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Algorithm of Common Solutions to the Cayley Inclusion and Fixed Point Problems

Aadil Hussain Dar*
Department of Mathematical Sciences, Baba Ghulam Shah Badshah University, Rajouri, Jammu and Kashmir, 185234, India
e-mail : daraadil555@gmail.com

Mohammad Kalimuddin Ahmad
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India and Department of Mathematics, Faculty of Science, Islamic University of Madinah, Saudi Arabia
e-mail : ahmad_kalimuddin@yahoo.co.in

Javid Iqbal and Waseem Ali Mir
Department of Mathematical Sciences, Baba Ghulam Shah Badshah University, Rajouri, Jammu and Kashmir, 185234, India
e-mail : javid2iqbal@gmail.com *and* waseemm181@gmail.com

Abstract. In this paper, we develop an iterative algorithm for obtaining common solutions to the Cayley inclusion problem and the set of fixed points of a non-expansive mapping in Hilbert spaces. A numerical example is given for the justification of our claim.

1. Introduction

Everywhere in the paper $\mathcal{V}$ is assumed to be a real Hilbert space with inner product and norm $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively and $F(\mathcal{S}) = \{v \in \mathcal{V} : \mathcal{S}v = v\}$ to be the fixed point set of the mapping $\mathcal{S}$. If $\mathcal{A} : \mathcal{V} \to \mathcal{V}$ and $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ are single and multi-valued mappings, respectively then the variational inclusion problem consists of obtaining $v \in \mathcal{V}$ such that

$$0 \in \mathcal{A}(v) + \mathcal{N}(v). \tag{1.1}$$

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Problem (1.1) and related problems have been considered by many authors in papers such as [1, 2, 3, 7, 8, 10, 11], and has applications in economics, physics, and structural analysis. Another problem known as the fixed point problem is the problem of obtaining $v^* \in \mathcal{V}$ such that

$$v^* = \mathcal{S}(v^*). \tag{1.2}$$

Here $\mathcal{S} : \mathcal{V} \to \mathcal{V}$. This problem (1.2) was considered in [6, 9, 12, 14], and is used for mathematical models of real problems. For the last several years, many researchers, see for example [12, 13, 14], have found common solutions to the problems (1.1) and (1.2). In this paper we find a common solution to the Cayley inclusion problem and the fixed point problem.

2. Preliminaries

In this section, we go through the basic definitions and results used in the paper.

Definition 2.1.([14]) A mapping $\mathcal{S} : \mathcal{V} \to \mathcal{V}$ is called *non-expansive* if

$$\|\mathcal{S}(v) - \mathcal{S}(w)\| \leq \|v - w\| \quad \forall\ v, w \in \mathcal{V}.$$

Definition 2.2.([14]) A mapping $\mathcal{A} : \mathcal{V} \to \mathcal{V}$ is called *α-inverse strongly monotone* if there exists $\alpha \in R^+$ such that

$$\langle \mathcal{A}(v) - \mathcal{A}(w), v - w\rangle \geq \alpha \|\mathcal{A}(v) - \mathcal{A}(w)\|^2\ \forall\ v, w \in \mathcal{V}.$$

Definition 2.3.([14]) Let $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ be a multi-valued mapping, then it is said to be

(i) *monotone* if for all $v, w \in \mathcal{V}$, $x \in \mathcal{N}(v)$, $y \in \mathcal{N}(w)$ such that

$$0 \leq \langle v - w, x - y\rangle\,;$$

(ii) *strongly monotone* if for all $v, w \in \mathcal{V}$, $x \in \mathcal{N}(v)$, $y \in \mathcal{N}(w)$ there exists $\theta \in R^+$ such that

$$\theta\|v - w\|^2 \leq \langle v - w, x - y\rangle\,;$$

(iii) *maximal monotone* if $\mathcal{N}$ is monotone and $(I + \eta\mathcal{N})(\mathcal{V}) = \mathcal{V}$ for all $\eta > 0$, where I is the identity mapping on $\mathcal{V}$.

Lemma 2.1.([14]) *Let $\{e_m\}, \{f_m\}$ and $\{g_m\}$ be three non-negative real sequences satisfying the following condition:*

$$e_{m+1} \leq (1 - \lambda_m)e_m + f_m + g_m \quad \forall\ m \geq m_0,$$

where m_0 is some non-negative integer, $\{\lambda_m\}$ is a sequence in $(0, 1)$ with $\sum_{m=0}^{\infty} \lambda_m = \infty$, $f_m = o(\lambda_m)$ and $\sum_{m=0}^{\infty} g_m < \infty$, then $\lim_{m\to\infty} e_m = 0$.

Lemma 2.2.([5]) *Let $\mathcal{E}$ be a real Banach space, $J : \mathcal{E} \to 2^{\mathcal{E}^*}$ be the normalized duality mapping, then for any $x, y \in \mathcal{E}$, the following conclusion holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

In particular, If $\mathcal{E} = \mathcal{V}$ is a real Hilbert space, then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in \mathcal{V}.$$

Definition 2.4.([14]) Let $\mathcal{K}$ be a nonempty closed and convex subset of a Hilbert space $\mathcal{V}$, then for any $v \in \mathcal{V}$, there exists a unique nearest point in $\mathcal{K}$, designated by $P^{\mathcal{K}}(v)$, such that

$$\|v - P^{\mathcal{K}}(v)\| \leq \|v - w\|, \quad \forall w \in \mathcal{K}.$$

This mapping $P^{\mathcal{K}}$ from $\mathcal{V}$ to $\mathcal{K}$ is known as metric projection.

Remark 2.1. The metric projection $P^{\mathcal{K}}$ has the following properties:

(i) $P^{\mathcal{K}} : \mathcal{V} \to \mathcal{K}$ is non-expansive, i.e.,

$$\|P^{\mathcal{K}}(v) - P^{\mathcal{K}}(w)\| \leq \|v - w\|, \quad \forall v, w \in \mathcal{V};$$

(ii) $P^{\mathcal{K}}$ is firmly non-expansive, i.e.,

$$\|P^{\mathcal{K}}(v) - P^{\mathcal{K}}(w)\|^2 \leq \langle P^{\mathcal{K}}(v) - P^{\mathcal{K}}(w), v - w\rangle \quad \forall v, w \in \mathcal{V};$$

(iii) for each $v \in \mathcal{V}$,

$$u = P^{\mathcal{K}}(v) \Leftrightarrow \langle v - u, u - w\rangle \geq 0, \quad \forall w \in \mathcal{K}.$$

Definition 2.5.([14]) Let $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ be a multi-valued maximal monotone mapping, then the single valued *resolvent operator* is defined as:

$$\mathcal{J}_{\eta}^{\mathcal{N}}(v) = [I + \eta\mathcal{N}]^{-1}(v), \quad \forall v \in \mathcal{V}.$$

Here $\eta \in R^+$ and I is the identity mapping.

Remark 2.2. The resolvent operator $\mathcal{J}_{\eta}^{\mathcal{N}}$ has the following properties:

(i) it is single valued and non-expansive, i.e.,

$$\|\mathcal{J}_{\eta}^{\mathcal{N}}(v) - \mathcal{J}_{\eta}^{\mathcal{N}}(w)\| \leq \|v - w\|, \quad \forall v, w \in \mathcal{V} \textit{ and for } \eta \in R^+;$$

(ii) it is 1-inverse strongly monotone, i.e.,

$$\|\mathcal{J}_{\eta}^{\mathcal{N}}(v) - \mathcal{J}_{\eta}^{\mathcal{N}}(w)\|^2 \leq \langle v - w, \mathcal{J}_{\eta}^{\mathcal{N}}(v) - \mathcal{J}_{\eta}^{\mathcal{N}}(w)\rangle, \quad \forall v, w \in \mathcal{V}.$$

Definition 2.6. Let $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ be a multi-valued maximal monotone mapping and $\mathcal{J}_\eta^{\mathcal{N}}$ be the resolvent operator associated with it, then the Cayley operator $C_\eta^{\mathcal{N}}$ is defined as:

$$C_\eta^{\mathcal{N}}(v) = [2\mathcal{J}_\eta^{\mathcal{N}}(v) - I], \quad \forall v \in \mathcal{V}. \tag{2.1}$$

Remark 2.3. Using Remark 2.2, it can be easily seen that the Cayley operator $C_\eta^{\mathcal{N}}$ is 3-Lipschitz continuous.

In short, we denote it by $\mathcal{C}$, i.e., $\mathcal{C}(v) = C_\eta^{\mathcal{N}}(v)$. Let $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ be a multi-valued maximal monotone mapping, $\mathcal{J}_\eta^{\mathcal{N}}$ be the resolvent operator associated with it and $C_\eta^{\mathcal{N}}$ be the Cayley operator, then Cayley inclusion problem is to find $v \in \mathcal{V}$ such that

$$0 \in C_\eta^{\mathcal{N}}(v) + \mathcal{N}(v).$$

Or in short it can be written as

$$0 \in \mathcal{C}(v) + \mathcal{N}(v). \tag{2.2}$$

Lemma 2.3.([4]) *Let $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ be a maximal monotone mapping and $\mathcal{B} : \mathcal{V} \to \mathcal{V}$ be a Lipschitz continuous mapping. Then a mapping $\mathcal{B} + \mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ is a maximal monotone mapping.*

In view of Remark 2.3 and Lemma 2.3, we can see that $\mathcal{C} + \mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$, where $\mathcal{C}$ is a Cayley operator given by (2.1) is a maximal monotone. So a new resolvent operator can be defined as follows.

Definition 2.7. Let $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ be a maximal monotone mapping and $\mathcal{C} : \mathcal{V} \to \mathcal{V}$ be a cayley operator given by equation (2.1) which is Lipschitz continuous, so that $\mathcal{C} + \mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ is also a maximal monotone mapping. A new resolvent operator associated with $\mathcal{C} + \mathcal{N}$ is defined as:

$$\mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v) = [I + \eta(\mathcal{C} + \mathcal{N})]^{-1}(v) \quad \forall v \in \mathcal{V}. \tag{2.3}$$

Remark 2.4. The resolvent operator $\mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}$ is also non-expansive and 1-inverse strongly monotone.

3. Main Result

In this section, we will discuss an algorithm for obtaining common solutions to the problems (1.2) and (2.2). Before going to the main result, we first state the Lemma which is used in the main result.

Lemma 3.1. *$v \in \mathcal{V}$ is a solution of variational inclusion problem (2.2) iff $v = \mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v), \quad \forall \quad \eta \in R^+$.*

Proof. If $v \in \mathcal{V}$ is a solution of problem (2.2), then for $\eta \in R^+$,

$$\begin{aligned} 0 \in (\mathcal{C}+\mathcal{N})v &\Leftrightarrow 0 \in \eta(\mathcal{C}+\mathcal{N})v \\ &\Leftrightarrow v \in [1+\eta(\mathcal{C}+\mathcal{N})v] \\ &\Leftrightarrow v = [1+\eta(\mathcal{C}+\mathcal{N})]^{-1}(v) = \mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v). \end{aligned}$$ □

Using Lemma 3.1, we develop the following iterative algorithm for obtaining common solutions to problems (1.2) and (2.2).

Algorithm : Let $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ be a multi-valued maximal monotone mapping, $\mathcal{J}_\eta^{\mathcal{N}}$ be the resolvent operator associated with it, $\mathcal{C}$ be the Cayley operator and $\mathcal{S} : \mathcal{V} \to \mathcal{V}$ be a non-expansive mapping, then let

$$\begin{cases} v_{m+1} = \beta_m v + (1-\beta_m)\mathcal{S}(w_m), \\ w_m = \mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v_m), \quad m = 0,1,2,\cdots . \end{cases} \tag{3.1}$$

Now we state and prove our main result in which we show that the sequence $\{v_m\}$ generated by (3.1) under certain conditions converges strongly to common solutions to the problems (1.2) and (2.2).

Theorem 3.1. *Let $\mathcal{V}$ be a Hilbert space, $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ be a multi-valued maximal monotone mapping, $\mathcal{C}$ be the Cayley operator given by* (2.1)*, which is Lipschitz continuous so that $\mathcal{C}+\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}}$ is a maximal monotone mapping by Lemma 2.3 and $\mathcal{S} : \mathcal{V} \to \mathcal{V}$ be a non-expansive mapping. Let $F(\mathcal{S}) \cap VI(I, \mathcal{C}+\mathcal{N}) \neq \phi$. Suppose $v_0 \in \mathcal{V}$ and $\{v_m\}$ be the sequence given by* (3.1) *with the following conditions:*

(i) $\lim_{m\to\infty} \beta_m = 0$; $\sum_{m=0}^{\infty} \beta_m = \infty$;

(ii) $\sum_{m=0}^{\infty} |\beta_{m+1} - \beta_m| < \infty$.

Then $\{v_m\}$ converges strongly to $F(\mathcal{S}) \cap VI(I, \mathcal{C}+\mathcal{N})$.

Proof. We prove the theorem in six steps.
Step 1 : First we show that the sequences $\{v_m\}$ and $\{w_m\}$ are bounded.
For $z \in F(\mathcal{S}) \cap VI(I, \mathcal{C}+\mathcal{N})$ and from Lemma 3.1, we have

$$z = \mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(z).$$

So, we calculate

$$\begin{aligned} \|w_m - z\| &= \|\mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v_m) - \mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(z)\| \\ &\le \|v_m - z\| \quad \forall m \ge 0. \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we can write

$$
\begin{aligned}
\|v_{m+1} - z\| &= \|\beta_m(v - z) + (1-\beta_m)(\mathcal{S}_{w_m} - z)\| \\
&\le \beta_m\|v - z\| + (1-\beta_m)\|w_m - z\| \\
&\le \beta_m\|v - z\| + (1-\beta_m)\|v_m - z\| \\
&\le \max\{\|v - z\|, \|v_m - z\|\} \\
&\le \cdots \\
&\le \max\{\|v - z\|, \|v_0 - z\|\} \\
&= \|v - z\|. \qquad (3.3)
\end{aligned}
$$

From above we conclude that the sequences $\{v_m\}$ and $\{w_m\}$ are bounded. Since $\mathcal{S}$ is non-expansive and $\mathcal{C}$ is Lipschitz continuous so $\{\mathcal{S}_{v_m}\}$ and $\{\mathcal{C}_{w_m}\}$ are also bounded in $\mathcal{V}$.

Step 2 : Here we prove that

$$\|v_{m+1} - v_m\| \to 0 \ \ and \ \ \|w_{m+1} - w_m\| \to 0 \ as \ m \to \infty. \qquad (3.4)$$

Since resolvent operator given by (2.3) is non-expansive, we calculate

$$
\begin{aligned}
\|w_{m+1} - w_m\| &= \|\mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v_{m+1}) - \mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v_m)\| \\
&\le \|v_{m+1} - v_m\|. \qquad (3.5)
\end{aligned}
$$

Hence from (3.1) and (3.5), we obtain

$$
\begin{aligned}
\|v_{m+1} - v_m\| &= \|\beta_m v + (1-\beta_m)\mathcal{S}_{w_m} - (\beta_{m-1} v + (1-\beta_{m-1})\mathcal{S}_{w_{m-1}})\| \\
&= \|(\beta_m - \beta_{m-1})(v - \mathcal{S}_{w_{m-1}}) + (1-\beta_m)(\mathcal{S}_{w_m} - \mathcal{S}_{w_{m-1}})\| \\
&\le |\beta_m - \beta_{m-1}|\|v - \mathcal{S}_{w_{m-1}}\| + (1-\beta_m)\|\mathcal{S}_{w_m} - \mathcal{S}_{w_{m-1}}\| \\
&\le |\beta_m - \beta_{m-1}|M + (1-\beta_m)\|w_m - w_{m-1}\| \\
&\le |\beta_m - \beta_{m-1}|M + (1-\beta_m)\|v_m - v_{m-1}\|. \qquad (3.6)
\end{aligned}
$$

Here $M = \sup_{m\ge 1}\|v - \mathcal{S}_{w_{m-1}}\|$. We see that all the conditions of Lemma 2.1 are satisfied by taking $e_m = \|v_m - v_{m-1}\|$, $f_m = 0$ and $g_m = |\beta_m - \beta_{m-1}|M$ and so $\|v_{m+1} - v_m\| \to 0$ as $m \to \infty$. From (3.5) $\|w_{m+1} - w_m\| \to 0$ as $m \to \infty$.

Step 3 : Here we prove that for $z \in F(\mathcal{S}) \cap VI(I, \mathcal{C} + \mathcal{N})$,

$$\|v_m - \mathcal{S}_{w_m}\| \to 0 \ \ as \ \ m \to \infty. \qquad (3.7)$$

$$
\begin{aligned}
\|v_m - \mathcal{S}_{w_m}\| &\le \|v_m - \mathcal{S}_{w_{m-1}}\| + \|\mathcal{S}_{w_{m-1}} - \mathcal{S}_{w_m}\| \\
&\le \beta_{m-1}\|v - \mathcal{S}_{w_{m-1}}\| + \|w_{m-1} - w_m\|. \qquad (3.8)
\end{aligned}
$$

Since $\beta_m \to 0$ and $\|w_{m-1} - w_m\| \to 0$, therefore $\|v_m - \mathcal{S}_{w_m}\| \to 0$.

Step 4 : Here we prove that

$$\|v_m - w_m\| \to 0; \quad \|\mathcal{S}_{w_m} - w_m\| \to 0. \qquad (3.9)$$

For $z \in F(\mathcal{S}) \cap VI(I, \mathcal{C}+\mathcal{N})$ and using Remark 2.4, we obtain

$$\begin{aligned}
\|w_m - z\|^2 &= \|\mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v_m) - \mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(z)\|^2 \\
&\leq \langle v_m - z, w_m - z\rangle \\
&= \frac{1}{2}\left\{\|v_m - z\|^2 + \|w_m - z\|^2 - \|v_m - z - (w_m - z)\|^2\right\} \\
&\leq \frac{1}{2}\left\{\|v_m - z\|^2 + \|v_m - z\|^2 - \|v_m - w_m\|^2\right\}.
\end{aligned}$$

So, we get

$$\|w_m - z\|^2 \leq \|v_m - z\|^2 - \frac{1}{2}\|v_m - w_m\|^2. \tag{3.10}$$

So, using (3.1) and (3.10), we have

$$\begin{aligned}
\|v_{m+1} - z\|^2 &= \|\beta_m(v - z) - (1-\beta_m)(\mathcal{S}_{w_m} - z)\|^2 \\
&\leq \beta_m\|v - z\|^2 + (1-\beta_m)\|\mathcal{S}_{w_m} - z\|^2 \\
&\leq \beta_m\|v - z\|^2 + (1-\beta_m)\|w_m - z\|^2 \\
&\leq \beta_m\|v - z\|^2 + (1-\beta_m)\left\{\|v_m - z\|^2 - \frac{1}{2}\|v_m - w_m\|^2\right\}.
\end{aligned}$$

This implies that

$$\frac{(1-\beta_m)}{2}\|v_m - w_m\|^2 \leq \beta_m\|v - z\|^2 + (\|v_m - z\|^2 - \|v_{m+1} - z\|^2). \tag{3.11}$$

Since $\beta_m \to 0$ and

$$|\|v_m - z\|^2 - \|v_{m+1} - z\|^2| \leq \|v_{m+1} - v_m\|(\|v_m\| + \|v_{m+1}\|) \to 0.$$

So, from (3.11), $\|v_m - w_m\| \to 0$. Also from (3.7), we obtain

$$\|\mathcal{S}_{w_m} - w_m\| \leq \|\mathcal{S}_{w_m} - v_m\| + \|v_m - w_m\| \to 0.$$

Step 5 : Here we prove that

$$\limsup_{m\to\infty}\langle v - q, \mathcal{S}_{w_m} - q\rangle \leq 0, \tag{3.12}$$

here $q = \mathcal{P}^{F(\mathcal{S})\cap VI(I,\mathcal{C}+\mathcal{N})}v$.
Since $\{w_m\}$ is a bounded sequence in $\mathcal{V}$, so there exists a subsequence $\{w_{m_i}\} \subset \{w_m\}$ such that $w_{m_i} \rightharpoonup w \in \mathcal{V}$ and

$$\limsup_{m\to\infty}\langle v - q, \mathcal{S}_{w_m} - q\rangle = \lim_{m_i\to\infty}\langle v - q, \mathcal{S}_{w_{m_i}} - q\rangle. \tag{3.13}$$

Since $\|\mathcal{S}_{w_m} - w_m\| \to 0$, $\|\mathcal{S}_{w_{m_i}} - w_{m_i}\| \to 0$, $\mathcal{S}$ is non-expansive hence $I - \mathcal{S} : \mathcal{V} \to \mathcal{V}$ is semi-closed, so $\mathcal{S}(w) = w$, i.e., $w \in F(\mathcal{S})$.
Now we prove that

$$w \in VI(I, \mathcal{C} + \mathcal{N}). \tag{3.14}$$

Since Cayley operator $\mathcal{C}$ is Lipschitz continuous and $\mathcal{N}$ is maximal monotone, therefore by Lemma 2.3 $\mathcal{C} + \mathcal{N}$ is maximal monotone. Let $(a, b) \in Graph(\mathcal{C} + \mathcal{N})$, i.e., $b \in (\mathcal{C} + \mathcal{N})(a)$. Since $w_{m_i} = \mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v_{m_i})$, we have $v_{m_i} \in [I + (\mathcal{C} + \mathcal{N})](w_{m_i})$, i.e.,

$$\frac{1}{\eta}(v_{m_i} - w_{m_i}) \in (\mathcal{C} + \mathcal{N})(w_{m_i}).$$

So, by maximal monotonicity $(\mathcal{C} + \mathcal{N})$, we have

$$\left\langle a - w_{m_i}, b - \frac{1}{\eta}(v_{m_i} - w_{m_i}) \right\rangle \geq 0.$$

So

$$\langle a - w_{m_i}, b \rangle \geq \left\langle a - w_{m_i}, \frac{1}{\eta}(v_{m_i} - w_{m_i}) \right\rangle. \tag{3.15}$$

Since $\|v_{m_i} - w_{m_i}\| \to 0$ and $w_{m_i} \rightharpoonup w$, we get

$$\lim_{m_i \to \infty} \langle a - w_{m_i}, b \rangle = \langle a - w, b \rangle \geq 0.$$

Because $\mathcal{C} + \mathcal{N}$ is maximal monotone, this implies that $0 \in (\mathcal{C} + \mathcal{N})(w)$, i.e., $w \in VI(I, \mathcal{C} + \mathcal{N})$. So $w \in F(\mathcal{S}) \cap VI(I, \mathcal{C} + \mathcal{N})$.

Since $\|\mathcal{S}_{w_m} - w_m\| \to 0$ and $w_{m_i} \rightharpoonup w \in F(\mathcal{S}) \cap VI(I, \mathcal{C} + \mathcal{N})$, so from (3.13) and Remark 2.1, we get

$$\begin{aligned}
\limsup_{m \to \infty} \langle v - q, \mathcal{S}_{w_m} - q \rangle &= \lim_{m_i \to \infty} \langle v - q, \mathcal{S}_{w_{m_i}} - q \rangle \\
&= \lim_{m_i \to \infty} \langle v - q, \mathcal{S}_{w_{m_i}} - w_{m_i} + w_{m_i} - q \rangle \\
&= \lim_{m_i \to \infty} \langle v - q, w - q \rangle \leq 0.
\end{aligned}$$

Hence (3.12) is proved.
Step 6 : Finally we prove that

$$v_m \to q = \mathcal{P}^{F(\mathcal{S}) \cap VI(I, \mathcal{C}+\mathcal{N})}(v_0). \tag{3.16}$$

Using (3.1), (3.2) and Lemma 2.2, we obtain

$$\begin{aligned}
\|v_{m+1} - q\|^2 &= \|\beta_m(v - q) + (1 - \beta_m)(\mathcal{S}_{w_m} - q)\|^2 \\
&\leq (1 - \beta_m)^2 \|(\mathcal{S}_{w_m} - q)\|^2 + 2\beta_m \langle v - q, v_{m+1} - q \rangle \\
&\leq (1 - \beta_m)^2 \|w_m - q)\|^2 + 2\beta_m \langle v - q, v_{m+1} - q \rangle \\
&\leq (1 - \beta_m)^2 \|v_m - q)\|^2 + 2\beta_m \langle v - q, v_{m+1} - q \rangle. \qquad (3.17)
\end{aligned}$$

Let

$$\gamma_m = \max\{0, \langle v - q, v_{m+1} - q\rangle\}.$$

Then $\gamma_m \geq 0$.

Now we prove that $\gamma_m \to 0$.

From (3.12), it follows that for given $\delta > 0$, there exists m_0 such that

$$\langle v - q, v_{m+1} - q\rangle < \delta.$$

So, we have

$$0 \leq \gamma_m < \delta, \quad \forall m \geq m_0.$$

By the arbitrariness of $\delta > 0$, we get $\gamma_m \to 0$. So we can write (3.17) as follows;

$$\|v_{m+1} - q\|^2 \leq (1 - \beta_m)^2 \|v_m - q)\|^2 + 2\beta_m\gamma_m. \tag{3.18}$$

By taking $e_m = \|v_{m+1} - q\|^2$, $f_m = 2\beta_m\gamma_m$ and $g_m = 0$, all the conditions of the Lemma 2.1 are satisfied. Hence $v_m \to q$ as $m \to \infty$. This proves our theorem. □

4. Numerical Example

Example 4.1. Let $\mathcal{V} = \mathbb{R}$, the set of reals and let $\mathcal{N} : \mathbb{R} \to 2^{\mathbb{R}}$, be defined as $\mathcal{N}(v) = \{\frac{1}{5}(v)\}\ \forall\ v \in \mathbb{R}$, then we calculate resolvent operator $\mathcal{J}_\eta^{\mathcal{N}}(v)$, Cayley operator $C_\eta^{\mathcal{N}}(v)$ and new resolvent operator $\mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v)$ for $\eta = 1$ as

$$\mathcal{J}_\eta^{\mathcal{N}}(v) = [I + \eta\mathcal{N}]^{-1}(v) = \frac{5}{6}v.$$

$$C_\eta^{\mathcal{N}}(v) = [2\mathcal{J}_\eta^{\mathcal{N}}(v) - I] = \frac{4}{6}v.$$

$$\mathcal{J}_\eta^{\mathcal{C}+\mathcal{N}}(v) = [I + \eta(\mathcal{C} + \mathcal{N})]^{-1}(v) = \frac{15}{28}v.$$

Let $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ be defined as $\mathcal{S}(v) = v$ and $\beta_m = \frac{1}{m}$. Then all the conditions of Theorem 3.1 are satisfied and we can calculate

$$v_{m+1} = \frac{1}{m}v_0 + \frac{(m-1)}{m}\frac{15}{28}v.$$

All codes are written in MATLAB 2012. We have taken different initial values $V_0 = 1, 3.5, 5.0$, which show that the sequence $\{v_m\}$ converges to the solution of the problem. The convergence graph is shown Figure 1.

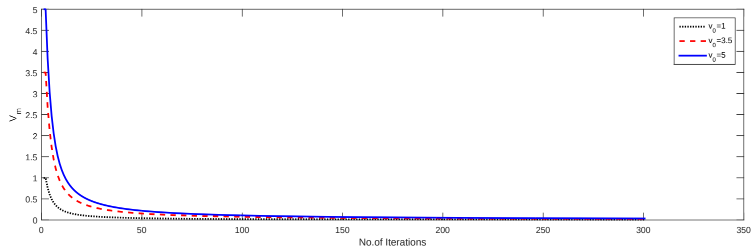

Figure 1: Convergence of $\{v_m\}$ by using Algorithm 3.1

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