 1_{\alpha^t(J)}(n) : \alpha \in A, \beta \in T\} = \{\overline{\chi_{\alpha^{-1}\beta}(n)} 1_{\alpha^t(J)}(n) : \alpha \in A, \beta \in T\}$$

is a Parseval frame for $L^2(G) = L^2(\widehat{G})$, where $\widehat{G}$ is a dual group of an infinite LCA group G. Let $\widehat{h} = 1_{\{0\}} \in L^2(G)$ be a indicator function for the set $\{0\}$. Then

$$\begin{aligned} 1 = \|\widehat{h}\|^2 &= \sum_{\alpha \in A, \beta \in T} |\langle h, \sigma_\alpha \tau_\beta \Psi\rangle|^2 \\ &= \sum_{\alpha \in A, \beta \in T} |\langle \widehat{h}, \widehat{\sigma_\alpha \tau_\beta \Psi}\rangle|^2 \\ &= \sum_{\alpha \in A, \beta \in T} |\sum_{n \in G} \widehat{h}(n) 1_{\alpha^t(J)}(n) \overline{\chi_{\alpha^* n}(\beta)}|^2 \\ &= \sum_{\alpha \in A, \beta \in T} |1_{\alpha^t(J)}(0)|^2 \\ &= \sharp(T) \sum_{\alpha \in A, \beta \in T} |1_{\alpha^t(J)}(0)|^2 \end{aligned} \tag{3.4}$$

where $\sharp$ stands for number of elements. Two cases are taken into consideration here:

Case one: If $0 \in J$, then $0 \in \alpha^t(J)$ for all $\alpha \in A$. The above formula thus indicates that $1 = \sharp(A)\sharp(T)$. So there must be only one element of the wavelet system, let $A = \{\alpha\}$ and $T = \{\beta\}$. Then all the vectors in $L^2(G)$ must be constant multiple of $\sigma_\alpha \tau_\beta \Psi$. We show that this is not the case if $J \neq G$. Let $g \neq 0$ in $L^2(G)$ such that $(\widehat{g}) \cap \alpha^t(J)$ is empty. There is no such function since J is not the whole group G. Then there is no constant ϵ such that $g = \epsilon \sigma_\alpha \tau_\beta \Psi$. This indicates that there is not just one element to the Parseval wavelet frame when $J \neq G$, thus $\sharp(A)\sharp(T) > 1$.

Case two: If 0 does not belong to J. By the equation in (3.4) we get $1 = 0$, which is impossible. This completes the proof. □

Therefore, the above Proposition 3.6. shows that there does not exist any Parseval wavelet frame for $L^2(G)$ generated by $\Psi = \check{1}_J$.

Definition 3.7. Let J and K be two subsets of an LCA group G, and we say (J, K) is a *tight wavelet frame spectral pair* if the set of characteristic $\{\chi_k : k \in K\}$ is a tight wavelet frame for $L^2(G)$.

Proposition 3.8. *Let J and K be two subsets of G and (J, K) be a spectral pair. Suppose that J is a multiplicative tiling set concerning a set of automorphisms $A \subset Aut(G)$. The following hold true:*

(i) *(J,K) is a tight wavelet frame spectral pair with the frame bound $0 < P \leq Q < \infty$.*

(ii) *For all $\alpha \in A$, $(\alpha(J), (\alpha^{-1})^t(K))$ is a tight wavelet frame spectral pair with the frame bound $0 < P \leq Q < \infty$.*

(iii) *The family $\{1_{\alpha(J)}\chi_{(\alpha^{-1})^t(k)} : \alpha \in A, k \in K\}$ is a tight wavelet frame for $L^2(G)$ with the frame bound $0 < P \leq Q < \infty$.*

(iv) *If 0 not in J, then $\{(J)^{-1/2}1_{\alpha(J)}\chi_{(\alpha^{-1})^t(k)} : \alpha \in A, k \in K\}$ is an orthonormal basis for $L^2(G)$.*

Proof. Suppose J has a spectrum K. Then $\{(J)^{-1/2}\chi_k : k \in K\}$ is an orthonormal basis for $L^2(G)$.

Proof of (i): Note that a projection map $f \to f1_J$ of $L^2(G)$ onto $L^2(G)$, therefore the image of the orthonormal basis $\{(J)^{-1/2}\chi_k : k \in K\}$ by this projection map is a Parseval frame for $L^2(G)$. This shows the statement (i).

Proof of (ii): Let H be a subgroup of LCA group G and $\{\chi_k : k \in K\}$ be a frame for $L^2(H)$ with the frame bound $0 < P \leq Q < \infty$. Then $\{\chi_{(\alpha^{-1})^t(k)} : k \in K\}$ is a frame for $L^2(\alpha(H))$ with the unified frame bounds P and Q. To prove this, we define the map $F_\alpha : L^2(H) \to L^2(\alpha(H))$ by $f \to f \circ \alpha^{-1}$. The image of $\{\chi_k : k \in K\}$ under F_α is $\{1_{\alpha(H)}\chi_{(\alpha^{-1})^t(k)} : k \in K\}$ and this map is unitary. Therefore, $\{1_{\alpha(H)}\chi_{(\alpha^{-1})^t(k)} : k \in K\}$ forms a frame for $L^2(\alpha(H))$ with the same frame bounds.

Proof of (iii): By the supposition on the multiplicative tiling property of J we get

$$L^2(G) = \bigcup_{\alpha \in A} L^2(\alpha(J))$$

To complete (iii), we shall prove the following instead.

Let Y be a measurable set and I be an index set such that $Y = \bigcup_{\alpha \in A} Y_\alpha$ are disjoint. Suppose that for all $\alpha \in A$, $L^2(Y_\alpha)$ has a tight wavelet frame $\{f_{m,\alpha} : m \in I_\alpha\}$ with the frame bound S. We are saying that the family $\{f_{m,\alpha} : \alpha \in A, m \in I_\alpha\}$ is a tight wavelet frame for $L^2(Y)$ with the frame bound S. To show this, let $h \in L^2(Y)$. Since $\{Y_\alpha : \alpha \in A\}$ is a partition for Y, then $h = \oplus_{\alpha \in A} h_\alpha$, $h_\alpha = h1_{Y_\alpha}$ and we get

$$\begin{aligned} \|h\|^2 &= \sum_{\alpha \in A} \|h_\alpha\|^2_{L^2(Y_\alpha)} \\ &= \sum_{\alpha \in A} \left(S^{-1} \sum_{m \in I} |\langle h_\alpha, f_{m,\alpha} \rangle_{L^2(Y_\alpha)}|^2 \right) \\ &= S^{-1} \sum_{\alpha \in A, m \in I_\alpha} |\langle h, 1_{Y_\alpha} f_{m,\alpha} \rangle_{L^2(Y)}|^2. \end{aligned}$$

This completes the proof of (iii).

Proof of (iv): If J is a spectral set in with 0 does not belong to J; therefore, facts can be derived explicitly from the fact that any Parseval frame with normalized frame components is an orthonormal basis. □

4. Open Problem

Now, we propose an open problem about the extension principle for Riesz wavelets in LCA groups. Let M be a non-singular matrix with dilation factor $c = |detM|^{1/2}$ and L a subset of an LCA group G, then for $j \geq 0$, we define the $2\pi M^{-j}l$ shift operator

$$T_j^l : L^2(G) \to L^2(G)$$
$$\text{such that}, T_j^l f = f(. - 2\pi M^{-j}l), \quad f \in L^2(G), l \in L.$$

Our aim is to construct scaling functions Φ_j^r $r = 1, 2, \cdots m$ in $L^2(G)$ and for the Riesz wavelets Ψ_j^r, $r = 1, 2, \cdots \rho_j$ in $L^2(G)$, where m and ρ_j are positive integers, such that the collection

$$\{\Phi_j^r : j \geq 0, r = 1, 2, \cdots m\} \bigcup \{T_j^l \Psi_j^r : j \geq 0, r = 1, 2, \cdots \rho_j, l \in L\}$$

forms a Riesz wavelet orthonormal basis for $L^2(G)$.

In the way defined above, what type of scaling function can show the extension principle for Riesz wavelets in LCA groups?

References

[1] D. Barbieri, E. Hernandez and A. Mayeli, *Tiling by lattices for locally compact abelian groups*, C. R. Math., **355**(2017), 193–199.

[2] J. J. Benedetto and R. L. Benedetto, *A wavelet theory for local fields and related groups*, J. Geom. Anal., **14(3)**(2004), 423–456.

[3] J. J. Benedetto and M. T. Leon, *The construction multiple dyadic minimally supportrd frequency wavelets on R^d*, The functional and harmonic analysis of wavelets and frames, 43-–74, Contemp. Math. **247**, Amer. Math. Soc., 1999.

[4] M. Bownik, *Riesz wavelets and generalized multiresolution analysis*, Appl. Comput. Harmonic. Anal., **14(3)**(2003), 181–194.

[5] M. Bownik and M. S. Jakobsen, *On Wilson bases in $L^2(R^d)$*, SIAM J. Math. Anal., **49(5)**(2017), 3999–4023.

[6] M. Bownik and J. Jasper, *Existence of frames with prescribed norms and frame operator*, Appl. Numer, Harmon. Anal., **4**(2015), 103–117.

[7] D. Chen, *On the splitting trick and wavelet frame packets*, SIAM J. Math. Anal., **31(4)**(2000), 726–739.

[8] B. Currey and A. Mayeli, *Gabor fields and wavelet sets for the Heisenberg group*, Monatsh. Math., **162**(2011), 119–142.

[9] I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., **27**(1986), 1271–1283.

[10] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72**(1952), 341–366.

[11] X. Fang and X. H. Wang, *Construction of minimally supported frequency wavelets*, J. Fourier Anal. Appl., **2**(1996), 315–327.

[12] S. S. Goh, B. Han and Z. Shen, *Tight periodic wavelet frames and approximation orders*, Appl. Comput. Harmon. Anal., **31**(2011), 228–248.

[13] S. S. Goh and K. M. Teo, *Extension principles for tight wavelet frames of periodic functions*, Appl. Comput. Harmon. Anal., **25(2)**(2008), 168–186.

[14] S. Grepstad and N. Lev, *Multi-tiling and Riesz bases*, Adv. Math., **252**(2014), 1–6.

[15] S. Grepstad and N. Lev, *Riesz bases, Meyer's quasicrystals, and bounded remainder sets*, Trans. Amer. Math. Soc., **370**(2018), 4273–4298.

[16] B. Han and R.-Q. Jia, *Characterization of Riesz bases of wavelets generated from multiresolution analysis*, Appl. Comput. Harmon. Anal., **23**(2007), 321–345.

[17] E. Hernandez, X. Wang and G. Weiss, *Smoothing minimally supported frequency wavelets, I*, J. Fourier Anal. Appl., **2**(1996), 329–340.

[18] E. Hernandez, X. H. Wang and G. Weiss, *Smoothing minimally supported frequency wavelets, II*, J. Fourier Anal. Appl., **3**(1997), 23–41.

[19] A. Iosevich, C. K. Lai, and A. Mayeli, *Tight wavelet frame sets in finite vector spaces*, Appl. Comput. Harmon. Anal., **46**(2019), 192–205.

[20] E. A. Lebedeva and J. Prestin, *Periodic wavelet frames and time-frequency localization*, Appl. Comput. Harmon. Anal., **37(2)**(2014), 347–359.

[21] A. Mayeli, *Riesz wavelets, tiling and spectral sets in LCA groups*, Complex Anal. Oper. Theory, **13**(2019), 1177–1195.

[22] C. Monico and M. Elia, *Note on an additive characterization of quadratic residues modulo p*, J. Comb. Inf. Syst. Sci., **31**(2006), 209–215.

[23] L. S. Pontryagin, *Topological groups*, Princeton Univ. Press, 1946.

[24] S. P. Raja, *Bezier and B-spline curves–a study and its application in wavelet decomposition*, Int. J. Wavelets Multiresolut. Inf. Process., **18(4)**(2020), 2050030, 38 pp.

[25] F. A. Shah, O. Ahmad and N. A. Sheikh, *Orthogonal gabor systems on local fields*, Filomat, **31**(2017), 5193–5201.

[26] F. A. Shah and L. Debnath, *Tight wavelet frames on local fields*, Analysis, **33**(2013), 293–307.

[27] F. A. Shah and A. Wahid, *Wavelet packets on locally compact abelian groups*, An. St. Univ. Ovidius Constanta Ser. Mat., **18(2)**(2010), 223–239.

[28] S. H. Srivastava and F. A. Shah, *AB-wavelet frames in* $L^2(R^n)$, Filomat, **33**(2019), 3587–3597.

[29] Y. Wang, *Wavelets, tiling and spectral sets*, Duke Math. J., **114(1)**(2002), 43–57.