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Baer–Kaplansky Theorem for Modules over Non-commutative Algebras

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Abstract. In this paper we investigate the Baer-Kaplansky theorem for module classes on algebras of finite representation types over a field. To do this we construct finite dimensional quiver algebras over any field.

1. Introduction

We consider associative rings R with identity; all modules considered are unitary left R-modules. Throughout this paper K will be any field.

For a vertex x of a quiver Q, $S(x)$ denotes the simple representation corresponding to the vertex x. Moreover, $P(x)$ (resp. $I(x)$) denotes the indecomposable projective (resp. injective) representation corresponding to the vertex x. For short, $S(x)$ is replaced by x. With this convention a chain of length $n \geq 2$ of the form

$$\begin{array}{c} 1 \\ 2 \\ . \\ . \\ . \\ n \end{array}$$

describes a uniserial module of length n with composition factors $n, \cdots, 2, 1$. Moreover, a picture of the form $\begin{smallmatrix} & 1 & \\ 1 & & 2 \end{smallmatrix}$ (resp. $\begin{smallmatrix} 1 & & 2 \\ & 2 & \end{smallmatrix}$) describes an indecomposable module M of length three such that the socle of M is isomorphic to $1 \oplus 2$ (resp.

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2), while the factor module $M/\mathrm{soc}M$ is isomorphic to 1 (resp. $1 \oplus 2$). For more background on quivers we refer to [1] and [11].

The aim of this paper is to construct classes of modules which satisfy or do not satisfy the Baer-Kaplansky theorem defined on K-algebras, where K is any field. When we look at the literature, we see that the Baer-Kaplansky theorem states that any two torsion abelian groups having isomorphic endomorphism rings are isomorphic [4, Theorem 108.1]. Finding other classes of abelian groups, and more generally, of modules, for which a Baer-Kaplansky-type theorem is still true remains an interesting problem. In [6], Ivanov and Vámos called such classes *Baer-Kaplanksy classes*. For example, the class of finitely generated abelian groups is Baer-Kaplansky (e.g., see [7, Example 1.3]). Over commutative rings, there are several Baer-Kaplansky classes of modules, but there are relatively few known over non-commutative rings. In particular, we know from Morita's paper ([8, Lemma 7.4]) that the class of all modules over a primary artinian uniserial ring is Baer-Kaplansky. Moreover, we know from Ivanov's paper ([5, Theorem 9]) that the class of all modules over a non-singular artinian serial ring is Baer-Kaplansky.

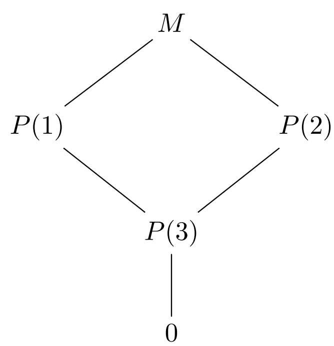
These are the motivating and leading ideas in our investigation of Baer-Kaplansky classes over non-commutative algebras.

This paper is organized as follows. In Section 1 we recall some definitions and conventions. In Section 2 we collect all the results. We begin with some negative results. As we shall see, rather few classes of modules over non-commutative algebras fail to be Baer-Kaplansky. In Example 2.1, we construct a class of simple injective left R-modules which is not Baer-Kaplansky over a hereditary K-algebra R of finite representation type. In Example 2.3, we construct a class of simple left R-modules which is not Baer-Kaplansky over a non-hereditary K-algebra R of finite representation type. Also we obtain some positive results by dealing with classes of modules with a rigid structure, containing two indecomposable modules and closed under finite direct sums. Indeed the endomorphism rings of the two indecomposable modules always have dimension 1 and 2, and the vector spaces of the morphisms between two indecomposable non-isomorphic modules have dimension ≤ 2. As for the invariants, some of these classes admit the number of indecomposable direct summands and the dimension of the endomorphism ring as a complete set of invariants (Example 2.8 and Example 2.10). However this property is not always true for a Baer-Kaplansky class of finitely generated projective (resp. injective) modules over a finite dimensional algebra (Example 2.6.)

2. Results

Example 2.1. There is a hereditary K-algebra R of finite representation type and a class of R-modules such that Baer-Kaplansky theorem fails.

Construction: Let R be the K-algebra given by the quiver $1 \longrightarrow 3 \longleftarrow 2$. Then $1, 2, 3$, $\begin{smallmatrix}1\\3\end{smallmatrix}$, $\begin{smallmatrix}2\\3\end{smallmatrix}$ and $\begin{smallmatrix}1 & & 2\\ & 3 & \end{smallmatrix}$ are the indecomposable left R-modules. Let M be the R-module $I(3) = \begin{smallmatrix} 1 & & 2 \\ & 3 & \end{smallmatrix}$. Note that $P(1) = \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}$, $P(2) = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ and $P(3) = 3$. The lattice of submodules of M is

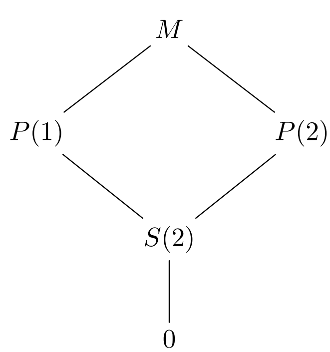


and we have $I(3)/P(1) \ncong I(3)/P(2)$. Also $\mathrm{End}_R(I(3)/P(1)) \cong \mathrm{End}_R(I(3)/P(2)) \cong K$ because $I(3)/P(1)$ and $I(3)/P(2)$ are one dimensional vector spaces. Therefore the class of simple injective left R-modules $\{I(3)/P(1), I(3)/P(2)\} = \{S(2), S(1)\}$ is not Baer-Kaplansky.

Remark 2.2. Let R be a K-algebra of finite representation type such that K is the endomorphism ring of any indecomposable left R-module. Then Baer-Kaplansky theorem fails for any class with more than one indecomposable module. Moreover Baer-Kaplansky theorem holds for any class of the form $\{M^n \mid n \geq 1\}$, where M is an indecomposable left R-module.

Example 2.3. There is a non-hereditary K-algebra R of finite representation type and a class of R-modules such that Baer-Kaplansky theorem fails.

Construction: Let R be the K-algebra given by the quiver $1 \xrightarrow{a} 2 \circlearrowleft b$ with relations $ba = b^2 = 0$. Then the indecomposable left R-modules are $1, 2, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$ and $\begin{smallmatrix} 1 & & 2 \\ & 2 & \end{smallmatrix}$. Let $M = I(2) = \begin{smallmatrix} 1 & & 2 \\ & 2 & \end{smallmatrix}$. Note that $P(1) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$, $P(2) = \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$ and $S(2) = 2$. Also in this case the lattice of submodules of M is of the form



with $I(2)/P(1) \ncong I(2)/P(2)$. Since $I(2)/P(1)$ and $I(2)/P(2)$ are one dimensional vector spaces, $\mathrm{End}_R(I(2)/P(1)) \cong \mathrm{End}_R(I(2)/P(2)) \cong K$. Then the class of simple left R-modules $\{I(2)/P(1), I(2)/P(2)\} = \{S(2), S(1)\}$ is not Baer-Kaplansky. Note that $S(1)$ is injective, while $S(2)$ has infinite injective dimension. Here $S(2)$ has a minimal injective resolution of the form

$$0 \longrightarrow S(2) = 2 \longrightarrow \begin{matrix} 1 & & 2 \\ & 2 & \end{matrix} \longrightarrow 1 \oplus \begin{matrix} 1 & & 2 \\ & 2 & \end{matrix} \longrightarrow 1 \oplus \begin{matrix} 1 & & 2 \\ & 2 & \end{matrix} \longrightarrow \cdots .$$

Moreover both simple modules have infinite projective dimension. The minimal projective resolutions of $S(1)$ and $S(2)$ are of the form

$$\cdots \longrightarrow \begin{matrix} 2 \\ 2 \end{matrix} \longrightarrow \begin{matrix} 2 \\ 2 \end{matrix} \longrightarrow \begin{matrix} 1 \\ 2 \end{matrix} \longrightarrow 1 = S(1) \longrightarrow 0$$

and

$$\cdots \longrightarrow \begin{matrix} 2 \\ 2 \end{matrix} \longrightarrow \begin{matrix} 2 \\ 2 \end{matrix} \longrightarrow \begin{matrix} 2 \\ 2 \end{matrix} \longrightarrow 2 = S(2) \longrightarrow 0.$$

Proposition 2.4. *Let R be a K-algebra admitting three non-isomorphic modules M with the following properties:*

(1) *M has exactly three non-zero proper submodules N_1, N_2 and $N_1 \cap N_2$.*

(2) *M/N_1 is not isomorphic to M/N_2.*

(3) $\mathrm{End}_R(M/N_i) \cong K$ *for $i = 1, 2$.*

Then R is not commutative and there is a class (of non-uniserial modules) which is not Baer-Kaplansky.

Proof. The existence of a module satisfying (1), (2) and (3) implies that R is not commutative [3, Remark 2.2]. On the other hand the endomorphism ring of a module satisfying (1), (2) and (3) is either isomorphic to K or isomorphic to $K[x]/(x^2)$ [2, Theorem 3.8]. Since there exist three non-isomorphic modules M_1, M_2 and M_3 satisfying (1), (2) and (3), without loss of generality we may assume that $\{M_1, M_2\}$ is not a Baer-Kaplansky class. □

We will use the next lemma to construct Baer-Kaplansky classes with infinitely many modules and exactly two indecomposable modules. In the sequel given a module M we will denote by $\mathrm{add}M$ the class of all finite direct sums of direct summands of M.

Lemma 2.5. *Let R be a K-algebra of finite dimension and let U and V be two finite dimensional left R-modules with the following properties:* $\mathrm{End}_R(U) \cong K$, $\mathrm{End}_R(V) \cong K[x]/(x^2)$, $\mathrm{Hom}_R(U, V) = 0$, $\mathrm{Hom}_R(V, U) \cong K$. *Then the class* $\mathrm{add}(U \oplus V)$ *is a Baer-Kaplansky class. Moreover the number of indecomposable direct summands and the dimension of the endomorphism ring are not a complete set of invariants for the modules in* $\mathrm{add}(U \oplus V)$.

Proof. Since U and V are indecomposable modules, it follows that $\mathrm{add}(U\oplus V)$ consists of finite direct sums of copies of U and V. Let X be a non-zero left R-module of the form $U^m \oplus V^n$ with $m, n \in \mathbb{N}$. Let $A = \mathrm{End}_R(U^m)$, $B = \mathrm{End}_R(V^n)$ and let $H = \mathrm{Hom}_R(V^n, U^m)$. Then A is isomorphic to the full matrix algebra $M_m(K)$ and B is isomorphic to the full matrix algebra $M_n(K[x]/(x^2))$. Finally H is an A-B-bimodule of dimension mn. Let $T = \mathrm{End}_R(X)$. Then our hypotheses on U, V and X imply that

(1) $T = \mathrm{End}_R(X)$ is isomorphic to the matrix algebra $\begin{bmatrix} A & H \\ 0 & B \end{bmatrix}$.

This means that the identity of T is the sum of primitive idempotents $e_1, \cdots, e_m$, $\epsilon_1, \cdots, \epsilon_n$ such that

(2) The regular module ${}_T T$ is the direct sum of the simple isomorphic modules $Te_1, \cdots, Te_m$ (of dimension m) and of the indecomposable non-simple isomorphic modules $T\epsilon_1, \cdots, T\epsilon_n$ (of dimension $m + 2n$).

Let Y be a module in $\mathrm{add}(U \oplus V)$ such that $\mathrm{End}_R(X) \cong \mathrm{End}_R(Y)$. Then there exist $p, q \in \mathbb{N}$ such that $Y \cong U^p \oplus V^q$, and so

(3) $\mathrm{End}_R(Y)$ is the direct sum of p simple isomorphic modules (of dimension p) and q indecomposable non-simple isomorphic modules (of dimension $p + 2q$).

Since T is a finite dimensional algebra, we know from [9, p. 66] that the category $\mathrm{mod}T$ of finitely generated T-modules is a Krull-Schmidt category, that is a category where the Krull-Remark-Schmidt theorem [10, p. 3] holds. Consequently we deduce from (2) and (3) that $m = p$ and $n = q$, and so $X \cong Y$.

We finally note that U^2 and $U \oplus V$ are non-isomorphic modules with endomorphism ring of dimension 4. More generally, if $m, n, s \in \mathbb{N}$ and $s \leq m$, then $U^m \oplus V^n$ and $U^{m-s} \oplus V^{n+s}$ have endomorphism ring of the same dimension $m^2 + mn + 2n^2$ if and only if $ms - 3ns - 2s^2 = 0$. The lemma is proved. □

Example 2.6. There is a non-commutative K-algebra R of finite representation type such that the class of finitely generated projective (resp. injective) modules is a Baer-Kaplansky class with the property described in Lemma 2.5.

Construction: Let R be the algebra considered in Example 2.3, given by the quiver $1 \stackrel{a}{\to} 2 \circlearrowleft b$ with relations $ba = b^2 = 0$. Then the classes of finitely generated projective and injective modules are $\mathrm{add}(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix})$ and $\mathrm{add}(1 \oplus \begin{smallmatrix}1 & & 2\\ & 2 & \end{smallmatrix})$ respectively. Moreover we clearly have $\mathrm{End}_R(\begin{smallmatrix}1\\2\end{smallmatrix}) \cong K \cong \mathrm{End}_R(1)$, $\mathrm{End}_R(\begin{smallmatrix}2\\2\end{smallmatrix}) \cong K[x]/(x^2) \cong \mathrm{End}_R(\begin{smallmatrix}1 & & 2\\ & 2 & \end{smallmatrix})$, $\mathrm{Hom}_R(\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\2\end{smallmatrix}) = 0 = \mathrm{Hom}_R(1, \begin{smallmatrix}1 & & 2\\ & 2 & \end{smallmatrix})$ and $\mathrm{Hom}_R(\begin{smallmatrix}2\\2\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}) \cong K \cong \mathrm{Hom}_R(\begin{smallmatrix}1 & & 2\\ & 2 & \end{smallmatrix}, 1)$. Hence the conclusion that $\mathrm{add}(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix})$ and $\mathrm{add}(1 \oplus \begin{smallmatrix}1 & & 2\\ & 2 & \end{smallmatrix})$ are Baer-Kaplansky classes with the desired property follows from Lemma 2.5.

We will use the next lemma to obtain Baer-Kaplansky classes $\mathcal{C}$ with the property that $\mathrm{Hom}_R(L, M) \neq 0$ if L and M are two non-zero modules in $\mathcal{C}$.

Lemma 2.7. *Let R be a K-algebra of finite dimension and let U and V be two finite dimensional left R-modules with the following properties:* $\mathrm{End}_R(U) \cong K, \mathrm{End}_R(V) \cong K[x]/(x^2), \mathrm{Hom}_R(U, V) \cong K \cong \mathrm{Hom}_R(V, U)$*. Then the class* $\mathrm{add}(U \oplus V)$ *is a Baer-Kaplansky class. Moreover the number of indecomposable direct summands and the dimension of the endomorphism ring are a complete set of invariants for the modules in* $\mathrm{add}(U \oplus V)$.

Proof. We first note that for any $m, n \in \mathbb{N}$ we have $\dim\mathrm{End}_R(U^m \oplus V^n) = m^2 + 2mn + 2n^2$. Assume that s is a natural number $\leq m$ such that $\dim\mathrm{End}_R(U^{m-s} \oplus V^{n+s}) = \dim(U^m \oplus V^n)$. Then we have $2ns + s^2 = 0$. Consequently $s = 0$. Hence $\mathrm{add}(U \oplus V)$ is a Baer-Kaplansky class with the desired property. □

Example 2.8. There is a non-hereditary K-algebra R of finite representation type, such that any indecomposable module is uniserial, with the following properties:

(1) The class of simple modules is not Baer-Kaplansky.

(2) Let $\mathcal{P}$ and $\mathcal{I}$ be the classes of finitely generated projective modules and finitely generated injective modules, respectively and let $\mathcal{C}$ be the class of finitely generated modules of projective and injective dimension at most one. Then $\mathcal{P}$, $\mathcal{I}$ and $\mathcal{C}$ are Baer-Kaplansky classes. Moreover the number of indecomposable direct summands and the dimension of the endomorphism ring are a complete set of invariants for the modules in $\mathcal{P}, \mathcal{I}$ and $\mathcal{C}$.

Construction: Let R be the K-algebra given by the quiver $1 \underset{b}{\overset{a}{\rightleftarrows}} 2$ with relation $ab = 0$. Then $1, 2, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}$ and $\begin{smallmatrix}1\\2\\1\end{smallmatrix}$ are the indecomposable modules. Since 1 and 2 are non-isomorphic one dimensional modules, (1) clearly holds. On the other hand we have $\mathcal{P} = \mathrm{add}(\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\1\end{smallmatrix})$, $\mathcal{I} = \mathrm{add}(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\1\end{smallmatrix})$ and $\mathcal{C} = \mathrm{add}(1 \oplus \begin{smallmatrix}1\\2\\1\end{smallmatrix})$. Consequently (2) follows from Lemma 2.7.

We will use the next lemma to obtain Baer-Kaplansky classes $\mathcal{C}$ with more complicated Hom spaces between indecomposable non-isomorphic modules in $\mathcal{C}$.

Lemma 2.9. *Let R be a K-algebra of finite dimension and let U and V be two finite dimensional left R-modules with the following properties:* $\mathrm{End}_R(U) \cong K, \mathrm{End}_R(V) \cong K[x]/(x^2), \mathrm{Hom}_R(U, V) = 0$ *and* $\dim\mathrm{Hom}_R(V, U) = 2$*. Then the class* $\mathrm{add}(U \oplus V)$ *is a Baer-Kaplansky class with the property described in Lemma 2.7.*

Proof. Our hypotheses imply that $\dim\mathrm{End}_R(U^m \oplus V^n) = m^2 + 2mn + 2n^2$ for any $m, n \in \mathbb{N}$. Hence the conclusion follows from the proof of Lemma 2.7. □

Example 2.10. There is a K-algebra R of finite dimension such that the classes $\mathcal{P}$ and $\mathcal{I}$ of finitely generated projective and finitely generated injective modules are Baer-Kaplansky, but the class $\mathcal{C}$ of finitely generated modules of projective and injective dimension at most one is not Baer-Kaplansky. Moreover the number of indecomposable direct summands and the dimension of the endomorphism ring are a complete set of invariants for the modules in $\mathcal{P}$ and $\mathcal{I}$.

Construction: Let R be the K-algebra given by the quiver $1 \xrightarrow{a} 2 \circlearrowleft b$ with relation $b^2 = 0$. Then we have $\mathcal{P} = \operatorname{add}\left(\begin{smallmatrix}1\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix}\right)$ and $\mathcal{I} = \operatorname{add}\left(1 \oplus \begin{smallmatrix}&&1\\1&&2\\&2&\end{smallmatrix}\right)$. This observation and Lemma 2.9 imply that $\mathcal{P}$ and $\mathcal{I}$ are Baer-Kaplansky classes with the desired property. We also note that there exist exact sequences of the form

$$0 \longrightarrow \begin{smallmatrix}2\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}&&1\\1&&2\\&2&\end{smallmatrix} \longrightarrow 1 \oplus 1 \longrightarrow 0,$$

$$0 \longrightarrow \begin{smallmatrix}1&&2\\&2&\end{smallmatrix} \longrightarrow \begin{smallmatrix}&&1\\1&&2\\&2&\end{smallmatrix} \longrightarrow 1 \longrightarrow 0$$

and

$$0 \longrightarrow \begin{smallmatrix}2\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&&2\\&2&\end{smallmatrix} \longrightarrow 0.$$

Hence $\begin{smallmatrix}2\\2\end{smallmatrix}$ and $\begin{smallmatrix}1&&2\\&2&\end{smallmatrix}$ are in $\mathcal{C}$. Since $\operatorname{End}_R\left(\begin{smallmatrix}2\\2\end{smallmatrix}\right) \cong K[x]/(x^2) \cong \operatorname{End}_R\left(\begin{smallmatrix}1&&2\\&2&\end{smallmatrix}\right)$, we conclude that $\mathcal{C}$ is not Baer-Kaplansky.

Proposition 2.11. *Let A and B be finite dimensional K-algebras such that there is an epimorphism from A to B. For any algebra R let $\mathcal{P}_R$ and $\mathcal{I}_R$ denote the classes of finitely generated projective and finitely generated injective left R-modules, respectively. Among others the following cases are possible:*

1. *$\mathcal{P}_A = \mathcal{I}_A$ is not a Baer-Kaplansky class, while $\mathcal{P}_B$ and $\mathcal{I}_B$ are Baer-Kaplansky classes. Moreover the number of indecomposable direct summands and the dimension of the endomorphism ring are a complete set of invariants for the modules in $\mathcal{P}_B$ and $\mathcal{I}_B$.*
2. *$\mathcal{P}_A, \mathcal{I}_A, \mathcal{P}_B$ and $\mathcal{I}_B$ are Baer-Kaplansky classes. Moreover the number of indecomposable direct summands and the dimension of the endomorphism ring are (resp. are not) a complete set of invariants for the modules in $\mathcal{P}_A$ and $\mathcal{I}_A$ (resp. $\mathcal{P}_B$ and $\mathcal{I}_B$).*

Proof. (1) Let A be the K-algebra given by the quiver $1 \underset{b}{\overset{a}{\rightleftarrows}} 2$ with relations $aba = bab = 0$. Then we have $\mathcal{P}_A = \mathcal{I}_A = \operatorname{add}\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix}\right)$. Since $\operatorname{End}_A\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right) \cong K[x]/(x^2) \cong$

$\mathrm{End}_A\left(\begin{smallmatrix}2\\1\\2\end{smallmatrix}\right)$, it follows that $\mathcal{P}_A = \mathcal{I}_A$ is not Baer-Kaplansky. Let B denote the algebra considered in Example 2.8, given by the quiver $1 \underset{b}{\overset{a}{\rightleftarrows}} 2$ with relation $ab = 0$. Then there is an epimorphism $A \to B$ and Lemma 2.7 implies that (1) holds.

(2) Let A be the algebra considered in Example 2.10, given by the quiver $1 \overset{a}{\to} 2 \circlearrowleft b$ with relation $b^2 = 0$. Next let B be the algebra considered in Example 2.6, given by the quiver $1 \overset{a}{\to} 2 \circlearrowleft b$ with relations $ba = b^2 = 0$. Also in this case there is an epimorphism $A \to B$. Hence (2) holds. □

We will use the next lemma to investigate classes of modules of finite projective or injective dimension.

Lemma 2.12. *Let R be the K-algebra of finite dimension and let U and V be two finite dimensional left R-modules such that* $\mathrm{End}_R(U) \cong K, \mathrm{End}_R(V) \cong K[x]/(x^2)$, $\mathrm{Hom}_R(U, V) \cong K$ *and* $\mathrm{Hom}_R(V, U) = 0$. *Then the class* $\mathrm{add}(U \oplus V)$ *is a Baer-Kaplansky class. Moreover the number of indecomposable direct summands and the dimension of the endomorphism ring are not a complete set of invariants for the modules in* $\mathrm{add}(U \oplus V)$.

Proof. The proof is similar to the proof of Lemma 2.5. More precisely, let X be a non-zero left R-module of the form $U^m \oplus V^n$ with $m, n \in \mathbb{N}$. Let $A = \mathrm{End}_R(U^m)$, $B = \mathrm{End}_R(V^n)$, $H = \mathrm{Hom}_R(U^m, V^n)$, $T = \mathrm{End}_R(X)$. Then A is isomorphic to the full matrix algebra $M_m(K)$, B is isomorphic to the full matrix algebra $M_n(K[x]/(x^2))$. H is a B-A-bimodule of dimension mn and T is isomorphic to the matrix algebra $\begin{bmatrix} A & 0 \\ H & B \end{bmatrix}$. Consequently, the following facts hold:

(1) $\dim T = m^2 + 2n^2 + mn$.

(2) The identity of T is the sum of $m + n$ primitive idempotents.

(3) The category of finitely generated left T-modules is a Krull-Schmidt category [9, page 66].

(4) The regular module ${}_T T$ is the direct sum of m indecomposable non-simple isomorphic left T-modules (of dimension $m + n$) and of n simple isomorphic left T-modules (of dimension $2n$).

From now on we continue as in the last part of the proof of Lemma 2.5.. □

Example 2.13. There exist finite dimensional K-algebras A and B with the following properties:

(1) The classes of finitely generated projective (resp. injective) left modules over A and B are Baer-Kaplansky classes.

(2) Any finitely generated left A-module of finite projective dimension is projective.

(3) Any finitely generated left B-module of finite injective dimension is injective.

(4) The class of finitely generated left A-modules of finite injective dimension is not a Baer-Kaplansky class.

(5) The class of finitely generated left B-modules of finite projective dimension is not a Baer-Kaplansky class.

Construction: Let A be the algebra of Example 2.6, given by the quiver $1 \overset{a}{\to} 2 \circlearrowleft b$ with relations $ba = b^2 = 0$. Next let B be the algebra, isomorphic to A^{op}, given by the quiver $a \circlearrowright 1 \overset{b}{\to} 2$ with relations $a^2 = ba = 0$. Then $1, 2, \begin{smallmatrix}1\\1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 2\end{smallmatrix}$ are the indecomposable left B-modules, while

$$\mathrm{add}(2 \oplus \begin{smallmatrix} & 1 & \\ 1 & & 2\end{smallmatrix}) \quad \text{and} \quad \mathrm{add}(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\end{smallmatrix})$$

are the classes of finitely generated projective and finitely generated injective left B-modules, respectively. Hence (1) immediately follows from Example 2.6 (or Lemma 2.5) and Lemma 2.12. Since the left A-modules $1, 2, \begin{smallmatrix}1 & & 2\\ & 2 & \end{smallmatrix}$ have infinite projective dimension, we conclude that (2) holds. Dually, the left B-modules $1, 2, \begin{smallmatrix} & 1 & \\ 1 & & 2\end{smallmatrix}$ have infinite injective dimension. Hence also (3) holds. Moreover the projective left A-module $\begin{smallmatrix}2\\2\end{smallmatrix}$ has injective dimension one, and we clearly have $\mathrm{End}_A(\begin{smallmatrix}2\\2\end{smallmatrix}) \cong K[x]/(x^2) \cong \mathrm{End}_A(\begin{smallmatrix}1 & & 2\\ & 2 & \end{smallmatrix})$. Consequently (4) holds. Finally the injective left B-module $\begin{smallmatrix}1\\1\end{smallmatrix}$ has projective dimension one, and we obviously have $\mathrm{End}_B(\begin{smallmatrix}1\\1\end{smallmatrix}) \cong K[x]/(x^2) \cong \mathrm{End}_B(\begin{smallmatrix} & 1 & \\ 1 & & 2\end{smallmatrix})$. Hence also (5) holds.

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