KYUNGPOOK Math. J. 61(2021), 309-322
https://doi.org/10.5666/KMJ.2021.61.2.309
pISSN 1225-6951 eISSN 0454-8124

Conservative Upwind Correction Method for Scalar Linear Hyperbolic Equations

Sang Dong Kim
Gyeongbuk Provincial College, Yecheon 36830, Korea, and
Department of Mathematics, University of Wisconsin-Whitewater, Whitewater WI, USA
e-mail : skim@knu.ac.kr

Yong Hun Lee*
Department of Mathematics and Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju 54896, Korea
e-mail : lyh229@jbnu.ac.kr

Byeong Chun Shin
Department of Mathematics, Chonnam National University, Gwangju 61186, Korea
e-mail : bcshin@jnu.ac.kr

Abstract. A conservative scheme for solving scalar hyperbolic equations is presented using a quadrature rule and an ODE solver. This numerical scheme consists of an upwind part, plus a correction part which is derived by introducing a new variable for the given hyperbolic equation. Furthermore, the stability and accuracy of the derived algorithm is shown with numerous computations.

1. Introduction

It is well known that lower order numerical methods, such as various monotone methods, behave well near discontinuities, while high order numerical methods, such as the Lax-Wendroff method, work well in smooth regions (see [14, 15] for example). Monotone methods are first-order accurate due to Godunov's theorem. They do not produce non-physical phenomena such as smearing of the solution or spurious oscillations. On the other hand, high-order methods yield non-physical

* Corresponding Author.
Received April 19, 2018; revised June 1, 2021; accepted June 1, 2021.
2010 Mathematics Subject Classification: 65M55, 65N30, 49J20, 49K20.
Key words and phrases: conservative method, hyperbolic scalar equation, ODE solver, quadrature rule, upwind method.

phenomena when shocks are presented.

The total variation diminishing (TVD) technique has long been used as one of the various techniques for avoiding the spurious or non-physical oscillations exhibited by high-order schemes. It requires one to modify a high-order method such as the Lax-Wendroff method (see [2, 13, 15, 19] and etc., for example) using flux- or slope- limiter techniques.

The main goal of this paper is to introduce a correction technique to the upwind method, another known method for eliminating undesired non-physical phenomena, to solve a simple scalar linear hyperbolic equation without the help of any limiter techniques. The first step towards realising this goal is to split the hyperbolic equation $u_t + au_x = 0$ by introducing a new flux variable $v = au_x$ (see [10]). As a result, we have a system of ordinary differential equations consisting of $v = au_x$ and $u_t + v = 0$. Our conservative method then consists of two-steps: the first-step is to set up a correction scheme for $v = au_x$ and the second-step is a scheme for $u_t+v = 0$. Once we deal with the two equations separately and then combine them, we have a new conservative scheme which is composed of the upwind scheme plus a correction term.

The *correction term* behaves like a high-resolution correction (see p.163 in [15]) while the upwind method of the form $U_j^m - \nu^2(U_j^m - U_{j-1}^m)$ obtains a low-resolution behavior. As a result, we obtain a stable scheme which is second-order accurate for u. It has been also been frequently observed [4, 11, 12, 13, 14, 15, 17, 18, 19] that a high resolution method using a flux-limiter has a restricted TVD region, and so the limiter functions must be chosen within this TVD region. Our proposed method does not use any limiter functions, so this is not a concern. Moreover it is not only second-order accurate, but also has the almost l_1 contracting property.

Our other goal is to computationally compare the proposed algorithm with the well-known upwind, Lax-Wendroff and flux-limiter methods, and to show the second-order local truncation error of the proposed algorithm. The order of convergence of the upwind correction scheme in comparible to that of min-mode type schemes reported in [8] and [16] and that of the fully-discrete high-resolution schemes with van Leer's flux limiter reported in [7], for example.

This paper is layed out as follows. In Section 2, we will present how the algorithm can be derived. In Section 3, the almost l_1 contracting property and truncation error estimates are shown. In Section 4, numerous examples are presented by comparing the present method with other well-known methods using flux-limiters. In the concluding section, we mention and discuss a possibility for expanding the present method to a nonlinear scalar hyperbolic equations.

2. Error Correction Upwind Method

We consider the following simple linear hyperbolic equation

$$u_t + au_x = 0 \tag{2.1}$$

with initial data

$$u(x,0) = u_0(x). \tag{2.2}$$

By introducing a new variable $v = au_x$, (2.1) can be written as

$$u_t + v = 0, \quad v = au_x, \quad u(x,0) = u_0(x), \quad v(x,0) = au_0'(x). \tag{2.3}$$

Lemma 2.1. *Assume that the solution of (2.1) is sufficiently smooth. Then the system of ordinary differential equations (2.3) is equivalent to*

$$u_t + v = 0, \quad v_t + av_x = 0, \quad u(x,0) = u_0(x), \quad v(x,0) = au_0'(x). \tag{2.4}$$

Proof. From (2.3), it follows that

$$v_t + av_x = a(u_t + au_x)_x = 0,$$

which implies (2.4). On the other hand, assuming (2.4) we have

$$0 = v_t + av_x = (v - au_x)_t = 0.$$

Hence, we have $v - au_x = c$. Then, the initial condition shows $v = au_x$. This implies (2.3). □

Now, let us discuss the discretization for which we assume that the domain $(-\infty, \infty) \times [0, T]$ has the discrete mesh points (x_j, t_m) given by

$$x_j = jh, \quad j = \cdots, -1, 0, 1, 2, \cdots, \quad \text{and} \quad t_m = m\tau, \quad m = 0, 1, 2, \cdots$$

where h and τ denote a mesh width and a time step, respectively.

We will present a conservative method for $u_t = -v$ by applying 3^{rd} order Runge-Kutta method which leads us

$$u(x,t_{m+1}) = u(x,t_m) - \frac{\tau}{6}\Big(v(x,t_m) + 4v(x,t_{m+1/2}) + v(x,t_{m+1})\Big) + O(\tau^3) \tag{2.5}$$

where, for an approximation of $v(x, t_{m+1/2})$, we consider $v(x, t_{m+1/2})$ as a convex combination of $v(x, t_m)$ and $v(x, t_{m+1})$ such that

$$v(x,t_{m+1/2}) := \gamma v(x,t_m) + (1-\gamma)v(x,t_{m+1}), \quad \gamma > 0. \tag{2.6}$$

Later, a parameter γ in (2.6) will be taken to adjust the convergence of the proposed algorithm. With this, (2.5) can be written as

$$u(x,t_{m+1}) \sim u(x,t_m) - \frac{\tau}{6}\Big((1+4\gamma)v(x,t_m) + (5-4\gamma)v(x,t_{m+1})\Big). \tag{2.7}$$

For the approximation of the values of the solution $u(x,t)$ at (x_j, t_m), we use the notation U_j^m, but for the conservation laws we approximate the cell average of $u(x, t_m)$, rather than $u(x_j, t_m)$, i.e.,

$$U_j^m \approx \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_m)dx. \tag{2.8}$$

Also we use the notation V_j^m to approximate the cell average of $v(x, t_m)$, i.e.,

$$V_j^m \approx \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} v(x,t_m)dx. \tag{2.9}$$

Taking the cell-average of the both sides of (2.7), we have

$$U_j^{m+1} = U_j^m - \frac{\tau}{6}\Big[(1+4\gamma)V_j^m + (5-4\gamma)V_j^{m+1}\Big]. \tag{2.10}$$

For a conservative scheme for $v = au_x$, we employ the Simpson's numerical quadrature

$$\int_{t_m}^{t_{m+1}} f(t)dt = \frac{\tau}{6}\Big[f(t_m) + 4f(t_m + \frac{\tau}{2}) + f(t_{m+1})\Big] + O(\tau^5), \tag{2.11}$$

to approximate the integral of $v = u_x$ on time interval $[t_m, t_{m+1}]$

$$\int_{t_m}^{t_{m+1}} v(x,t)\,dt \;=\; a\int_{t_m}^{t_{m+1}} u_x(x,t)\,dt.$$

As a result, it follows that

$$v(x,t_{m+1}) + 4v(x,t_m + \frac{\tau}{2}) + v(x,t_m) = \frac{6a}{\tau}\int_{t_m}^{t_{m+1}} u_x(x,t)dt + O(\tau^4), \tag{2.12}$$

in which we approximate $v(x, t_m + \frac{\tau}{2})$ with the linear combination of $v(x, t_m)$ and $v(x, t_{m+1})$ given by

$$v\Big(x, t_m + \frac{\tau}{2}\Big) := \delta v(x,t_m) + (1-\delta)v(x,t_{m+1}). \tag{2.13}$$

Here, the positive constants δ, which differ from γ in (2.6), will be chosen later. Using (2.13), we can rewrite (2.12) as

$$v(x,t_{m+1}) \sim -\frac{1+4\delta}{5-4\delta}v(x,t_m) + \frac{1}{5-4\delta}\frac{6a}{\tau}\int_{t_m}^{t_{m+1}} u_x(x,t)dt + \cdots \tag{2.14}$$

Taking the cell-average for (2.14) on $I_j := [x_{j-1/2}, x_{j+1/2}]$ (see [14, 15], for example)

$$\begin{aligned}\frac{1}{h}\int_{I_j} v(x,t_{m+1})dx &\sim -\frac{1+4\delta}{5-4\delta}\frac{1}{h}\int_{I_j} v(x,t_m)dx \\ &\quad + \frac{1}{5-4\delta}\frac{6a}{h}\frac{1}{\tau}\int_{t_m}^{t_{m+1}}\Big[u(x_{j+1/2},t) - u(x_{j-1/2},t)\Big]dt + \cdots .\end{aligned} \tag{2.15}$$

leads to the conservative numerical method for $v = u_x$. Let us set

$$\mathcal{F}(U_j^m, U_{j+1}^m) \sim \frac{1}{\tau}\int_{t_m}^{t_{m+1}} u(x_{j+1/2}, t)dt. \tag{2.16}$$

Then, (2.15) becomes

$$V_j^{m+1} = -\frac{1+4\delta}{5-4\delta}V_j^m + \frac{6a}{h}\frac{1}{5-4\delta}\Big[\mathcal{F}(U_j^m, U_{j+1}^m) - \mathcal{F}(U_{j-1}^m, U_j^m)\Big]. \tag{2.17}$$

Hence, using (2.10) for $u_t = -v$ and (2.17), the new conservative method with

$$\mathcal{F}(U_j^m, U_{j+1}^m) = U_j^m \tag{2.18}$$

can be written as

$$V_j^{m+1} = -\frac{1+4\delta}{5-4\delta}V_j^m + \frac{6a}{h}\frac{1}{5-4\delta}\Big[U_j^m - U_{j-1}^m\Big] \tag{2.19}$$

$$U_j^{m+1} = U_j^m - \frac{\tau}{6}\Big[(1+4\gamma)V_j^m + (5-4\gamma)V_j^{m+1}\Big] \tag{2.20}$$

where the parameters δ and γ will be chosen with proper accuracy in Section 3. Actually, the two constants

$$\delta = \frac{1}{4}\Big(5 - \frac{3h}{a\tau}\Big), \quad \text{and} \quad \gamma = \frac{1}{2}, \tag{2.21}$$

will be chosen so that (2.20) and the correction term V_j^m from (2.19) lead to the following algorithm.

$$\begin{aligned} U_j^{m+1} &= U_j^m \; - \nu^2\,(U_j^m - U_{j-1}^m) \; - \nu\left(\frac{h}{a} - \tau\right)V_j^m, \quad \nu = \frac{a\tau}{h} \\ &= \text{ upwind method } + \text{ correction} \end{aligned} \tag{2.22}$$

$$V_j^{m+1} = \Big(1 - 2\nu\Big)V_j^m + \frac{2\nu^2}{\tau}\Big[U_j^m - U_{j-1}^m\Big]. \tag{2.23}$$

It will be shown in Section 3 that the local truncation error of (2.22) is of second-order and that of (2.23) is of first-order. Due to Lemma 2.1, it may be suggested to allow the correction term V_j^m in (2.22) updated by (2.23) as the exact solution to $v_t + av_x = 0$. Since the exact solution for the equation $v_t + av_x = 0$ is $v(x,t) = v_0(x - at)$, we get the revised algorithm as following:

Algorithm 2.1. *For the linear problem* $u_t + au_x = 0$, *with* $U_j^0 = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx$ *and* $V_j^0 = \frac{a}{h}\Big[u_0(x_{j+1/2}) - u_0(x_{j-1/2})\Big]$ *as initial data, iterates for* $m = 0, 1, \cdots$, *and* $j = 0, 1, \cdots$

$$\begin{aligned} U_j^{m+1} &= U_j^m \; - \nu^2\,(U_j^m - U_{j-1}^m) \; - \nu\left(\frac{h}{a} - \tau\right)V_k^0 \\ &= \textit{ upwind method } + \textit{ correction}, \end{aligned} \tag{2.24}$$

where the integer index k is chosen to be

$$x_{k+\frac{1}{2}} = x_{j+\frac{1}{2}} - at_m.$$

3. Local Truncation Error and Stability

Let us denote (2.19) by $V^{m+1} = \mathcal{N}(U^m, V^m)$ and (2.20) by $U^{m+1} = \mathcal{N}(U^m, V^m, V^{m+1})$ for the accuracy of convergence where V^m and U^m are replaced with the exact solution $u(x,t)$ and $v(x,t)$ at (x_j, t_m) respectively in the method (2.19) and (2.20). The local truncation errors $E(v^m) := \frac{1}{\tau}\Big[\mathcal{N}(u^m,\, v^m) - v^{m+1}\Big]$ and $E(u^m) := \frac{1}{\tau}\Big[\mathcal{N}(u^m,\, v^m, v^{m+1}) - u^{m+1}\Big]$ for (2.19) and (2.20), respectively, can be shown using the Taylor expansion of u and v at the point (x_j, t_m).

Assuming that its exact solution is smooth enough, the following relations are hold:

$$u_{tt} = -v_t, \quad u_{ttt} = -v_{tt}, \quad v_t = au_{xt} = -av_x = -a^2u_{xx}, \quad a^2u_{xx} = u_{tt},$$
$$-a^2u_{xxx} = v_{xt} = -av_{xx} = -u_{ttx}, \; v_{tt} = -u_{ttt} = -a^2u_{xxt} = au_{xtt} = a^3u_{xxx}.$$

In fact, since $v = au_x$ and $v_t = -a^2u_{xx}.\, v_{tt} = a^3u_{xxx}$, we have

$$-\tau E(v^m) = \Big(-a^2\tau + \frac{3ah}{5-4\delta}\Big)(u_{xx})_j^m + \Big(\frac{a^3\tau^2}{2} - \frac{ah^2}{5-4\delta}\Big)(u_{xxx})_j^m + \cdots \tag{3.1}$$

and, using $u_t + v = 0$, $a^2u_{xx} = u_{tt} = -v_t$ and $v_{tt} = a^3u_{xxx} = -u_{ttt}$, we have

$$-\tau E(u^m) = \tau^2\Big(\frac{1}{2} - \frac{5-4\gamma}{6}\Big)(u_{tt})_j^m + \tau^3\Big(\frac{1}{6} - \frac{5-4\gamma}{12}\Big)(u_{ttt})_j^m + \cdots . \tag{3.2}$$

Hence, if one takes the constants δ and γ in (3.1) and (3.2) as

$$\delta = \frac{1}{4}\Big(5 - \frac{3h}{a\tau}\Big), \quad \text{and} \quad \gamma = \frac{1}{2}, \tag{3.3}$$

then it follows that for a fixed $\frac{a\tau}{h} =: \nu$

$$E(v^m) = O(\tau), \quad \text{and} \quad E(u^m) = O(\tau^2) \tag{3.4}$$

With parameters from (3.3), the algorithm (2.19) and (2.20) becomes

$$V_j^{m+1} = \Big(1 - 2\nu\Big)V_j^m + \frac{2\nu^2}{\tau}\Big[U_j^m - U_{j-1}^m\Big] \tag{3.5}$$

$$U_j^{m+1} = U_j^m - \frac{\tau}{2}\Big[V_j^{m+1} + V_j^m\Big], \tag{3.6}$$

whose combination leads to (2.22) and (2.23).

Summarizing the above arguments, we have

Theorem 3.1. *Suppose that the solution is smooth enough. Then the local truncation errors for $\{U^m\}$ and $\{V^m\}$ generated by the algorithm (3.5) and (3.6) for the problem $u_t + au_x = 0$ are*

$$E(v^m) = O(\tau), \quad \textit{and} \quad E(u^m) = O(\tau^2).$$

Hence, (3.5) and (3.6) are first and second-order accuracy respectively.

Let us denote (2.24) as $U^{m+1} = \mathcal{N}(U^m, V^0)$. The local truncation error $E(u^m) := \frac{1}{\tau}\Big[\mathcal{N}(u^m, v^0) - u^{m+1}\Big]$ will be analyzed in the following theorem. For the initial V_k^0, note that the exact solution to $v_t + av_x = 0$ is $v(x,t) = v_0(x - at) = au_0'(x - at)$ and that, due to the relation of the index k and j in algorithm (2.24), we have

$$\begin{aligned} V_j^m &= \frac{1}{h}\int_{I_j} v(x, t_m)dx \\ &= \frac{a}{h}\Big(u_0(x_{j+\frac{1}{2}} - at_m) - u_0(x_{j-\frac{1}{2}} - at_m)\Big) = \frac{a}{h}\Big(u_0(x_{k+\frac{1}{2}}) - u_0(x_{k-\frac{1}{2}})\Big) \\ &= \frac{1}{h}\int_{I_k} au_0'(x)dx = \frac{1}{h}\int_{I_k} v(x,0)dx = V_k^0. \end{aligned} \tag{3.7}$$

Theorem 3.2. *Suppose that the solution is smooth enough. Then the local truncation error for $\{U^m\}$ generated by Algorithm (2.24) for the problem $u_t + au_x = 0$ is*

$$E(u^m) = \frac{a^2\tau}{6}(a\tau - h)u_{xxx} = O(\tau^2).$$

Proof. First note that the correction term V_k^0 is V_m^j in algorithm (2.24) due to (3.7). Therefore, using $u_t + v = 0$, $au_x - v = 0$, $u_{tt} = a^2u_{xx}$ and $u_{ttt} - a^3u_{xxx} = 0$, it follows that

$$\begin{aligned} -\tau E(u^m) &= (u)_j^{m+1} - (u)_j^m + \nu^2((u)_j^m - (u)_{j-1}^m) + \nu\Big(\frac{h}{a} - \tau\Big)(v)_j^m \\ &= \tau(u_t + v)_j^m + \frac{a\tau^2}{h}(au_x - v)_j^m + \frac{\tau^2}{2}(u_{tt} - a^2u_{xx})_j^m \\ &\quad + \frac{a^2\tau^2}{6}(h - a\tau)(u_{xxx})_j^m + \cdots \\ &= \frac{a^2\tau^2}{6}(h - a\tau)(u_{xxx})_j^m + \cdots . \end{aligned} \tag{3.8}$$

This completes the proof. □

We note that using (3.8) one may have a modified equation for $u_t + au_x = 0$ (see [1, 15, 20] for example). Since $v = au_x$ and $u_{tt} = a^2u_{xx}$, the modified equation turns out to be

$$u_t + au_x = \frac{a^2\tau}{6}(h - a\tau)\ u_{xxx} \tag{3.9}$$

which is compared to the modified equation $u_t + au_x = -\frac{1}{6}ah^2(1-\nu^2)u_{xxx}$ for Lax-Wendroff scheme (see [2, 3, 15] for example).

Theorem 3.3. *Let $0 \le \nu \le 1$. The sequence $\{U^m\}$ generated by Algorithm for the problem $u_t + au_x = 0$ satisfies*

$$\|U^{m+1}\|_1 \le \|U^m\|_1 + \nu\left|\frac{h}{a} - \tau\right| \|V^0\|_1$$

where $\|\ \cdot\ \|_1$ denotes the l_1 norm of a vector $W := (w_1, w_2, \cdots, w_k)$.

Proof. From the algorithm

$$U_j^{m+1} = (1-\nu^2)U_j^m + \nu^2 U_{j-1}^m - \nu\left(\frac{h}{a} - \tau\right) V_k^0,$$

it follows that

$$|U_j^{m+1}| \le (1-\nu^2)|U_j^m| + \nu^2 |U_{j-1}^m| + \nu\left|\frac{h}{a} - \tau\right| |V_k^0|.$$

Hence, one has the conclusion by taking summation.. □

Note that, according to the above theorems, the new method has almost l_1 contracting property on $\|U^m\|_1$ with second-order accuracy.

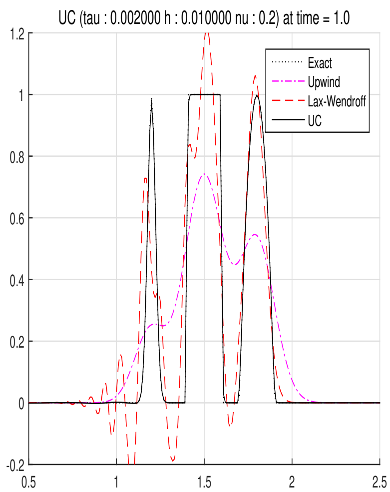
4. Numerical Example

We will take a typical linear hyperbolic equation $u_t + au_x = 0$ $(a = 1)$ with several initial data. The numerical solution from the proposed Algorithm will be compared by upwind method, Lax-Wendroff method and several flux methods.

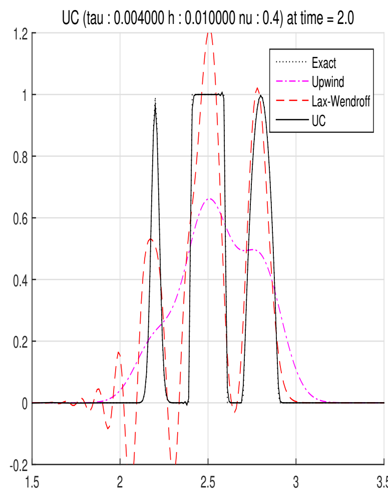
Example 4.1. The first initial condition is given by

$$u_0(x) = \begin{cases} e^{-1000(x-0.2)^2}, & \text{for} \quad 0.1 < x < 0.3 \\ 1, & \text{for} \quad 0.4 < x < 0.6 \\ 1 - 100(x-0.8)^2, & \text{for} \quad 0.7 < x < 0.9. \end{cases} \tag{4.1}$$

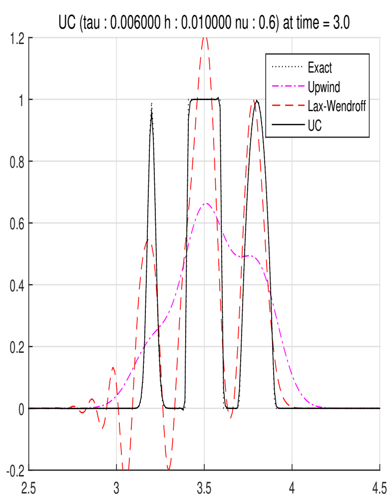
By comparing the numerical solutions of the proposed upwind correction scheme (UC) with the classical upwind method and the Lax-Wendroff method, we see the effects of the correction term V_k^0 with the weight $\nu(h - \tau)$. While the numerical behavior of the UC scheme without the correction term are same as those of upwind method with ν^2, the numerical behavior do reveal little spurious oscillations or smearing of the solution (see Figure 4.1). Next, we compare the numerical solutions



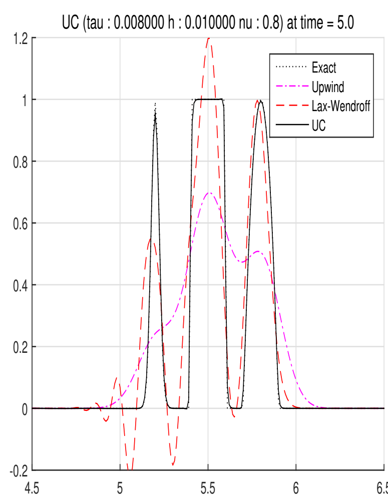
(a) at time $t = 1$, $\nu = 0.2$



(b) at time $t = 2$, $\nu = 0.4$

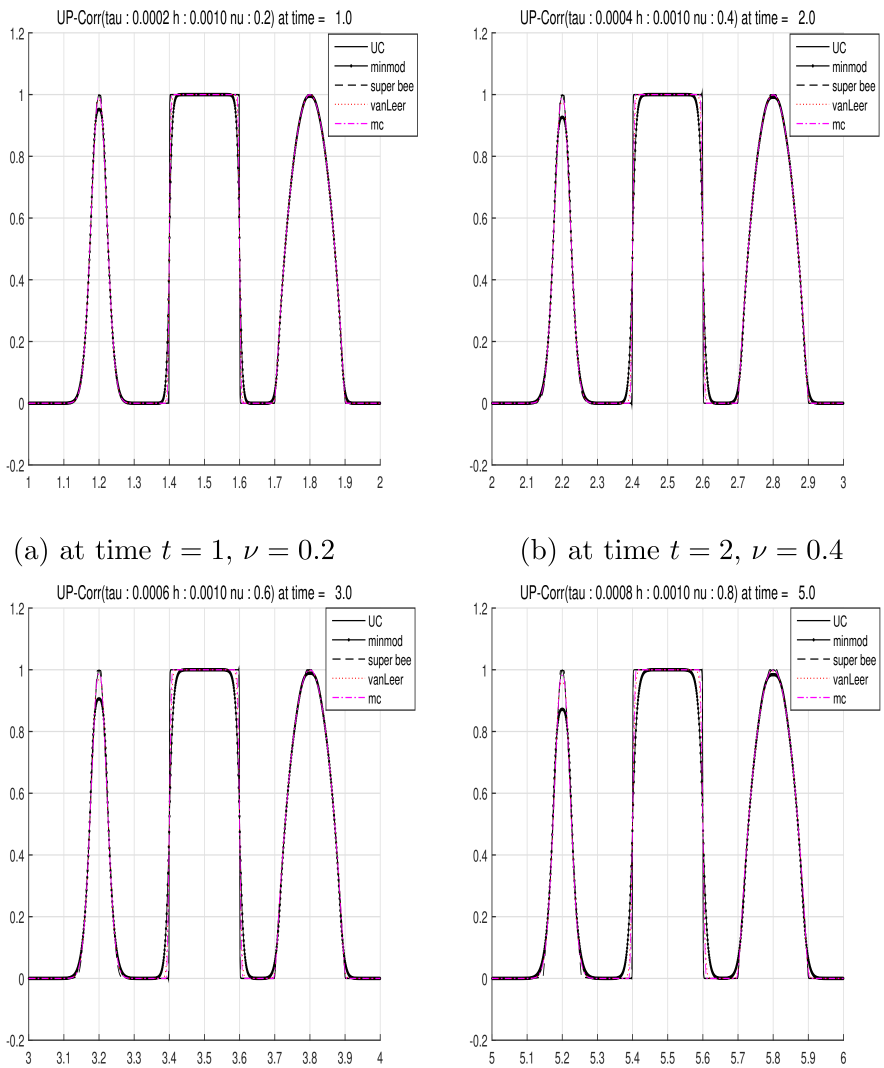


(c) at time $t = 3$, $\nu = 0.6$



(d) at time $t = 5$, $\nu = 0.8$

Figure 4.1: The numerical solutions for the linear equation for the initial data (4.1) at $t = 1$, 2, 3, and 5 with the step size $h = 1/100$.



(a) at time $t = 1$, $\nu = 0.2$ (b) at time $t = 2$, $\nu = 0.4$

(c) at time $t = 3$, $\nu = 0.6$ (d) at time $t = 5$, $\nu = 0.8$

Figure 4.2: The numerical solutions for the linear equation for the initial data (4.1) at $t = 1$, 2, 3, and 5 with the step size $h = 1/100$.

of the UC scheme with the Godunov's method for several flux limiters such as minmod, superbee, van Leer and mc. As shown in Figure 4.2, one may notice that the proposed scheme yields much better approximations to exact solution than the Godunov's method yields.

Example 4.2. The second initial condition will be chosen as the smooth

$$u_0(x) = -x^2 \sin(3\pi x). \tag{4.2}$$

Comparing the numerical solutions of the UC scheme with the classical upwind method and the Lax-Wendroff method, one may see that the smooth exact solution can be almost exactly approximated comparing to second-order Lax-Wendroff method even if both two methods keep the same second-order accuracy.

Example 4.3. The third initial condition is given by

$$u_0(x) = \begin{cases} 1, & \text{for} \quad 0 < x < 0.5 \\ 0, & \text{elsewhere} \end{cases}. \tag{4.3}$$

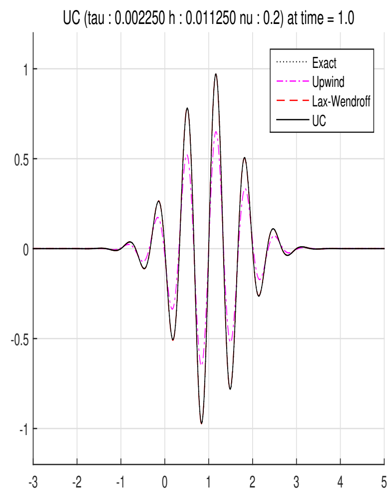
In this example, the initial data has only one singularity. The numerical solutions of the UC scheme are compared with the Godunov's method for several flux limiters.

The UC scheme shows a better approximation comparing to those of the Godunov's method(see [2, 15] , etc.). Throughout numerous demonstrated examples, one can verify that such better approximations can be obtained. The reason is that the developed method (UC) is of second-order accurate with almost l_1 contracting property.

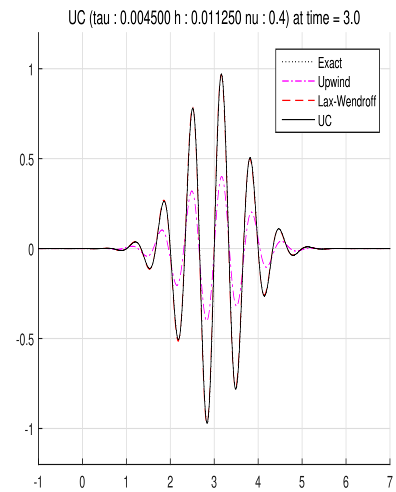
5. Further Discussion

The newly developed method which works for linear scalar hyperbolic equations uses the upwind algorithm plus a correction term whose weight is chosen as $\nu(\frac{h}{a}-\tau)$. This particular weight is derived by combining the first-order correction algorithm for $v = u_x$ and the second-order algorithm for $u_t = -v$. It is shown that the proposed upwind correction method has second-order accurate local truncation error with almost l_1 contracting property while a high-order method using flux-limiter function preserves the TVD property.

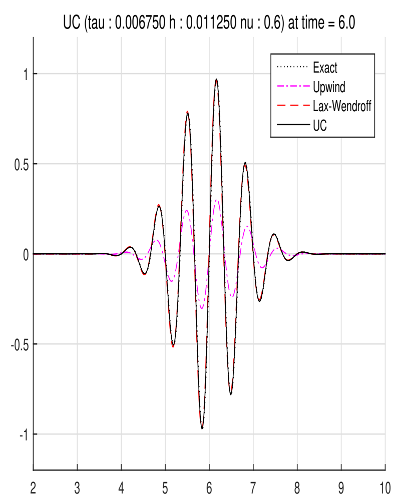
As done for a linear hyperbolic equation, one may apply the developed algorithm to a nonlinear scalar hyperbolic equation $u_t + f_x(u) = 0$ with given initial data $u(x, 0) = u_0(x)$ following known techniques (see [14, 15] for example). However, in an attempt to expand the developed techniques in this paper to a nonlinear scalar equation directly, one will likely have to find a nice weight to make the computations feasible. Such issues will be discussed in a upcoming paper.



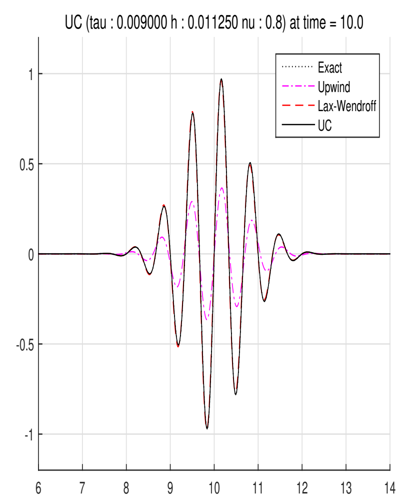
(a) at time $t = 1$, $\nu = 0.2$



(b) at time $t = 3$, $\nu = 0.4$

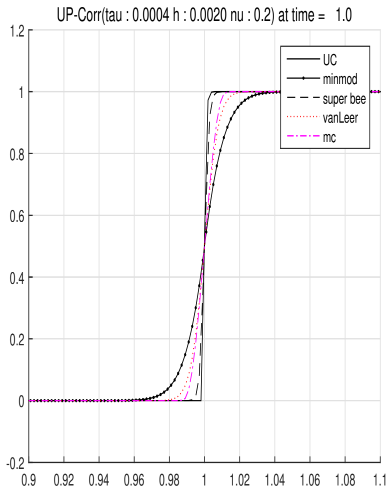


(c) at time $t = 6$, $\nu = 0.6$

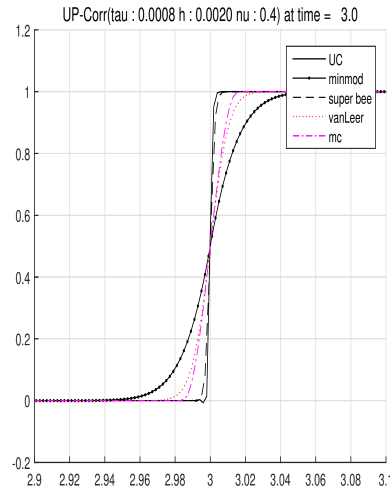


(d) at time $t = 10$, $\nu = 0.8$

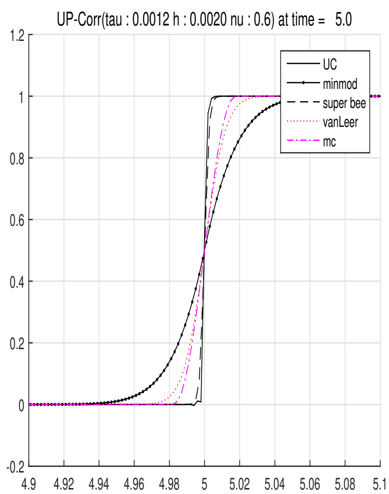
Figure 4.3: The numerical solutions for the linear equation for the initial data (4.2) at $t = 1$, 3, 6, and 10 with the step size $h = 1/20$.



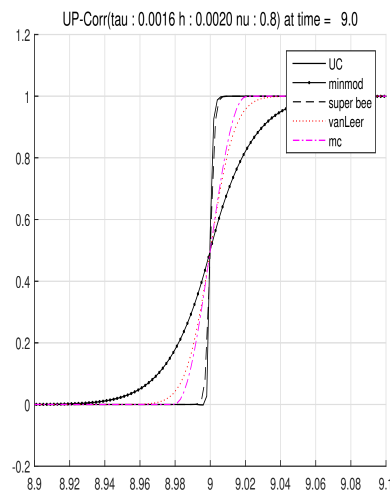
(a) at time $t = 1$, $\nu = 0.2$



(b) at time $t = 3$, $\nu = 0.4$



(c) at time $t = 5$, $\nu = 0.6$



(d) at time $t = 9$, $\nu = 0.8$

Figure 4.4: The numerical solutions for the linear equation for the initial data (4.3) at $t = 1$, 3, 5, and 9 with the step size $h = 1/50$.

References

[1] D. Estep, *A modified equation for dispersive difference schemes*, Appl. Number. Math., **17**(1995), 299–309.

[2] S. K. Godunov, *A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics*, Mat. Sb., **47**(1959), 271–306.

[3] D. F. Griffiths and J. M. Sanz-Serna, *On the scope of the method of modified equations*, SIAM J. Sci. Statist. Comput., **7**(1986), 994–1008.

[4] A. Harten, *High resolution schemes for hyperbolic conservation laws*, J. Comput. Phys., **49**(1983), 357–393.

[5] G. Hedstrom, *Models of difference schemes for* $u_t + u_x = 0$ *by partial differential equations*, Math. Comp., **29**(1975), 969–977.

[6] C. Hirsch, *Numerical computation of internal and external flows*, **2**, Wiley, 1990.

[7] N. Jiang, *On the convergence of fully-discrete high-resolution schemes with van Leer's flux limiter for conservation laws*, Methods Appl. Anal., **16(3)**(2009), 403–422.

[8] S. Konyagin, B. Popov and O. Trifonov, *On convergence of minmod-type schemes*, SAIM J. Numer. Anal., **42(5)**(2005), 1978–1997.

[9] C. B. Laney, *Computational gasdynamics*, Cambridge University Press, Cambridge, 1998.

[10] Y. H. Lee and S. D. Kim, *Note on a classical conservative method for scalar hyperbolic equations*, Kyungpook Math. J., **56**(2016), 1179–1189.

[11] B. van Leer, *Towards the ultimate conservative difference scheme V. A second order sequel to Godunov's method*, J. Comput. Phys., **32**(1979), 101–136.

[12] B. van Leer, *On the relation between the upwind-differencing schemes of Godunov, Engquist-Oscher and Roe*, SIAM J. Sci. Statist. Comput., **5**(1984), 1–20.

[13] B. van Leer, *Towards the ultimate conservative difference scheme II. Monotonicity and conservation combined in a second order scheme*, J. Comput. Phys., **23**(1997), 361–370.

[14] R. J. LeVeque, *Numerical methods for conservation laws*, Lectures in Mathematics ETH Zurich, Birkhauser Verlag, 1992.

[15] R. J. LeVeque, *Finite volume methods for hyperbolic problems,* Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002.

[16] B. Popov, and O. Trifonov, *Order of convergence of second order schemes based on the minmod limiter*, Math. Comp., **75**(2006), 1735–1753.

[17] P. L. Roe, *Some contributions to the modelling of discontinuous flows*, Large-scale computations in fluid mechanics, Part 2 (La Jolla, Calif., 1983), 163–193, Lectures in Appl. Math. **22-2**, Amer. Math. Soc., Providence, RI, 1985.

[18] C.-W. Shu, *TVB uniformly high-order schemes for conservation laws*, Math. Comp., **49**(1987), 105–121.

[19] P. K. Sweby, *High resolution schemes using flux limiters for hyperbolic conservation laws*, SAIM J. Numer. Anal., **21**(1984), 995–1011.

[20] R. F. Warming and B. J. Hyett, *The modified equation approach to the stability and accuracy analysis of finite-difference methods*, J. Comput. Phys., **14**(1974), 159–179.