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Left Translations and Isomorphism Theorems for Menger Algebras of Rank n

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Abstract. Let n be a fixed natural number. Menger algebras of rank n can be regarded as a canonical generalization of arbitrary semigroups. This paper is concerned with studying algebraic properties of Menger algebras of rank n by first defining a special class of full n-place functions, the so-called a left translation, which possess necessary and sufficient conditions for an $(n+1)$-groupoid to be a Menger algebra of rank n. The isomorphism parts begin with introducing the concept of homomorphisms, and congruences in Menger algebras of rank n. These lead us to establish a quotient structure consisting a nonempty set factored by such congruences together with an operation defined on its equivalence classes. Finally, the fundamental homomorphism theorem and isomorphism theorems for Menger algebras of rank n are given. As a consequence, our results are significant in the study of algebraic theoretical Menger algebras of rank n. Furthermore, we extend the usual notions of ordinary semigroups in a natural way.

1. Introduction

The study of algebraic properties of the composition of multiplace functions was initiated by K. Menger in 1946 [7]. The essential property of composition, which is called *superassociative law*, was studied in both primary and advanced ways. Following the suggestion of K. Menger, the concept of Menger algebras of rank n is presented. A nonempty set G with an $(n+1)$-ary operation $\circ$ defined on G where $n \geq 1$ is called a *Menger algebra of rank n* and denoted by $(G, \circ)$ if for all $x, y_1, \ldots, y_n, z_1, \ldots, z_n \in G$ the following superassociative law holds:

$$\circ(\circ(x, y_1, \ldots, y_n), z_1, \ldots, z_n) = \circ(x, \circ(y_1, z_1, \ldots, z_n), \ldots, \circ(y_n, z_1, \ldots, z_n)).$$

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For $n = 1$, it is an arbitrary semigroup. By a Menger subalgebra of rank n of G, we mean a nonempty subset A of G which is closed with respect to the restriction of $\circ$ to A.

Example 1.1.([3]) Some examples of Menger algebras of rank n are provided.

(1) The set $\mathbb{R}^+$ of all positive real numbers with the operation $\circ : (\mathbb{R}^+)^{n+1} \to \mathbb{R}^+$, defined by $\circ(x_0, \dots, x_n) = x_0 \sqrt[n]{x_1 \cdots x_n}$, forms a Menger algebra of rank n.

(2) The set of all real numbers $\mathbb{R}$ with the following $(n+1)$-ary operation $\circ$, which is defined by $\circ(x, y_1 \dots, y_n) = x + \frac{y_1 + \dots + y_n}{n}$ for all $x, y_1 \dots, y_n \in \mathbb{R}$ is a Menger algebra of rank n.

If there exist elements $e_1, \dots, e_n \in G$, called *selectors*, such that

$$\circ(x, e_1, \dots, e_n) = x \text{ and } \circ(e_i, x_1 \dots, x_n) = x_i$$

for all $x, x_1, \dots, x_n \in G, i = 1, \dots, n$, then a Menger algebra of rank n $(G, \circ)$ is called *unitary.* It is obviously evident that the definition of selectors is an extension of an identity element in semigroups by considering $n = 1$ so that there exists $e_1 \in G$ and hence $\circ(x, e_1) = x$ and $\circ(e_1, x_1) = x_1$. So e_1 acts as a right identity and a left identity, respectively.

The development of the theory of Menger algebras of rank n and their applications continued with the work of W. A. Dudek and V. S. Trokhimenko, which are studied nowadays by many mathematicians in various topics. Particularly, a Menger algebra of rank n of n-ary operation is one of the popular topic for studying the algebraic structural properties of n-place functions in this decade. For further results on this area, see W. A. Dudek and V. S. Trokhimenko [3, 4, 5].

In various branches of mathematics, espectially in modern algebra, the set of functions of fixed type and composition operations on functions is actually important. For example, the theory of transformation semigroup is the heart of contemporary semigroups. Generally, the set of n-place operations (full functions) defined on a fixed set A, i.e., n-place functions defined for each element of the set A^n is an extension idea of the usual functions. We now present some basic notions about n-ary operations as follows:

Let A^n be the n-th Cartesian product of a nonempty set A. Any mapping from A^n to A is called a *full n-place functions* or an *n-ary operations* if it is defined for all elements of A. The set of all such mapping is denoted by $T(A^n, A)$. One can consider the *Menger's superposition* on the set $T(A^n, A)$, i.e., an $(n+1)$-operation $\mathcal{O} : T(A^n, A)^{n+1} \to T(A^n, A)$ defined by

$$\mathcal{O}(f, g_1, \dots, g_n)(a_1, \dots, a_n) = f(g_1(a_1, \dots, a_n), \dots, g_n(a_1, \dots, a_n)),$$

where $f, g_1, \dots, g_n \in T(A^n, A), a_1, \dots, a_n \in A$. The set $T(A^n, A)$ is said to be an *algebra of full functions* or *algebra of operations* if the composition of $n+1$ functions from this set is also in this set, i.e., closed with respect to Menger's superposition.

We can remark here that the Menger's superposition can be reduced to the usual composition of functions if $n = 1$.

A *Menger algebra of all full n-ary functions* or *Menger algebra of all n-ary operations* is a pair of the set $T(A^n, A)$ of all full n-place functions defined on A and the Menger composition of full n-place functions satisfying the superassociative law. Each subalgebra of this algebra will be called a *Menger algebra of full n-place functions* or *Menger algebra of n-ary operations*.

Let X a nonempty set. The transformations on X together with the usual composition forms a semigroup, called a *transformations semigroup*, and denoted by $T(X)$. Another important class of transformations is that of the so-called *translations*, which introduced by A. H. Clifford [1] in 1950 and then studied by a number of other mathematicians. Some essential concepts related to translations and their properties in semigroups will be recalled. For convenient, we may write a semigroup S and xy instead of $(S, \cdot)$ and the product $x \cdot y$, respectively. A transformation $\lambda : S \to S$ of a semigroup S is called a *left translation* of S if $\lambda(xy) = \lambda(x)y$ for all x, y in S. A transformation ρ of S is called a *right translation* of S if $\rho(xy) = x\rho(y)$ for all x, y in S.

For each element a of a groupoid S, we associate a mapping $\lambda_a : S \to S$ defined by $\lambda_a(x) = ax$ for all $x \in S$. We call λ_a the *inner left translation of* S corresponding to the element a of S. Similarly, the mapping $\rho_a : S \to S$ defined by $\rho_a(x) = xa$ for all $x \in S$. We call ρ_a the *inner right translation of* S corresponding to the element a of S. If a semigroup S has a left identity element e then every left translation of S is inner, for $\lambda(x) = \lambda(ex) = \lambda(e)x = ax$ where $a = \lambda(e)$. For the right, this situation is also valid. Denote the set of all left translations of a semigroup S by $\Lambda(S)$, and the set of all inner left translations by $\Lambda_0(S)$. Similarly, we will use for the right translations the notations $P(S)$ and $P_0(S)$, respectively. Since translations are element of the transformation semigroup $T(S)$, there is defined for them a multiplication. It is immediately evident that $\Lambda(S), \Lambda_0(S), P(S), P_0(S)$ are semigroups with respect to this operation of usual composition of transformations. As a consequcence, A. H. Clifford proved the characterization of semigroups via their translations that a groupoid S is a semigroup if and only if any inner left(right) translation of S is a left(right) translation. This means that the concept of translations can be applied to characterize a semigroup. Moreover, it also plays an essential role in the ideal extension of semigroups. For further reading on translations of semigroup, see [2].

The main results of the paper concern the structure of Menger algebras of rank n with respect to one $(n+1)$-superassociative operation. In this work, in Section 3, we generalize the notion of translations in Semigroups to Menger algebras of rank n and study their structure by using the full n-place functions studied by many algebraists in [4, 5] for classial algebraic algebras. We also extend some well-known results related to an inner left translation from arbitrary semigroups to its extension. We continue the study of Menger algebras of rank n in Section 4 concerning a binary relation, especially a congruence relation. This posses a corresponding quotient structure from the original one. The relationship between all structures in this section in sense of the homomorphic image is investigated.

In Section 5, we complete the paper with a summary discussion on the interesting problems and a suggestion for the future work in this research direction.

2. Preliminary Results in Semigroups

For an extensive introduction to the theory and history of semigroups, we refer the reader to A. H. Clifford [2] and J. M. Howie [6]. We will begin by recalling some definitions and interesting results in semigroups.

A binary relation ρ on S is a subset of $S \times S$. Let ρ be an equivalence relation on a nonempty set S.

Definition 2.1.([6]) Let S be a semigroup. A binary relation ρ on the set S is called *left compatible* (with the operation on S) if

$$(\forall a, b, x \in S) \ \ (a, b) \in \rho \Rightarrow (xa, xb) \in \rho,$$

and *right compatible* if

$$(\forall a, b, x \in S) \ \ (a, b) \in \rho \Rightarrow (ax, bx) \in \rho.$$

It is called *compatible* if satisfied left and right compatible, i.e.,

$$(\forall a, b, c, d \in S) \ \ (a, b) \in \rho \text{ and } (c, d) \in \rho \Rightarrow (ac, bd) \in \rho.$$

A left (right) compatible equivalence relation is called *left (right) congruence*. A compatible equivalence relation is called *congruence.*

For every $x \in S$, the set $x\rho$ or $\overline{x}$ or $[x]_\rho$ induced by the partition determined by the equivalence relation is called a *ρ-class*, or an *equivalence class*. The set of ρ-classes is called the *quotient set of S by ρ*, and is denoted by S/ρ.

If ρ is a congruence on a semigroup S then a binary operation $\star$ on the quotient set S/ρ can be defined in a canonical way as follows;

$$(a\rho) \star (b\rho) = (ab)\rho. \tag{2.1}$$

It is well known that the quotient set S/ρ together with a binary operation $\star$ defined by (2.1) forms a semigroup, and called *a quotient semigroup*.

Let ρ be congruence on a semigroup S. Then a mapping $\rho^\sharp : S \to S/\rho$, defined by $a\rho^\sharp = a\rho$ is a surjective homomorphism from S to S/ρ and called a *natural homomorphism* from S onto S/ρ. If S and T are semigroups and $\phi : S \to T$ is a homomorphism, then the *kernel of ϕ* written by $ker\phi$ defined by $ker\phi = \{(a, b) \in S \times S \mid a\phi = b\phi\}$. It is clearly evident that $ker\phi$ is congruence on S.

An important connection between homomorphism and quotient semigroup is established as follows: Let S and T be semigroups and $\phi : S \to T$ is a homomorphism, and let ρ be congruence on S with $\rho \subseteq ker\phi$. Then, there exists a unique homomorphism $\varphi : S/\rho \to T$ such that the following diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & T \\ {\scriptstyle \rho^\sharp}\downarrow & \nearrow_{\varphi} & \\ S/\rho & & \end{array}$$

is commutative. Many authors prefer to call the above statement the fundamental theorem of semigroup homomorphism. Furthermore, if $\phi : S \to T$ is a homomorphism, then $ker\phi$ is congruence, and there exists a mapping $\psi : S/ker\phi \to im\phi$ with $\psi([x]_{ker\phi}) = \phi(x)$ is an isomorphism for all $x \in S$, and hence $S/\ker\phi \cong im\phi$.

One structural application of the fundamental homomorphism theorem is to the situation where ρ and σ are two congruences on S with $\rho \subseteq \sigma$. A binary relation σ/ρ on S/ρ is defined by

$$(a\rho, b\rho) \in \sigma/\rho \Leftrightarrow (a, b) \in \sigma.$$

Then σ/ρ is congruence on S/ρ and so $(S/\rho)/(\sigma/\rho) \cong S/\sigma$.

After we completed this section concerning the basic definitions and background results in semigroup theory, the main results of this paper will be started in the next section.

3. The Left Translation of Menger Algebras of Rank n

In order to study the characterization of Menger algebras of rank n through translations, we first introduce the notion of translations in Menger algebras of rank n and study their structural properties.

Definition 3.1. Let $(G, \circ)$ be a Menger algebra of rank n. A mapping $\lambda : G^n \to G$ is called the *left translation* of G^n if it is satisfied the following equation

$$\lambda(\circ(y_1, z_1, \ldots, z_n), \ldots, \circ(y_n, z_1, \ldots, z_n)) = \circ(\lambda(y_1, \ldots, y_n), z_1, \ldots, z_n). \tag{3.1}$$

According to Definition 3.1, this definition can be regarded as a natural generalization of translations in ordinary semigroups by putting $n = 1$, then λ acts as a left translation.

The characterization theorem of Menger algebras of rank n states that an $(n+1)$-ary groupoid is a Menger algebra of rank n if and only if every inner left translation on G is a left translation of G. Before we prove this fact, we first give a several results concerning the algebraic properties of a left translation, and introduce a definition of an inner left translation later.

Lemma 3.2. *Let $(G, \circ)$ be a Menger algebra of rank n and*

$$\Lambda(G) = \{\lambda : G^n \to G \mid \lambda \quad \textit{is a left translation of } G\}.$$

Then $\Lambda(G)$ forms a Menger algebra of rank n with respect to the Menger's superposition.

Proof. It is cleary verified that $\Lambda(G) \neq \emptyset$. Indeed, let a full n-place function $\lambda : G^n \to G$ be defined by $\lambda(x_1, \dots, x_n) = x_1$. Then λ which defined above is a left translation of G. Hence $\lambda \in \Lambda(G)$ and thus $\Lambda(G) \neq \emptyset$. Now let $\lambda, \lambda_1, \dots, \lambda_n \in \Lambda(G)$. Then

$$\begin{aligned}
&\mathcal{O}(\lambda, \lambda_1, \dots, \lambda_n)(\circ(y_1, z_1, \dots, z_n), \dots, \circ(y_n, z_1, \dots, z_n))\\
&= \lambda(\lambda_1(\circ(y_1, z_1, \dots, z_n), \dots, \circ(y_n, z_1, \dots, z_n)), \dots,\\
&\quad \lambda_n(\circ(y_1, z_1, \dots, z_n), \dots, \circ(y_n, z_1, \dots, z_n)))\\
&= \lambda(\circ(\lambda_1(y_1, \dots, y_n), z_1, \dots, z_n), \dots, \circ(\lambda_n(y_1, \dots, y_n), z_1, \dots, z_n)))\\
&= \circ(\lambda(\lambda_1(y_1, \dots, y_n), \dots, \lambda_n(y_1, \dots, y_n)), z_1, \dots, z_n)\\
&= \circ(\mathcal{O}(\lambda, \lambda_1, \dots, \lambda_n)(x_1, \dots, x_n), z_1, \dots, z_n).
\end{aligned}$$

This shows that $\mathcal{O}(\lambda, \lambda_1, \dots, \lambda_n) \in \Lambda(G)$. Hence we conclude that $(\Lambda(G), \mathcal{O})$ is a Menger algebra of rank n. □

Corollary 3.3. *The structure $(\Lambda(G), \mathcal{O})$ is a Menger subalgebra of rank n of a Menger algebra of full n-place functions $(T(G^n, G), \mathcal{O})$.*

Proof. The proof of this corollary follows immediately from Lemma 3.2. □

Subsequently, we present the definition of an inner left translation using the concept of full n-place functions.

Definition 3.4. For each element a of a Menger algebra of rank n $(G, \circ)$, a mapping $\lambda_a : G^n \to G$ of G^n defined by

$$\lambda_a(x_1, \dots, x_n) = \circ(a, x_1, \dots, x_n) \tag{3.2}$$

for all $x_1, \dots, x_n \in G$, where $\circ$ is an $(n+1)$ operation defined on G, is called an *inner left translation* correspond to the element a of G.

We can remark here that if $n = 1$ the inner left translation λ_a defined by (3.2) is reduced to the usual inner left translation in an ordinary semigroup.

Let $\Lambda_0(G)$ be the set of all inner left translations in G. Clearly, it is a subset of $T(G^n, G)$. Applying the Menger's composition of full n-place functions, it follows that $\Lambda_0(G)$ forms a Menger algebra of rank n.

The following result gives a necessary condition for an inner left translation.

Proposition 3.5. *Let $(G, \circ)$ be a Menger algebra of rank n. If G contains selectors, then every left translation of G is an inner left translation.*

Proof. Let $x_1, \dots, x_n$ be arbitrary elements of G. Assume that $\lambda : G^n \to G$ is a left translation. By the definition of selectors, then we have

$$\begin{aligned}\lambda(x_1,\dots,x_n) &= \lambda(\circ(e_1,x_1,\dots,x_n),\dots,\circ(e_n,x_1,\dots,x_n))\\ &= \circ(\lambda(e_1,\dots,e_n),x_1,\dots,x_n)\\ &= \circ(a,x_1,\dots,x_n) \text{ where } a=\lambda(e_1,\dots,e_n)\\ &= \lambda_a(x_1,\dots,x_n).\end{aligned}$$

Hence λ is a inner left translation of G. □

Below we give an important result that shows the characterization of Menger algebras of rank n in terms of translations.

Theorem 3.6. *An $(n+1)$-groupoid $(G,\circ)$ is a Menger algebra of rank n if and only if every inner left translation on G is a left translation of G.*

Proof. For each element a of G induces a full n-place function $\lambda_a : G^n \to G$ which is an inner left translation. The fact that this inner left translation is a left translation because

$$\begin{aligned}&\lambda_a(\circ(y_1,z_1,\dots,z_n),\dots,\circ(y_n,z_1,\dots,z_n))\\ &= \circ(a,\circ(y_1,z_1,\dots,z_n),\dots,\circ(y_n,z_1,\dots,z_n))\\ &= \circ(\circ(a,y_1,\dots,y_n),z_1,\dots,z_n)\\ &= \circ(\lambda_a(y_1,\dots,y_n),z_1,\dots,z_n).\end{aligned}$$

Hence λ_a is a left translation. For the converse, suppose that λ_a is a left translation for all $a \in G$. To prove that an $(n+1)$-operation $\circ$ on G is superassociative, let $a, y_1,\dots,y_n,z_1,\dots,z_n \in G$. Then

$$\begin{aligned}&\circ\,(\circ(a,y_1,\dots,y_n),z_1,\dots,z_n)\\ &= \circ(\lambda_a(y_1,\dots,y_n),z_1,\dots,z_n)\\ &= \lambda_a(\circ(y_1,z_1,\dots,z_n),\dots,\circ(y_n,z_1,\dots,z_n))\\ &= \circ(a,\circ(y_1,z_1,\dots,z_n),\dots,\circ(y_n,z_1,\dots,z_n)).\end{aligned}$$ □

4. Quotient Menger Algebras of Rank n and Their Corresponding Isomorphism Theorems

In this section the notion of a congruence relation on Menger algebras of rank n is introduced and their properties are dealt with in detail. After some preliminaries defining, we form a quotient Menger algebra of rank n in natural ways. We will supplement these results by establishing further properties of their corresponding homomorphism.

Before we begin the results, we will use the following notation: for nonnegative integers i, j, the sequence $x_i,\dots,x_j$ is well defined if $i < j$. Otherwise, if $i > j$, $x_i,\dots,x_j$ is the empty symbol.

Definition 4.1. Let $(G, \circ)$ be a Menger algebra of rank n and ρ be an equivalence relation on G. Then ρ is called *i-congruence* on G, if for each $i = 1, \ldots, n+1$ and $(a, b) \in \rho$ implies that the following assertion holds:

$$(\circ(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n+1}), \circ(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_{n+1})) \in \rho \tag{4.1}$$

for all $a, b, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1} \in G$.

Remark 4.2. It is commonly seen that Definition 4.1 is a natural generalization of congruence in an ordinary semigroup by considering $n = 1$ and if $i = 1$, then $(\circ(a, x_2), \circ(b, x_2)) \in \rho$. So ρ acts as right congruence. Similary with $i = 2$, we have $(\circ(x_1, a), \circ(x_1, b)) \in \rho$. So ρ acts as left congruence.

Definition 4.3. Let $(G, \circ)$ be a Menger algebra of rank n and ρ be an equivalence relation on G. Then ρ is called *congruence* on G if it is i-congruence on G for all $i = 1, \ldots, n+1$.

The characterization of congruence on Menger algebras of rank n will be provided in the next theorem, see [3, Proposition 2.1.11].

Theorem 4.4.([3]) *Let $(G, \circ)$ be a Menger algebra of rank n and ρ be an equivalence relation on G. Then ρ is congruence on G if and only if $(a_1, b_1) \in \rho, (a_2, b_2) \in \rho, \ldots, (a_{n+1}, b_{n+1}) \in \rho$ implies that $(\circ(a_1, \ldots, a_{n+1}), \circ(b_1, \ldots, b_{n+1})) \in \rho$.*

Note that Theorem 4.4 can be considered as a generalization of the same situation in arbitrary semigroups. The following theorem can be easily proved by applying Theorem 4.4.

Theorem 4.5. *Let ρ be congruence on a Menger algebra of rank n $(G, \circ)$. Then so is $\rho \circ \rho$.*

We now construct a quotient set G/ρ for some congruence ρ on a Menger algebra of rank n $(G, \circ)$ as follows:

$$G/\rho = \{[a]_\rho \mid a \in G\}$$

where

$$[a]_\rho = \{b \in G \mid (a, b) \in \rho\}.$$

In order to form the algebraic structure consisting the set of all equivalence class and its operation, we first define the generally product of equivalence classes in the following definition.

Definition 4.6. On the quotient set G/ρ, an $(n+1)$-ary operation $\otimes$ define by

$$\otimes([a_1]_\rho, [a_1]_\rho, \ldots, [a_{n+1}]_\rho) = [\circ(a_1, a_1, \ldots, a_{n+1})]_\rho.$$

At first, we need to ensure that this is a well-defined operation. Suppose first that $[a_1]_\rho = [b_1]_\rho, \ldots, [a_{n+1}]_\rho = [b_{n+1}]_\rho$. This means that $(a_1, b_1) \in \rho, (a_2, b_2) \in$

$\rho, \ldots, (a_{n+1}, b_{n+1}) \in \rho$. Since ρ is congruence, and then by Theorem implies that $(\circ(a_1, \ldots, a_{n+1}), \circ(b_1, \ldots, b_{n+1})) \in \rho$ and so $[\circ(a_1, \ldots, a_{n+1})]_\rho = [\circ(b_1, \ldots, b_{n+1})]_\rho$. This follows immediately that $\otimes([a_1]_\rho, \ldots, [a_{n+1}]_\rho) = \otimes([b_1]_\rho, \ldots, [b_{n+1}]_\rho)$. Hence an $(n+1)$-operation $\otimes$ is well-defined.

The fact that the operation $\otimes$ satisfies superassociative law follows from the superassociativity of the usual operation $\circ$ defined on a Menger algebra of rank n $(G, \circ)$. Then we have the following theorem.

Theorem 4.7. *Let ρ be congruence on a Menger algebra of rank n $(G, \circ)$. The quotient set G/ρ together with one $(n+1)$-ary operation $\otimes$ defined in Definition 4.6 forms a Menger algebra of rank n.*

If ρ is a congruence relation on a Menger algebra of rank n G, then the Menger algebra of rank n G/ρ, in Theorem 4.7 is called a *quotient Menger algebra of rank n of G by ρ.*

To construct another structures, the concept of a homomorphism in Menger algebras of rank n is firstly defined.

Definition 4.8. If $(G, \circ)$ and $(K, *)$ are Menger algebras of rank n, then a map $\alpha : G \to K$ is said to be a *homomorphism of Menger algebras of rank n* if

$$\alpha(\circ(x_1, \ldots, x_{n+1})) = *(\alpha(x_1), \ldots, \alpha(x_{n+1})),$$

for all $x_1, \ldots, x_{n+1} \in G$. A homomorphism $\alpha : G \to K$ is called a *monomorphism*, an *epimorphism*, and an *isomorphism* if it is injective, surjective, and both injective and surjective, respectively. If α is isomorphism from G to K, we say G and K are *isomorphic* and we write $G \cong K$. A homomorphism of a Menger algebra of rank n into itself is called an *endomorphism*, while an isomorphism upon itself is called an *automorphism*.

Proposition 4.9. *Let G_1, G_2 and G_3 be any Menger algebras of rank n and the maps $\alpha_1 : G_1 \to G_2, \alpha_2 : G_2 \to G_3$ be two homomorphisms. Then $\alpha_2 \circ \alpha_1$ is a homomorphism from G_1 to G_3.*

Proof. The proof is straightforward. □

Next, we will present the connection between congruence relations and homomorphisms on Menger algebras of rank n.

Theorem 4.10. *Let $(G, \circ)$ and $(K, *)$ be two Menger algebras of rank n, $\alpha : G \to K$ be a homomorphism and ρ be a congruence relation on G. Then the relation*

$$\alpha(\rho) = \{(\alpha(x), \alpha(y)) \in K \times K \mid (x, y) \in \rho\}$$

is a congruence relation on K.

Proof. Obviously, $\alpha(\rho)$ is an equivalence relation on K. For every $j = 1, \ldots, n+1$, let $x_j, y_j \in G$ be such that $(\alpha(x_j), \alpha(y_j)) \in \alpha(\rho)$. Then $(x_j, y_j) \in \rho$ for all $j = 1, \ldots, n+1$. Since ρ is congruence, by Theorem 4.4, $(\circ(x_1, \ldots, x_{n+1}), \circ(y_1, \ldots, y_{n+1})) \in \rho$.

By defining $\alpha(\rho)$, we obtain that $(\alpha(\circ(x_1,\ldots,x_{n+1})),\alpha(\circ(y_1,\ldots,y_{n+1})))\in\alpha(\rho)$. Since α is a homomorphism, $(*(\alpha(x_1),\ldots,\alpha(x_{n+1})),*(\alpha(y_1),\ldots,\alpha(y_{n+1})))\in\alpha(\rho)$. This shows that $\alpha(\rho)$ is a congruence on K. □

We give some common relationships between homomorphism and Menger subalgebras of rank n.

Theorem 4.11. *Let $(G,\circ)$ and $(K,*)$ be Menger algebras of rank n. Assume that $\alpha: G\to K$ is a homomorphism. Then, for any Menger subalgebra of rank n G' of G, the image $\alpha(G')=\{\alpha(a)\mid a\in G'\}$ of G' under α is a Menger subalgebra of rank n of K. Similarly, for any Menger subalgebra of rank n K' of K, the pre image $\alpha^{-1}(K')=\{a\in G\mid \alpha(a)\in K'\}$ of K' under α is a Menger subalgebra of rank n of G, if nonempty.*

Proof. The proof is straightforward. □

Let $(G,\circ)$ and $(K,*)$ be Menger algebras of rank n and $\phi: G\to K$ be a homomorphism. Define the relation

$$ker\phi=\phi\circ\phi^{-1}=\{(a,b)\in G\times G\mid \phi(a)=\phi(b)\}$$

which is called the *kernel of ϕ*.

Proposition 4.12. *The relation $ker\phi$ of a homomorphism from G to K is a congruence relation on G.*

Proof. Clearly, $ker\phi$ is equivalence. Suppose now that $(a_j,b_j)\in ker\phi$ for every $j=1,\ldots,n+1$. Then $\phi(a_j)=\phi(b_j)$ for all $j=1,\ldots,n+1$. So

$$\begin{aligned}\phi(\circ(a_1,\ldots,a_{n+1}))&=*(\phi(a_1),\ldots,\phi(a_{n+1}))\\&=*(\phi(b_1),\ldots,\phi(b_{n+1}))\\&=\phi(\circ(b_1,\ldots,b_{n+1})).\end{aligned}$$

Thus $(\circ(a_1,\ldots,a_{n+1}),\circ(b_1,\ldots,b_{n+1}))\in\ker\phi$, and hence $\ker\phi$ is a congruence on G. □

Lemma 4.13. *Let ρ be a congruence on a Menger algebra of rank n $(G,\circ)$. Then a mapping $\rho^\sharp: G\to G/\rho$ defined by $\rho^\sharp(a)=[a]_\rho$ is a surjective homomorphism from G to G/ρ.*

Proof. It is obviuos that $\rho^\sharp$ is surjective. Next, suppose that $x_1,\ldots,x_{n+1}\in G$. Then we have

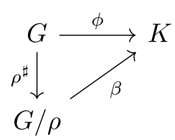
$$\begin{aligned}\rho^\sharp(\circ(x_1,\ldots,x_{n+1}))&=[\circ(x_1,\ldots,x_{n+1})]_\rho\\&=\otimes([x_1]_\rho,\ldots,[x_{n+1}]_\rho)\\&=\otimes(\rho^\sharp(x_1),\ldots,\rho^\sharp(x_{n+1})).\end{aligned}$$

This shows that $\rho^\sharp$ is a surjective homomorphism from G to G/ρ. □

The mapping $\rho^\sharp$ which is defined in Lemma 4.13 is called the *natural homomorphism*. We now want to examine the kernel of $\rho^\sharp$ and so $(x,y) \in \rho^\sharp \circ (\rho^\sharp)^{-1} = ker\rho^\sharp \Leftrightarrow \rho^\sharp(x) = \rho^\sharp(y) \Leftrightarrow [x]_\rho = [y]_\rho \Leftrightarrow (x,y) \in \rho$. Therefore $\rho = ker\rho^\sharp$ and thus every congruence is the kernel of a homomorphism.

The foregoing shows that every quotient Menger algebra of rank n of a Menger algebra of rank n G is a homomorhic image of G. The following theorem is the generalization of the fundamental semigroup homomorphism theorem and shows conversely that every homomorphic image of G is isomorphic with a quotient Menger algebra of rank n of G.

Theorem 4.14.(Fundamental Homomorphism Theorem) *Let ϕ be a homomorphism of a Menger algebra of rank n $(G, \circ)$ into a Menger algebra of rank n $(K, *)$ and $\rho \subseteq ker\phi$. Then there exists a unigue homomorphism β of G/ρ into K such that the following diagram*



is commutatative, i.e., $\beta \circ \rho^\sharp = \phi$ where $\rho^\sharp$ is a natural homomorphism.

Proof. For each element $[a]_\rho$ of G/ρ, define $\beta : G/\rho \to K$ by $\beta([a]_\rho) = \phi(a)$ where $a \in G$. To see that β is well-defined, we note that if $[a]_\rho, [b]_\rho \in G/\rho$ and $[a]_\rho = [b]_\rho$, then $(a,b) \in \rho \subseteq ker\phi$. Thus $\phi(a) = \phi(b)$ and that $\beta([a]_\rho) = \beta([b]_\rho)$. It is clear that β maps G/ρ onto K. To show that β is homomorphism, let $[a_1]_\rho, \ldots, [a_{n+1}]_\rho \in G/\rho$. Then

$$\begin{aligned}\beta(\otimes([a_1]_\rho, \ldots, [a_{n+1}]_\rho)) &= \beta([\circ(a_1, \ldots, a_{n+1})]_\rho)\\ &= \phi(\circ(a_1, \ldots, a_{n+1}))\\ &= *(\phi(a_1), \ldots, \phi(a_{n+1}))\\ &= *(\beta([a_1]_\rho), \ldots, \beta([a_{n+1}]_\rho)).\end{aligned}$$

In order to prove the diagram commutes, let $a \in G$. Then $(\beta \circ \rho^\sharp)(a) = \beta([a]_\rho) = \phi(a)$. Hence $\beta \circ \rho^\sharp = \phi$. Finally, assume that a mapping $\gamma : G/\rho \to K$ is a homomorphism such that $\gamma \circ \rho^\sharp = \phi$. If $[a]_\rho \in G/\rho$, then $\gamma([a]_\rho) = \gamma(\rho^\sharp(a)) = (\gamma \circ \rho^\sharp)(a) = \phi(a) = (\beta \circ \rho^\sharp) = \beta(\rho^\sharp(a)) = \beta([a]_\rho)$. This proved the uniqueness of β. □

Theorem 4.14 is very useful for proving that the first isomorphism theorem of Menger algebras of rank n exists. By $im\phi$ we denoted the image of a mapping ϕ.

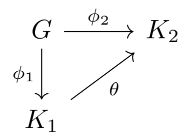
Corollary 4.15.(First Isomorphism Theorem) *Let $(G, \circ)$ and $(K, *)$ be Menger algebras of rank n and $\phi : G \to K$ is a homomorphism. Then there exists an isomorphism, such that $S/\ker\phi \cong im\phi$.*

Proof. From Theorem 4.14, put $\rho = ker\phi$. Then there exists $\beta : G/\rho \to im\beta$. It is evidently clear that $im\beta = im\phi$. Indeed, let $t \in im\beta$, then there exists $[a]_\rho \in G/\rho$

such that $\beta([a]_\rho) = t$. From $\beta \circ \rho^\sharp = \phi$ implies that $\phi(a) = t$ and so $t \in im\phi$. The converse is also valid. If $\phi(a) = \phi(b)$, then $(a, b) \in ker\phi$. It follows that $[a]_\rho = [b]_\rho$. Hence $\beta : G/\rho \to im\beta = im\phi$ is an isomorphism. □

We can say that Corollary 4.15 is the extension of the first isomorphism theorem of an arbitrary semigroup, i.e., in case $n = 1$.

Theorem 4.16.(Induced Homomorphism Theorem) *Let ϕ_1 and ϕ_2 be homomorphisms from a Menger algebra of rank n $(G, \circ)$ to Menger algebras of rank n $(K_1, *_1)$ and $(K_2, *_2)$, respectively, such that ϕ_1 is surjective and $ker\phi_1 \subseteq ker\phi_2$. Then there exists a unique homomorphism θ of K_1 and K_2 such that the following diagram commutes*



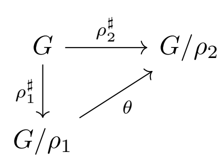
i.e., $\theta \circ \phi_1 = \phi_2$.

Proof. Let k be an arbitrary element in K_1. Since ϕ_1 is surjective, there exists $a_k \in G$ such that $\phi_1(a_k) = k$. Define $\theta : K_1 \to K_2$ by $\theta(k) = \phi_2(a_k)$. This is well-defined, if $x, y \in K_1$ and $x = y$. Then $\phi_1(a_x) = \phi_1(a_y)$ and thus $(a_x, a_y) \in ker\phi_1 \subseteq ker\phi_2$. It implies that $\phi_2(a_x) = \phi_2(a_y)$ and so $\theta(x) = \theta(y)$. It is obvious that $\theta \circ \phi_1 = \phi_2$ since $(\theta \circ \phi_1)(a) = \theta(\phi_1(a)) = \theta(a_1) = \phi_2(a)$. Next, let $k_j \in K_1$ for every $j = 1, \ldots, n+1$. Then we have

$$\begin{aligned}
\theta(*_1(k_1, \ldots, k_{n+1})) &= \theta(*_1(\phi_1(a_{k_1}), \ldots, \phi_1(a_{k_{n+1}}))) \\
&= \theta(\phi_1(\circ(a_{k_1}, , \ldots, a_{k_{n+1}}))) \\
&= \phi_2(\circ(a_{k_1}, , \ldots, a_{k_{n+1}})) \\
&= *_2(\phi_2(a_{k_1}), \ldots, \phi_2(a_{k_{n+1}})) \\
&= *_2(\theta(\phi_1(a_{k_1})), \ldots, \theta(\phi_1(a_{k_{n+1}}))) \\
&= *_2(\theta(k_1), \ldots, \theta(k_{n+1})).
\end{aligned}$$

Finally, suppose that $\eta : K_1 \to K_2$ satisfies $\eta \circ \phi_1 = \phi_2$. Then $\eta(k) = \eta(\phi_1(a_k)) = (\eta \circ \phi_1)(a_k) = \phi_2(a_k) = (\theta \circ \phi_1)(a_k) = \theta(\phi_1(a_k)) = \theta(k)$ for all $k \in K_1$. □

One direct application of the fundamental homomorphism theorem is to the situation where ρ_1 and ρ_2 are congruences on G with $\rho_1 \subseteq \rho_2$. Then there exists a homomorphism from G/ρ_1 to G/ρ_2 such that the diagram



commutes. This homomorphism is defined by the following: For any arbitrary element $[a]_{\rho_1}$ in G/ρ_1, then a mapping $\theta : G/\rho_1 \to G/\rho_2$ can be defined by

$$\theta([a]_{\rho_1}) = [a]_{\rho_2} \tag{4.2}$$

where $a \in G$.

Moreover, the kernel of θ on G/ρ_1 is given by

$$ker\theta = \{([a]_{\rho_1}, [b]_{\rho_1}) \in G/\rho_1 \times G/\rho_1 \mid (a, b) \in \rho_2\}. \tag{4.3}$$

For convenient, it is suitable to write $ker\theta$ as ρ_2/ρ_1. We will give a remark later that these two sets are coincide.

Lemma 4.17. *The relation ρ_2/ρ_1 defined in* (4.3) *is congruence on G/ρ_1.*

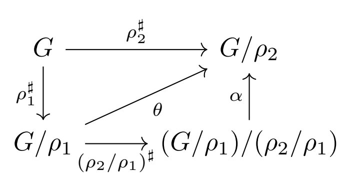
Proof. It is obviously clear that ρ_2/ρ_1 is equivalence. Suppose that $([a_j]_{\rho_1}, [b_j]_{\rho_1})$ for all $j = 1, \ldots, n+1$. Then we have $(a_j, b_j) \in \rho_2$ for all $j = 1, \ldots, n+1$. Since ρ_2 is a congruence relation, by Theorem 4.4, $(\circ(a_1, \ldots, a_{n+1}), \circ(b_1, \ldots, b_{n+1})) \in \rho_2$. It follows directly that $([\circ(a_1, \ldots, a_{n+1})]_{\rho_1}, [\circ(b_1, \ldots, b_{n+1})]_{\rho_1}) \in \rho_2/\rho_1$ and thus by defining the operation $\otimes$ on G/ρ_1 we conclude that ρ_2/ρ_1 is congruence. $\square$

On a quotient set of G/ρ_1, we can construct a new quotient set by using a congruence ρ_2/ρ_1 from Lemma 4.17, i.e., a quotient set of the form $(G/\rho_1)/(\rho_2/\rho_1)$. Then a natural homomorphism $(\rho_2/\rho_1)^\sharp$ from G/ρ_1 to $(G/\rho_1)/(\rho_2/\rho_1)$ exists certainly.

Finally, The connection between the structure $(G/\rho_1)/(\rho_2/\rho_1)$ and G/ρ_2 will be provided. This first way to study this connection derived from directly application of the first isomorphism theorem. In fact, a mapping $\theta : G/\rho_1 \to G/\rho_2$ is defined by an equation (4.2), is a homomorphism. The fact that the kernel of θ and ρ_2/ρ_1 are the same thing is a true result, follows immediately from the above defining. In fact, let a, b be two arbitrary elements of G. Then

$$\begin{aligned} ([a]_{\rho_1}, [b]_{\rho_1}) \in \rho_2/\rho_1 &\Leftrightarrow (a, b) \in \rho_2 \\ &\Leftrightarrow [a]_{\rho_2} = [b]_{\rho_2} \\ &\Leftrightarrow \theta([a]_{\rho_1}) = \theta([b]_{\rho_1}) \\ &\Leftrightarrow ([a]_{\rho_1}, [b]_{\rho_1}) \in ker\theta. \end{aligned}$$

We can also use Corollary 4.15 to obtain significant results, $(G/\rho_1)/(\rho_2/\rho_1)$ and G/ρ_2 are isomorphic. Furthermore, the following diagram is commutative.



The final theorem shows significant different processes to prove the relationship between the quotient structure $(G/\rho_1)/(\rho_2/\rho_1)$ and the original quotient G/ρ_2.

Theorem 4.18.(Second Isomorphism Theorem) *Let ρ_1 and ρ_2 be two congruences on a Menger algebra of rank n $(G, \circ)$ such that $\rho_1 \subseteq \rho_2$. Then*

$$(G/\rho_1)/(\rho_2/\rho_1) \cong G/\rho_2.$$

Proof. Firstly, by Lemma 4.17, ρ_2/ρ_1 is congruence on G/ρ_1. Define a mappping $\alpha : (G/\rho_1)/(\rho_2/\rho_1) \to G/\rho_2$ by $\alpha([[a]_{\rho_1}]_{\rho_2/\rho_1}) = [a]_{\rho_2}$ where $a \in G$. Then α is both well-defined and injection,

$$\begin{aligned}
[[a]_{\rho_1}]_{\rho_2/\rho_1} = [[b]_{\rho_1}]_{\rho_2/\rho_1} &\Leftrightarrow ([a]_{\rho_1}, [b]_{\rho_1}) \in \rho_2/\rho_1 \\
&\Leftrightarrow (a, b) \in \rho_2 \\
&\Leftrightarrow [a]_{\rho_2} = [b]_{\rho_2} \\
&\Leftrightarrow \alpha([[a]_{\rho_1}]_{\rho_2/\rho_1}) = \alpha([[b]_{\rho_1}]_{\rho_2/\rho_1}).
\end{aligned}$$

Obviously, α is surjective. It is actually a homomorphism, since, for all $a_1, \ldots, a_{n+1} \in G$,

$$\begin{aligned}
\alpha(\otimes([[a_1]_{\rho_1}]_{\rho_2/\rho_1}, \ldots, [[a_{n+1}]_{\rho_1}]_{\rho_2/\rho_1})) &= \alpha([\otimes([a_1]_{\rho_1}, \ldots, [a_{n+1}]_{\rho_1})]_{\rho_2/\rho_1}) \\
&= \alpha([[\circ(a_1, \ldots, a_{n+1})]_{\rho_1}]_{\rho_2/\rho_1}) \\
&= [\circ(a_1, \ldots, a_{n+1})]_{\rho_2} \\
&= \otimes([a_1]_{\rho_2}, \ldots, [a_{n+1}]_{\rho_2}) \\
&= \otimes(\alpha([[a_1]_{\rho_1}]_{\rho_2/\rho_1}), \ldots, \alpha([[a_{n+1}]_{\rho_1}]_{\rho_2/\rho_1})).
\end{aligned}$$

We conclude that a mapping α is an isomorphism from $(G/\rho_1)/(\rho_2/\rho_1)$ to G/ρ_2. This completed the proof. □

As an immediate consequence of Lemma 4.17 and Theorem 4.18, we have the following generalization:

Corollary 4.19 *Let G be a Menger algebra of rank n and let $\rho_1, \rho_2, \ldots, \rho_{m+1}$ be congruences on G such that $\rho_1 \subseteq \rho_2 \subseteq \ldots \subseteq \rho_{m+1}$. Then for each $i = 1, \ldots, m$, the relation*

$$\rho_{i+1}/\rho_i = \{([a]_{\rho_i}, [b]_{\rho_i}) \in G/\rho_i \times G/\rho_i \mid (a, b) \in \rho_{i+1}\}$$

is a congruence relation on G/ρ_i and

$$(G/\rho_i)/(\rho_{i+1}/\rho_i) \cong G/\rho_{i+1}.$$

In this situation, if we set a fixed natural number $m = 1$, then Corollary 4.19 and Theorem 4.18 are the same thing. Furthermore, Corollary 4.19 can be considered as a canonical generalization of the second isomorphism theorem for semigroups if $n = m = 1$.

5. Conclusion

Menger algebras of rank n are the core of the study in this paper. Every Menger algebra can be reduced to arbitrary semigroup in a natural way. In order to investigate the characterization of Menger algebras of rank n in sense of their translations, it is necessary to understood the concept of full n-place functions, a left translation, an inner left translation. So, in Section 3, the most significant knowledges and some interesting results concerning such tools in Menger algebras of rank n are presented. In the fourth section, we first explained what is meant by a congruence relation on Menger algebras of rank n and then discussed some of their properties. We also made an attention to form the quotient structure and define their multiplication in analog with those of ordinary semigroups and investigated several properties of such algebraic structure. The notion of isomorphisms is one of the most interesting concepts in the study of classical algebras that describe the relationship between quotients, homomorphisms, and subalgebras. Also, we constructed the isomorphism theorem for Menger algebras of rank n. It turns out that our main results are also noticeable extensions of semigroups if we set an arbitrary fixed natural number n is 1. Finally, the following problems are of interest to do in the near future:

(1) Define a right translation and an inner right translation for Menger algebras of rank n. Give a characterization of Menger algebras of rank n via these concepts.

(2) Construct a weakly reductive Menger algebras of rank n via the translations.

(3) Study the theory of (ideal) extensions of Menger algebras of rank n through translation as we already defined in this work.

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