KYUNGPOOK Math. J. 61(2021), 441-453
https://doi.org/10.5666/KMJ.2021.61.2.441
pISSN 1225-6951   eISSN 0454-8124

Coupled Fixed Point Theorems in Modular Metric Spaces Endowed with a Graph

Yogita Sharma*
Department of Computer Science, Shri Vaishnav Institute of Management, Gumashta Nagar, Indore
e-mail : yogitasharma2006@gmail.com

Shishir Jain
Department of Mathematics, Shri Vaishnav Vidyapeeth Vishwavidyalaya, Gram Baroli, Sanwer Road, Indore
e-mail : jainshishir11@rediffmail.com

Abstract. In this work, we define the concept of a mixed G-monotone mapping on a modular metric space endowed with a graph, and prove some fixed point theorems for this new class of mappings. Results of this paper extend coupled fixed point theorems from partially ordered metric spaces into the modular metric spaces endowed with a graph. An example is presented to illustrate the new result.

1. Introduction

The Banach contraction principle is one of the most useful principle in applied mathematics. Because of its simplicity and usefulness, and its compatibility in modeling various problems, it has been generalized and extended by several researchers in various directions. Many mathematicians have done remarkable work on fixed point results for partially ordered metric spaces. The very first foray in this direction was taken by Ran and Reurings [17], it was a combination of Banach contraction principle and Knaster-Tarski fixed point theorem. They proved fixed point results for monotone mapping $F\colon X \to X$ on a complete metric space (X, d) endowed with a partial order relation $\preceq$. The results of Ran and Reurings were extended by Neito et al. [16] to functions which are not necessarily continuous.

In 2009, Harjani and Sadarangani [11] considered the result of Rhoades [18] in

* Corresponding Author.
Received March 25, 2020; revised September 19, 2020; accepted January 6, 2021.
2010 Mathematics Subject Classification: 46A80, 47H9, 47H10, 54H25.
Key words and phrases: coupled fixed point, G-monotone mapping, connected graph, modular metric space, partial order relation.

the setting of partially ordered metric spaces. The concept of coupled fixed point was introduced by Geo and Lakshmikantham [10] in 1987. After that, Bhaskar and Lakshmikantham [5] studied applications of coupled fixed point theorems for binary mappings. They introduced the concept of the mixed monotone property, and proved certain coupled fixed point theorems. These theorems are among the most interesting coupled fixed point theorems for mappings in ordered metric spaces having the mixed monotone property. In particular, they manifested the existence of a unique solution for a periodic boundary value problem. Ansari et al. [4] proved some coupled coincidence point results for mixed g-monotone mappings in partially ordered metric spaces via new functions. Jachymski [13] and Jachymski and Lukawska [12] introduced the concept of graph theory in the study of fixed point results. They generalized the above mentioned results and presented applications to the theory of linear operators, They studied the class of generalized Banach contractions on a metric space with a directed graph. This work on the fixed point theory of a metric space endowed with a graph, has since been extended by Alfuraidan [1] and Alfuraidan and Khamsi [2]. Alfuraidan and Khamsi [3] also proved coupled fixed point results of monotone mappings in a metric space with a graph.

Many coupled fixed point theorems were extended to modular metric space, which was introduced by Chistyakov via F-modular mappings [6] in 2008. This theory was developed further in [7] and [8]. In 2012, Chistyakov [9] established some fixed point theorems for contractive maps in modular metric spaces. Many authors have since considered this space. Ali Mutlu et al. [15] extended to partially ordered modular metric spaces certain coupled fixed point theorems for mappings having the mixed monotone property, and proved the existence of a unique solution for a given nonlinear integral equation.

In this paper, we extend certain the coupled fixed point results of Ali Mutlu et al. [15] to a mapping having the mixed monotone property in modular metric spaces endowed with a graph.

2. Preliminaries

Let X be a nonempty set, λ be in $(0,\infty)$, and the function $\omega\colon (0,\infty)\times X\times X \to [0,\infty]$ will be written as $\omega_\lambda(a,b) = \omega(\lambda,a,b)$ for all $\lambda > 0$ and $a,b\in X$.

Definition 2.1.([8]) Let X be a nonempty set, a function $\omega\colon (0,\infty)\times X\times X \to [0,\infty]$ is said to be a *modular metric* on X if it satisfies the following axioms, for all $a,b,c\in X$:

(i) $\omega_\lambda(a,b) = 0$ for all $\lambda > 0$ if and only if $a = b$.

(ii) $\omega_\lambda(a,b) = \omega_\lambda(b,a)$ for all $\lambda > 0$.

(iii) $\omega_{\lambda+\mu}(a,b) \le \omega_\lambda(a,c) + \omega_\mu(c,b)$ for all $\lambda,\mu > 0$.

If instead of (i), we have the condition (i*)

(i*) $$\omega_\lambda(a,a)=0 \text{ for all } \lambda>0, a\in X$$

then ω is said to be pseudomodular on X. The main property of a (pseudo) modular function ω on a set X is that for given $a,b\in X$, the function $0<\lambda\mapsto\omega_\lambda(a,b)\in[0,\infty]$ is non-increasing on $(0,\infty)$.

In fact, if $0<\mu<\lambda$, then (iii), (i*) and (ii) imply

$$\omega_\lambda(a,b)\le\omega_{\lambda-\mu}(a,a)+\omega_\mu(a,b)=\omega_\mu(a,b). \tag{2.1}$$

Definition 2.2.([8]) Let ω be a psedomodular function on X. Fix $a_0\in X$, and set

$$X^*_\omega=X^*_\omega(a_0)=\{a\in X:\omega_\lambda(a,a_0)\to 0 \text{ as } \lambda\to\infty\}.$$

A modular (pseudomodular, strict modular) function ω on X is said to be *convex* if, instead of (iii), for all $\lambda,\mu>0$ and $a,b,c\in X$ it satisfies the inequality

$$(iv)\ \ \omega_{\lambda+\mu}(a,b)\le\frac{\lambda}{\lambda+\mu}\omega_\lambda(a,c)+\frac{\mu}{\lambda+\mu}\omega_\mu(c,b).$$

The set

$$X_\omega=X_\omega(a_0)=\{(a\in X:\exists\lambda=\lambda(a)>0) \text{ such that } \omega_\lambda(a,a_0)<\infty\}$$

is called a *modular metric space* (around a_0). It is clear that $X^*_\omega\subset X_\omega$, and it is known that this inclusion is proper in general. Also, if ω is a modular function on X, then the modular space X_ω can be equipped with a (nontrivial) metric d^*_ω, generated by ω and given by

$$d^*_\omega(a,b)=\inf\{\lambda>0:\omega_\lambda(a,b)\le\lambda\},\quad a,b\in X^*_\omega.$$

If ω is a convex modular function on X, then the two modular spaces coincide, $X_\omega=X_\omega$, and this common set can be endowed with a metric d_ω given by

$$d_\omega(a,b)=\inf\{\lambda>0:\omega_\lambda(a,b)\le 1\},\quad a,b\in X_\omega.$$

Even if ω is a nonconvex modular on X, then $d^*_\omega(a,a)=0$ and $d_\omega(a,b)=d_\omega(b,a)$.

Definition 2.3.([8]) let X_ω be a modular metric space, and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of X_ω. Then,

(i) $\{a_n\}_{n\in\mathbb{N}}$ in X_ω or X^*_ω, is said to be *modularly convergent* to an element $a\in X_\omega$ if $\omega_\lambda(a_n,a)\to 0$ as $n\to\infty$ for all $\lambda>0$, and any such element a will be called a modular limit of the sequence $\{a_n\}$.

(ii) $\{a_n\}_{n\in\mathbb{N}}\subset X_\omega$ is a *modular Cauchy sequence* (ω-*Cauchy*) if there exists a number $\lambda=\lambda(\{a_n\})>0$ such that $\omega_\lambda(a_n,a_m)\to 0$ as $n,m\to\infty$, i.e.,

$$\text{for all } \epsilon>0\ \exists\, n_0(\epsilon)\in\mathbb{N}\quad \text{such that for all } n,m\ge n_0(\epsilon),\omega_\lambda(a_n,a_m)\le\epsilon.$$

(iii) A modular space X_ω is called *modularly complete* if every modular Cauchy sequence $\{a_n\}$ in X_ω is modularly convergent in the following sense – if $\{a_n\} \subset X_\omega$ and there exists $\lambda = \lambda(\{a_n\}) > 0$ such that $\lim_{n,m\to\infty} \omega_\lambda(a_n, a_m) = 0$, then there exists an $a \in X_\omega$ such that $\lim_{n\to\infty} \omega_\lambda(a_n, a) = 0$.

Mongkolkeha et al. [14] introduced Banach contraction in modular metric spaces.

Definition 2.4.([14]) Let X_ω be a modular metric space. A self mapping F on X_ω is said to be a *contraction* if there exists $0 \le k < 1$ such that

$$\omega_\lambda(Fa, Fb) \le k\, \omega_\lambda(a, b)$$

for all $a, b \in X_\omega$ and $\lambda > 0$.

We use the following terminology for graphs (see,[13]).

Let (X, d) be a metric space and $\triangle$ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of vertices coincides with X and the set $E(G)$ of edges contains all loops, i.e. $(a, a) \in E(G)$ for every $a \in V(G)$. Such a digraph is called *reflexive*. Assume that G has no parallel edges, so we have $G = (V(G), E(G))$. Let G^{-1} denote the graph obtained from G by reversing the direction of edges. Thus we have $E(G^{-1}) = \{(b, a) \mid (a, b) \in E(G)\}$. Also, $\tilde{G}$ denotes the undirected graph defined by G by ignoring the direction of edges and we have,

$$E(\tilde{G}) = E(G) \bigcup E(G^{-1}).$$

If a and b are vertices in a graph G, then a (directed) path in G from a to b of length N is a sequence $\{a_i\}_{i=0}^N$ of $N+1$ vertices such that $a_0 = a$, $a_N = b$ and $(a_{n-1}, a_n) \in E(G)$ for $i = 1, 2, \cdots, N$. A graph G is connected if there is a directed path between any two vertices. G is weakly connected if $\tilde{G}$ is connected.

The operator $F : X \to X$ is called *continuous* if for all $a, b \in X$, there exist any sequences $\{a_n\}, \{b_n\} \in X$, for any $n \in \mathbb{N}$ such that,

$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad \lim_{n\to\infty} b_n = b,$$

implies that

$$\lim_{n\to\infty} F(a_n, b_n) = F(a, b).$$

Definition 2.5.([13]) Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, that is $\triangle \subseteq E(G)$. We say that a mapping $F\colon X \to X$ is a G-contraction if F preserves edges of G, i.e., for every $a, b \in X$,

$$(a, b) \in E(G) \Rightarrow (Fa, Fb) \in E(G)$$

and there exists $\alpha \in (0, 1)$ such that $a, b \in X$,

$$(a,b) \in E(G) \Rightarrow d(Fa, Fb) \leq \alpha\, d(a,b).$$

Remark 2.6. Elements are said to be *comparable* if for every $(a_1, b_1), (a_2, b_2) \in X \times X$ there exists $(c_1, c_2) \in X \times X$ such that

$$\begin{aligned}&(a_1, c_1) \in E(G),\ \ (c_2, b_1) \in E(G), \text{ and}\\ &(a_2, c_1) \in E(G),\ \ (c_2, b_2) \in E(G).\end{aligned}$$

Definition 2.7.([5]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. We say that F has the *mixed monotone property* if $F(a,b)$ is monotone nondecreasing in a and is monotone nonincreasing in b, that is, for any $a, b \in X$,

$$a_1, a_2 \in X, a_1 \preceq a_2 \Rightarrow F(a_1, b) \preceq F(a_2, b)$$

and

$$b_1, b_2 \in X, b_1 \preceq b_2 \Rightarrow F(a, b_1) \succeq F(a, b_2).$$

Definition 2.8.([5]) Let X be a nonempty set. An element $(a,b) \in X \times X$ is called a *coupled fixed point* of a mapping $F : X \times X \to X$ if $F(a,b) = a$ and $F(b,a) = b$.

Note that if G is a directed graph defined on X, we can construct another graph on $X \times X$, still denoted by G, by

$$((a,b),(u,v)) \in E(G) \Leftrightarrow (a,u) \in E(G) \text{ and } (v,b) \in E(G),$$

for any $(a,b),(u,v) \in X \times X$.

Remark 2.9. It is noted that if $(a_0, b_0) \in X_\omega$ such that $a_0 \preceq F(a_0, b_0)$ and $b_0 \succeq F(b_0, a_0)$ and let $a_1 = F(a_0, b_0)$ and $b_1 = F(b_0, a_0)$, then $a_0 \preceq a_1$ and $b_0 \succeq b_1$.
Again let $a_2 = F(a_1, b_1)$ and $b_2 = F(b_1, a_1)$, we denote

$$F^2(a_0, b_0) = F(F(a_0, b_0), F(b_0, a_0)) = F(a_1, b_1) = a_2$$

and

$$F^2(b_0, a_0) = F(F(b_0, a_0), F(a_0, b_0)) = F(b_1, a_1) = b_2$$

. Due to the mixed monotone property of F, we have

$$a_2 = F^2(a_0, b_0) = F(a_1, b_1) \succeq F(a_0, b_0) = a_1$$

and

$$b_2 = F^2(b_0, a_0) = F(b_1, a_1) \preceq F(b_0, a_0) = b_1.$$

Further for $n = 1, 2, \cdots$, we get

$$a_{n+1} = F^{n+1}(a_0, b_0) = F(F^n(a_0, b_0), F^n(b_0, a_0)) = F(a_n, b_n)$$

and

$$b_{n+1} = F^{n+1}(b_0, a_0) = (F^n(b_0, a_0), F^n(a_0, b_0)) = F(b_n, a_n).$$

3. Main Results

In this section, we assume that (X_ω, G) is a modular metric space endowed with a graph G such that $V(G) = X_\omega$, $\triangle \subseteq E(G)$ and G is transitive, i.e., $(a, b) \in E(G), (b, c) \in E(G)$ implies that $(a, c) \in E(G)$.

Definition 3.1. Let X be a nonempty set endowed with a graph G.

(i) A mapping $F : X_\omega \times X_\omega \to X_\omega$ has *G-preserving property* if

$$(a_1, a_2) \in E(G) \Rightarrow (F(a_1, b), F(a_2, b)) \in E(G),$$

for all $a_1, a_2, b \in X_\omega$ and

$$(b_1, b_2) \in E(G) \Rightarrow (F(a, b_1), F(a, b_2)) \in E(G),$$

for all $a, b_1, b_2 \in X_\omega$.

(ii) The mapping F has *G-inverting property* if

$$(a_1, a_2) \in E(G) \Rightarrow (F(a_2, b), F(a_1, b)) \in E(G),$$

for all $a_1, a_2, b \in X_\omega$ and

$$(b_1, b_2) \in E(G) \Rightarrow (F(a, b_2), F(a, b_1)) \in E(G),$$

for all $a, b_1, b_2 \in X_\omega$.

(iii) And we say that a mapping F has *mixed G-monotone property* if

$$(a_1, a_2) \in E(G) \implies (F(a_1, b), F(a_2, b)) \in E(G),$$

for all $a_1, a_2, b \in X_\omega$, and

$$(b_1, b_2) \in E(G) \implies (F(a, b_2), F(a, b_1)) \in E(G),$$

for all $a, b_1, b_2 \in X_\omega$.

Example 3.2. Let $X_\omega = [0, \infty)$ and G be a graph such that $V(G) = X_\omega$ and $E(G) = \{(a, b) \in X_\omega \times X_\omega \colon a \leq b\}$. Define mappings $F_1, F_2, F_3 \colon X_\omega \times X_\omega \to X_\omega$ by:

$$F_1(a, b) = a + b \text{ for all } a, b \in X_\omega$$
$$F_2(a, b) = e^{-a} + e^{-b} \text{ for all } a, b \in X_\omega$$
$$F_3(a, b) = a + \frac{1}{1 + b} \text{ for all } a, b \in X_\omega.$$

Then, F_1 has G-preserving property, F_2 has G-inverting property, while F_3 has mixed G-monotone property. Note that each of these three mappings has exactly one property, therefore we can say that these three properties are independent of each other.

Theorem 3.3. *Let (X_ω, G) be a complete modular metric space with a graph G. Suppose that $F : X_\omega \times X_\omega \to X_\omega$ is a continuous mapping which has mixed G-monotone property in X_ω and k, l be nonnegative constants such that $k + l < 1$. Suppose that the following condition is satisfied for all $a, b, p, q \in X_\omega$ and $\lambda > 0$:*

$$\omega_\lambda(F(a,b), F(p,q)) \leq k\ \omega_\lambda(a,p) + l\ \omega_\lambda(b,q), \tag{3.1}$$

where $((a,p),(q,b)) \in E(G)$. If there exist $a_0, b_0 \in X_\omega$ such that

$$((a_0,b_0),(F(a_0,b_0),F(b_0,a_0))) \in E(G),$$

then F has a coupled fixed point.

Proof. Let $a_0, b_0 \in X_\omega$ be such that $((a_0,b_0),(F(a_0,b_0),F(b_0,a_0))) \in E(G)$, i.e.,

$$(a_0, F(a_0,b_0)) \in E(G) \text{ and } (F(b_0,a_0), b_0) \in E(G).$$

We take $a_1, b_1 \in X_\omega$ with $a_1 = F(a_0,b_0)$ and $b_1 = F(b_0,a_0)$, then $(a_0,a_1) \in E(G)$, $(b_1,b_0) \in E(G)$. Let $a_2, b_2 \in X_\omega$, where $a_2 = F(a_1,b_1)$ and $b_2 = F(b_1,a_1)$. Then, by mixed monotone property of F we have

$$\begin{aligned}
&(F(a_0,b_0),F(a_1,b_0)) \in E(G), (F(a_1,b_0),F(a_1,b_1)) \in E(G) \\
&\implies (F(a_0,b_0),F(a_1,b_1)) \in E(G) \\
&\implies (a_1,a_2) \in E(G).
\end{aligned}$$

Similarly, we can obtain that $(b_2,b_1) \in E(G)$. By induction, we construct two sequences $\{a_n\}$ and $\{b_n\}$ in X_ω such that

$$a_{n+1} = F(a_n,b_n) \text{ and } b_{n+1} = F(b_n,a_n),\ n = 0,1,2,\ldots,$$

$$(a_{i-1},a_i),(b_i,b_{i-1}) \in E(G),\ i = 1,2,\ldots.$$

Then by (3.1), we get

$$\begin{aligned}
\omega_\lambda(a_n,a_{n+1}) &= \omega_\lambda(F(a_{n-1},b_{n-1}),F(a_n,b_n)) \\
&\leq k\omega_\lambda(a_{n-1},a_n) + l\omega_\lambda(b_{n-1},b_n),\ n \in \mathbb{N}.
\end{aligned} \tag{3.2}$$

Similarly

$$\begin{aligned}
\omega_\lambda(b_n,b_{n+1}) &= \omega_\lambda(F(b_{n-1},a_{n-1}),F(b_n,a_n)), \\
&\leq k\omega_\lambda(b_{n-1},b_n) + l\omega_\lambda(a_{n-1},a_n), \quad \text{for all} \quad n \in \mathbb{N}.
\end{aligned} \tag{3.3}$$

Thus, for any $n \in \mathbb{N}$ from (3.2) and (3.3), we get

$$\begin{aligned} \omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1}) &\leq (k+l)\omega_\lambda(a_{n-1}, a_n) + (k+l)\omega_\lambda(b_{n-1}, b_n) \\ &= (k+l)\left[\omega_\lambda(a_{n-1}, a_n) + \omega_\lambda(b_{n-1}, b_n)\right]. \end{aligned} \tag{3.4}$$

By successive applications of the above inequality we obtain

$$\begin{aligned} 0 &\leq \omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1}) \leq (k+l)\left[\omega_\lambda(a_{n-1}, a_n) + \omega_\lambda(b_{n-1}, b_n)\right] \\ &\leq \cdots \leq (k+l)^n \left[\omega_\lambda(a_0, a_1) + \omega_\lambda(b_0, b_1)\right]. \end{aligned} \tag{3.5}$$

It follows from (3.5) that

$$\lim_{n\to\infty} \left[\omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1})\right] = 0.$$

Therefore, if $\varepsilon > 0$ is given then there exists $n_0 \in \mathbb{N}$ such that

$$\omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1}) < \varepsilon \quad \text{for all} \quad n > n_0,\ \lambda > 0.$$

Without loss of generality, suppose $m, n \in \mathbb{N}$ and $n < m$, there exist $n_{\frac{\lambda}{m-n}} \in \mathbb{N}$ satisfying

$$\omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1}) < \frac{\varepsilon}{m-n} \text{ for all } n \geq n_{\frac{\lambda}{m-n}}.$$

We get

$$\omega_\lambda(a_n, a_m) \leq \omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(a_{n+1}, a_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(a_{m-1}, a_m) \tag{3.6}$$

and

$$\omega_\lambda(b_n, b_m) \leq \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_{n+1}, b_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(b_{m-1}, b_m),\ n < m. \tag{3.7}$$

Thus from inequalities (3.6) and (3.7), we get

$$\begin{aligned} \omega_\lambda(a_n, a_m) + \omega_\lambda(b_n, b_m) &\leq \omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1}) \\ &\quad + \cdots + \omega_{\frac{\lambda}{m-n}}(a_{m-1}, a_m) + \omega_{\frac{\lambda}{m-n}}(b_{m-1}, b_m) \\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \cdots + \frac{\varepsilon}{m-n} \\ &= \varepsilon \end{aligned}$$

for all $n > n_{\frac{\lambda}{m-n}}$. The above inequality shows that

$$\omega_\lambda(a_n, a_m) < \varepsilon,\ \omega_\lambda(b_n, b_m) < \varepsilon$$

for all $n > n_{\frac{\lambda}{m-n}}$.

This shows that $\{a_n\}$ and $\{b_n\}$ are modular Cauchy sequences in X_ω. Using completeness of X_ω, for $a, b \in X_\omega$ we have

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

Since F is continuous, we obtain:

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} F(a_n, b_n) = F(a, b)$$

and

$$b = \lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} F(b_n, a_n) = F(b, a).$$

Thus, $(a, b) \in X_\omega \times X_\omega$ is a coupled fixed point of F. □

The assumption of continuity of F may be relaxed by applying the condition of ω-regularity of graph G which is inspired by Neito and Rodŕiguez-López [16].

Definition 3.4. Let (X_ω, G) be a complete modular metric space with a graph G. Then, the graph G is called *ω-regular* if for every sequence $\{a_n\}$ in X_ω such that $\lim_{n\to\infty} a_n = a$ we have:

(i) if $(a_n, a_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ implies $(a_n, a) \in E(G)$;

(ii) if $(a_{n+1}, a_n) \in E(G)$ for all $n \in \mathbb{N}$ implies $(a, a_n) \in E(G)$.

The following theorem uses the ω-regularity of graph G instead the continuity of F.

Theorem 3.5. *Let (X_ω, G) be a complete modular metric space with a graph G. Suppose that $F : X_\omega \times X_\omega \to X_\omega$ is a mapping which has mixed monotone property in X_ω and k, l be nonnegative constants such that $k + l < 1$. Suppose that the following condition is satisfied for all $a, b, p, q \in X_\omega$ and $\lambda > 0$:*

$$\omega_\lambda(F(a, b), F(p, q)) \le k\, \omega_\lambda(a, p) + l\, \omega_\lambda(b, q), \tag{3.8}$$

where $((a, p), (q, b)) \in E(G)$. If there exist $a_0, b_0 \in X_\omega$ such that $((a_0, b_0), (F(a_0, b_0), F(b_0, a_0)) \in E(G)$, and the graph G is ω-regular, then F has a coupled fixed point.

Proof. The construction of the sequence $\{a_n\}$ and $\{b_n\}$, which converge to a and b respectively, following the similar process used in Theorem 3.3. Thus, we have two sequences $\{a_n\}$ and $\{b_n\}$ such that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and

$$a_{n+1} = F(a_n, b_n), b_{n+1} = F(b_n, a_n), (a_n, a_{n+1}), (b_{n+1}, b_n) \in E(G), \quad \text{for all} \quad n \ge 0.$$

Since the graph G is ω-regular, therefore $(a_n, a) \in E(G)$ and $(b, b_n) \in E(G)$ for all $n \ge 0$. Let $\epsilon > 0$ be given, then there exist $n_0 \in \mathbb{N}$ with $\omega_{\frac{\lambda}{2}}(a_n, a) < \frac{\epsilon}{2}$ and

$\omega_{\frac{\lambda}{2}}(b, b_n) < \frac{\epsilon}{2}$, for all $n \geq n_0$, $\lambda > 0$. So, from (iii) and using (3.1) we get

$$
\begin{aligned}
\omega_\lambda(a, F(a,b)) &\leq \omega_{\frac{\lambda}{2}}(a, a_{n+1}) + \omega_{\frac{\lambda}{2}}(a_{n+1}, F(a,b)) \\
&= \omega_{\frac{\lambda}{2}}(a_{n+1}, a) + \omega_{\frac{\lambda}{2}}(F(a_n, b_n), F(a,b)) \\
&\leq \omega_{\frac{\lambda}{2}}(a_{n+1}, a) + k\omega_{\frac{\lambda}{2}}(a_n, a) + l\omega_{\frac{\lambda}{2}}(b_n, b) \\
&< \frac{\epsilon}{2} + k\frac{\epsilon}{2} + l\frac{\epsilon}{2} \\
&= \frac{\epsilon}{2} + (k+l)\frac{\epsilon}{2} \\
&< \epsilon \quad as\ k + l < 1,
\end{aligned} \tag{3.9}
$$

for all $\lambda > 0$. Hence, $\omega_\lambda(a, F(a,b)) = 0$. So, $F(a,b) = a$. Similarly, we get $F(b,a) = b$. Thus, $(a,b) \in X_\omega \times X_\omega$ is a coupled fixed point of F. □

Remark 3.6. Since the contractivity assumption is made only on comparable elements in $X_\omega \times X_\omega$, Theorems 3.3 and 3.5, don't guarantee the uniqueness of the coupled fixed point. However, the uniqueness of the coupled fixed point can be establish with the following condition:

For the uniqueness of coupled fixed point we endow product space $X_\omega \times X_\omega$ with the graph G such that for every $(a,b), (a^*, b^*) \in X_\omega \times X_\omega$ there exists $(u,v) \in X_\omega \times X_\omega$ such that

$$(a,u) \in E(G), (v,b) \in E(G) \text{ and } (a^*, u) \in E(G), (v, b^*) \in E(G). \tag{3.10}$$

Here, we discuss the uniqueness of the coupled fixed point.

Theorem 3.7. *Suppose that all the conditions of Theorem 3.3 (respectively Theorem 3.5) are satisfied. In addition, suppose that the condition (3.10) is satisfied, then F has a unique coupled fixed point.*

Proof. It follows from Theorem 3.3 (respectively Theorem 3.5). Suppose that (a,b) and (a^*, b^*) are two distinct coupled fixed of F. We consider two cases:
Case I: If $((a, a^*), (b^*, b)) \in E(G)$. Then, we have from (3.1)

$$\omega_\lambda(F(a,b), F(a^*, b^*)) \leq k\omega_\lambda(a, a^*) + l\omega_\lambda(b, b^*)$$

and

$$\omega_\lambda(F(b^*, a^*), F(b,a)) \leq k\omega_\lambda(b^*, b) + l\omega_\lambda(a^*, a) \text{ with } k + l < 1.$$

Since (a,b) and (a^*, b^*) are coupled fixed points of F, we get

$$\omega_\lambda(a, a^*) = \omega_\lambda(F(a,b), F(a^*, b^*)) \leq k\omega_\lambda(a, a^*) + l\omega_\lambda(b, b^*)$$

and

$$\omega_\lambda(b, b^*) = \omega_\lambda(F(b,a), F(b^*, a^*)) \leq k\omega_\lambda(b, b^*) + l\omega_\lambda(a, a^*).$$

Therefore, we have

$$\begin{aligned}\omega_\lambda(a,a^*)+\omega_\lambda(b,b^*) &\leq (k+l)\omega_\lambda(a,a^*)+(k+l)\omega_\lambda(b^*,b)\\ &=(k+l)(\omega_\lambda(a,a^*)+\omega_\lambda(b,b^*))\\ &<\omega_\lambda(a,a^*)+\omega_\lambda(b,b^*). \end{aligned} \tag{3.11}$$

This is a contradiction as $k+l<1$ and yields the conclusion that the coupled fixed point is unique.

Case II: If (a,b) is not comparable to (a^*,b^*) such that $(a,b),(a^*,b^*)\notin E(G)$, then there exists $(u,v)\in X_\omega\times X_\omega$ such that

$(a,u)\in E(G)$, $(v,b)\in E(G)$ and $(a^*,u)\in E(G)$, $(v,b^*)\in E(G)$. From the monotonic property of F it follows that $F^n(u,v)$ is comparable to $F^n(a,b)=a, F^n(b,a)=b$ and $F^n(a^*,b^*)=a^*, F^n(b^*,a^*)=b^*$. Then, we have

$$\begin{aligned}\omega_\lambda((a,b),(a^*,b^*)) &= \omega_\lambda(F^n(a,b),F^n(b,a),F^n(a^*,b^*),F^n(b^*,a^*))\\ &\leq \omega_{\frac{\lambda}{2}}(F^n(a,b),(F^n(b,a),F^n(u,v),F^n(v,u))\\ &\quad+\ \omega_{\frac{\lambda}{2}}(F^n(u,v),F^n(v,u),F^n(a^*,b^*),F^n(b^*,a^*))\\ &\leq (k+l)^n(\omega_\lambda(a,u)+\omega_\lambda(b,v)+\omega_\lambda(u,a^*)+\omega_\lambda(v,b^*)). \end{aligned} \tag{3.12}$$

Taking $n\to\infty$, it follows that $\omega_\lambda((a,b),(a^*,b^*))\leq 0\Rightarrow (a,b)=(a^*,b^*)$. It follows that coupled fixed point is unique. Therefore for given $(a_0,b_0)\in X_\omega\times X_\omega$ such that $((a_0,b_0),(F(a_0,b_0),F(b_0,a_0)))\in E(G)$, there exist a unique coupled fixed point (a,b) of F. □

Corollary 3.8. *Let (X_ω,G) be a complete modular metric space with a graph, A continuous mapping $F:X_\omega\times X_\omega\to X_\omega$ has mixed monotone property in X_ω and $k\in[0,1)$. Suppose that we have the following condition for all $a,b,p,q\in X_\omega$ and $\lambda>0$.*

$$\omega_\lambda(F(a,b),F(p,q))\leq \frac{k}{2}(\omega_\lambda(a,p)+\omega_\lambda(b,q)). \tag{3.13}$$

Here $(p,a),(b,q)\in E(G)$. If there exist $a_0,b_0\in X_\omega$ such that $((a_0,b_0),(F(a_0,b_0),F(b_0,a_0)))\in E(G)$. In addition suppose that the condition (3.10) is satisfied, then F has a unique coupled fixed point.

Corollary 3.9. *let (X_ω,G) be a complete modular metric space with graph G. Suppose that X_ω satisfies the following conditions:*

(i) *if a non-decreasing sequence $\{a_n\}\to a$ then $(a_n,a)\in E(G)$ for all n,*

(ii) *if a non-increasing sequence $\{b_n\}\to b$ then $(b,b_n)\in E(G)$ for all n,*

let a mapping $F:X_\omega\times X_\omega\to X_\omega$ has mixed monotone property in X_ω and $k\in[0,1)$. Suppose that we have the following condition for all $a,b,p,q\in X_\omega$ and $\lambda>0$

$$\omega_\lambda(F(a,b),F(p,q))\leq \frac{k}{2}(\omega_\lambda(a,p)+\omega_\lambda(b,q)) \tag{3.14}$$

where $(a,p),(q,b)\in E(G)$. if there exist $a_0,b_0\in X_\omega$ with

$$((a_0, b_0), (F(a_0, b_0), F(b_0, a_0))) \in E(G),$$

In addition suppose that the condition (3.10) is satisfied, then F has a unique coupled fixed point.

Remark 3.10. Let for a complete modular metric space $X_\omega = \mathbb{R}$, we define a metric modular function $\omega : (0, \infty) \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $\omega_\lambda(a, b) = \frac{|a-b|}{\lambda}$ for all $a, b \in \mathbb{R}$ and $\lambda > 0$. Define a mapping $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $F(a, b) = \frac{a-2b}{4}, (a, b) \in X_\omega \times X_\omega$. Then F is continuous. Let G be the reflexive digraph defined on X_ω with $((a, p), (q, b)) \in E(G)$. Then we easily see that F has the mixed G-monotone property and satisfies condition (3.1) but does not satisfy the condition (3.13). Assume there exists $k \in [0, 1)$, such that (3.1) holds. Then, we must have

$$|\frac{a-2b}{4} - \frac{p-2q}{4}| \leq \frac{k}{2}[|a-p| + |b-q|], a \geq p, b \leq q, \tag{3.15}$$

by which, for $a = p$, we get

$$| b - q | \leq k | b - q |, b \leq q, \tag{3.16}$$

which is a contradiction, since $k \in [0, 1)$. Hence F does not satisfy the contractive condition (3.13).

Now, we prove that (3.1) holds. Indeed, for $a \geq p$ and $b \leq q$, we have

$$| \frac{a-2b}{4} - \frac{p-2q}{4} | \leq \frac{1}{4} | a - p | + \frac{1}{2} | b - q |,$$

$$| \frac{b-2a}{4} - \frac{q-2p}{4} | \leq \frac{1}{4} | b - q | + \frac{1}{2} | a - p |, \tag{3.17}$$

that is, the inequality (3.1) holds for $k = \frac{1}{4}$ and $l = \frac{1}{2}$, so by Theorem 3.3 we obtain that F has a coupled fixed point $(0, 0)$ but none of the Corollary 3.8 and 3.9 can be applied to F in this example.

References

[1] M. R. Alfuraidan, *Remarks on monotone multivalued mappings on a metric space with a graph*, J. Inequal. Appl., (2015), 2015:202, 7 pp.

[2] M. R. Alfuraidan and M. A. Khamsi, *Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph*, Fixed Point Theory Appl., (2015), 2015:44, 10 pp.

[3] M. R. Alfuraidan and M. A. Khamsi, *Coupled fixed points of monotone mappings in a metric space with a graph*, arXiv:1801.07675v1[math.FA].

[4] A. H. Ansari, M. Sangurlu and D. Turkoglu, *Coupled fixed point theorems for mixed G-monotone mappings in partially ordered metric spaces via new functions*, Gazi Univ. J. Sci., **29(1)**(2016), 149–158.

[5] T. G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65**(2006), 1379–1393.

[6] V. V. Chistyakov, *Modular metric spaces generated by F-modulars*, Folia Math., **15**(2008), 3–24.

[7] V. V. Chistyakov, *Modular metric spaces I, basic concepts*, Nonlinear Anal., **72**(2010), 1–14.

[8] V. V. Chistyakov, *A fixed point theorem for contractions in modular metric spaces*, arXiv:1112.5561 [math.FA].

[9] V. V. Chistyakov, *Fixed points of modular contractive mappings*, Dokl. Math., **86(1)**(2012), 515–518.

[10] D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal., **11**(1987), 623–632.

[11] J. Harjani and K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Analysis: Theory, Methods and Applications, **71**(2009), 3403–3410.

[12] L. G. Gwóźdź and J. Jachymski, *IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem*, J. Math. Anal. Appl., **356**(2009), 453–463.

[13] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136(4)**(2008), 1359–1373.

[14] C. Mongkolkeha, W. Sintunavarat and P. Kumam, *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl., (2011), 2011:93, 9 pp.

[15] A. Mutlu, K. Özkan and U. Gürdal, *Coupled fixed point theorem in partially ordered modular metric spaces and its an application*, J. Comput. Anal. Appl., **25(2)**(2018), 207–216.

[16] J. Neito, R. L. Poiuso and R. Rodéiguez-López, *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc., **135**(2007), 2505–2517.

[17] A. Ran and M. Reurings, *A fixed point theorem in partially ordered sets and some applications to metrix equations*, Proc. Amer. Math. Soc., **132**(2004), 1435–1443.

[18] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., **47(4)**(2001), 2683–2693.