KYUNGPOOK Math. J. 61(2021), 269-278
https://doi.org/10.5666/KMJ.2021.61.2.269
pISSN 1225-6951 eISSN 0454-8124

Coefficient Bounds for a Subclass of Harmonic Mappings Convex in One Direction

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Abstract. In this paper, we investigate harmonic univalent functions convex in the direction θ, for $\theta \in [0, \pi)$. We find bounds for $|f_z(z)|$, $|f_{\overline{z}}(z)|$ and $|f(z)|$, as well as coefficient bounds on the series expansion of functions convex in a given direction.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\Omega \subset \mathbb{C}$ if both u and v are real harmonic in Ω. In any simply connected domain $\Omega \subset \mathbb{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in Ω. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in Ω is that $|h^{'}(z)| > |g^{'}(z)|$ in Ω. (See [2]).

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for

* Corresponding Author.
Received August 6, 2020; revised December 14, 2020; accepted December 14, 2020.
2020 Mathematics Subject Classification: 30C45, 30C50.
Key words and phrases: harmonic, univalent, convex, shear construction.

$f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

If a univalent harmonic mapping $f = h + \overline{g}$ satisfies the condition

$$\left| \frac{g'(z)}{h'(z)} \right| \le k \le 1 \quad (z \in \mathbb{D}),$$

then f is called a harmonic K-quasiconformal mapping in $\mathbb{D}$ where $K = \frac{1+k}{1-k}$. Recently, several authors derived conditions for univalent harmonic mappings to be K-quasiconformal, see (for example) the works [1, 4, 5, 6, 8, 9, 10, 12] and the references therein.

A domain Ω is said to be convex in the direction θ for $\theta \in [0, \pi)$, if for all $a \in \mathbb{C}$, the set $\Omega \cap \{a + te^{i\theta} : t \in \mathbb{R}\}$ is either connected or empty. In particular, a domain is convex in the direction of the real (imaginary) axis if every line parallel to the real (imaginary) axis has either an empty intersection or a connected intersection with the domain. A function is said to be convex in the direction θ if it maps $\mathbb{D}$ univalently onto a domain convex in the direction θ.

2. Preliminaries

The *shear construction* introduced by Clunie and Sheil-Small in [2] provides a way of producing univalent harmonic functions in the unit disk which are convex in one direction. They proved the following interesting theorem.

Theorem 2.1.([2]) *A sense-preserving harmonic function $f = h + \overline{g}$ in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis if and only if $h - g$ is an analytic univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis.*

Moreover, they proved by the following theorem that Theorem 2.1 would be generalized to a convex domain in the direction θ.

Theorem 2.2.([2]) *A harmonic function $f = h + \overline{g}$, locally univalent in $\mathbb{D}$, is a univalent mapping of onto a domain convex in the direction θ if and only if $h - e^{2i\theta} g$ is an analytic univalent mapping in $\mathbb{D}$ onto a domain convex in the direction θ.*

Hengartner and Schober [3] studied analytic functions ψ that are convex in the direction of the imaginary axis. They used a normalization which basically depends on the right and left extremes of $\psi(\mathbb{D})$ being the images of 1 and -1. Actually, this method deals with existing the sequences $\{z'_n\}$ converging to $z = 1$ and $\{z''_n\}$ converging to $z = -1$ such that

$$\begin{aligned} \lim_{n\to\infty} \mathbf{Re}\{\psi(z'_n)\} &= \sup_{|z|<1} \mathbf{Re}\{\psi(z)\}, \\ \lim_{n\to\infty} \mathbf{Re}\{\psi(z''_n)\} &= \inf_{|z|<1} \mathbf{Re}\{\psi(z)\}. \end{aligned} \tag{2.1}$$

Let CIA be the class of domains Ω which are convex in direction the imaginary axis and admit a mapping ψ such that $\psi(\mathbb{D}) = \Omega$ and ψ satisfies the normalization (2.1), then we have the following result:

Theorem 2.3.([3]) *Suppose ψ is analytic and nonconstant for $|z| < 1$. Then we have $\mathbf{Re}\{(1 - z^2)\psi'(z)\} \geq 0$ for $|z| < 1$ if and only if*

(1) *ψ is univalent on $\mathbb{D}$,*

(2) *$\psi(\mathbb{D}) \in CIA$, and*

(3) *ψ is normalized by (2.1).*

Using this characterization of functions, Hengartner and Schober then proved the next theorem:

Theorem 2.4.([3]) *If ψ is analytic for $|z| < 1$ and satisfies $\mathbf{Re}\{(1 - z^2)\psi'(z)\} \geq 0$, then for $|z| \leq r < 1$,*

$$\frac{|\psi'(0)|(1-r)}{(1+r)(1+r^2)} \leq |\psi'(z)| \leq \frac{|\psi'(0)|}{(1-r)^2}. \tag{2.2}$$

The upper bound is sharp for $\psi(z) = \dfrac{z}{1-z}$, which maps $\mathbb{D}$ onto the right half-plane $\mathbf{Re}\{z\} > \frac{1}{2}$, and the lower bound is sharp for $\psi(z) = (\frac{i}{2})\log\left(\dfrac{(1-iz)^2}{(1-z^2)}\right)$, which maps $\mathbb{D}$ onto a vertical strip, slit from $i\log\sqrt{2}$ to infinity along the positive imaginary axis.

To be able to manipulate these consistent results for the specific functions that are convex in the proper direction of θ, $(0 \leq \theta < \pi)$, let us consider the following typical situation. Suppose that $\varphi(z)$ is a function that is analytic and convex in the direction of θ, $(0 \leq \theta < \pi)$. Furthermore, suppose that the $\varphi(z)$ is normalized a follows.

Let $\{w'_n\}$ and $\{w''_n\}$ be sequences such that $w'_n \to e^{-i\alpha}$ and $w''_n \to -e^{-i\alpha}$ and

$$\begin{aligned} \lim_{n\to\infty} \mathbf{Im}\left\{e^{-i\theta}\varphi(w'_n)\right\} &= \sup_{|w|<1} \mathbf{Im}\left\{e^{-i\theta}\varphi(w)\right\}, \\ \lim_{n\to\infty} \mathbf{Im}\left\{e^{-i\theta}\varphi(w''_n)\right\} &= \inf_{|w|<1} \mathbf{Im}\left\{e^{-i\theta}\varphi(w)\right\}. \end{aligned} \tag{2.3}$$

Then $z'_n = e^{i\alpha}w'_n \to 1$ and $z''_n = e^{i\alpha}w''_n \to -1$ as $n \to \infty$. Now define

$$\psi(z) := ie^{-i\theta}\varphi(e^{-i\alpha}z). \tag{2.4}$$

In this case, we have

$$\begin{aligned}\lim_{n\to\infty} \mathbf{Re}\left\{\psi(z'_n)\right\} &= \lim_{n\to\infty} \mathbf{Re}\left\{ie^{-i\theta}\varphi(e^{-i\alpha}z'_n)\right\} \\ &= \lim_{n\to\infty} \mathbf{Im}\left\{e^{-i\theta}\varphi(e^{-i\alpha}z'_n)\right\} \\ &= \sup_{|z|<1} \mathbf{Im}\left\{e^{-i\theta}\varphi(w'_n)\right\} \\ &= \sup_{|z|<1} \mathbf{Re}\left\{ie^{-i\alpha}\varphi(w'_n)\right\} \\ &= \sup_{|z|<1} \mathbf{Re}\left\{\psi(z'_n)\right\}.\end{aligned}$$

Similarly, one can shows that

$$\lim_{n\to\infty} \mathbf{Re}\left\{\psi(z''_n)\right\} = \inf_{|z|<1} \mathbf{Re}\left\{\psi(z''_n)\right\},$$

which shows that ψ defined in (2.4) satisfies (2.1). Therefore ψ is univalent and convex in the direction of the imaginary axis in Theorem 2.4, satisfying the conditions of Theorem 2.3. Replacing the definition of ψ as (2.4) in Theorem 2.3, we have

$$\mathbf{Re}\{(1-z^2)\psi'(z)\} = \mathbf{Re}\{ie^{-i\theta}(1-z^2)\varphi'(z)\} \geq 0$$

Therefore we can apply Theorem 2.4 to $\varphi(z)$. The result still holds with $\psi(z)$ replaced by $\varphi(z)$.

Let $\mathcal{S}^0_{\mathcal{H}}(k,\theta)$ denote the subclass of $\mathcal{S}^0_{\mathcal{H}}$ consisting of functions $f = h + \overline{g}$ that $\left|\frac{g'(z)}{h'(z)}\right| \leq k < 1$, convex in the direction of θ for $0 \leq \theta < \pi$ and $\varphi := h - e^{2i\theta}g$ which satisfy normalization (2.3).

3. Growth and Distortion Theorems

In this section we study the class $\mathcal{S}^0_{\mathcal{H}}(k,\theta)$ and find bounds for $|f_z(z)|$, $|f_{\overline{z}}(z)|$ and $|f(z)|$, as well as coefficient bounds on the series expansion of harmonic quasi conformal mappings that are univalent and convex in a given direction. As is done in the litrature for Theorems 2.1 and 2.2, let $\varphi = h - e^{2i\theta}g$ and $w = \frac{g'}{h'}$.

Theorem 3.1. *Let $f = h + \overline{g} \in \mathcal{S}^0_{\mathcal{H}}(k,\theta)$. For $|z| \leq r$, we have*

$$\frac{1-r}{(1+kr)(1+r)(1+r^2)} \leq |f_z(z)| \leq \frac{1}{(1-kr)(1-r)^2} \tag{3.1}$$

and

$$\frac{|\omega(z)|(1-r)}{(1+kr)(1+r)(1+r^2)} \leq |f_{\overline{z}}(z)| \leq \frac{kr}{(1-kr)(1-r)^2}. \tag{3.2}$$

Equality is obtained for both upper bounds when

$$\varphi(z) = e^{i\theta}z/(1-iz) \quad \textit{and} \quad \omega(z) = ke^{(\pi/2-2\theta)i}z,$$

and for the lower bounds when

$$\varphi(z) = (e^{i\theta}/2)\log((1+z)^2/(1+z^2)) \quad \textit{and} \quad \omega(z) = ke^{(\pi-2\theta)i}z.$$

Proof. Since $\varphi' = h' - e^{2i\theta}g'$ and $g' = \omega h'$ we have

$$\begin{aligned} f_z(z) = h'(z) &= \big(\varphi'(z)/(1-e^{2i\theta}\omega(z))\big) \\ \overline{f_{\overline{z}}(z)} = g'(z) &= \big(\omega(z)\varphi'(z)/(1-e^{2i\theta}\omega(z))\big) \end{aligned}$$

Now $(\omega(z)/k)$ is a Schwarz function, therefore

$$|f_z(z)| = \left|\frac{\varphi'(z)}{1-e^{2i\theta}\omega(z)}\right| \leq \frac{|\varphi'(z)|}{1-|e^{2i\theta}||\omega(z)|} \leq \frac{|\varphi'(z)|}{1-k|z|}.$$

Furthermore,

$$|f_z(z)| \geq \frac{|\varphi'(z)|}{1+|e^{2i\theta}||\omega(z)|} \geq \frac{|\varphi'(z)|}{1+k|z|}.$$

Using Theorem 2.4 gives inequality (3.1). Similarly,

$$|f_{\overline{z}}(z)| = \left|\frac{\omega(z)\varphi'(z)}{1-e^{2i\theta}\omega(z)}\right| \leq \frac{|\omega(z)||\varphi'(z)|}{1-|e^{2i\theta}||\omega(z)|} \leq \frac{k|z|}{1-k|z|}|\varphi'(z)|,$$

and

$$|f_{\overline{z}}(z)| \geq \frac{|\omega(z)||\varphi'(z)|}{1+|e^{2i\theta}||\omega(z)|} \geq \frac{|\omega(z)||\varphi'(z)|}{1+k|z|}.$$

Applying Theorem 2.4 again yields (3.2).

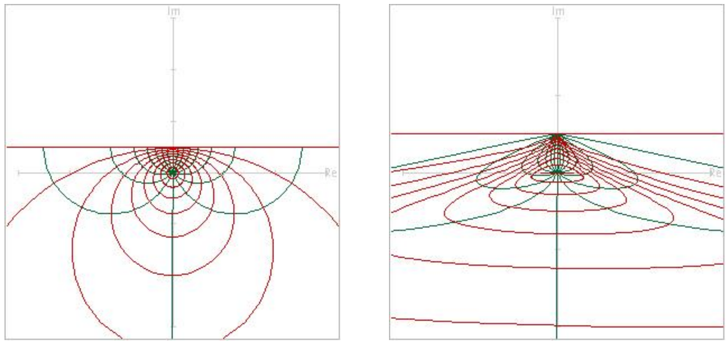
The sharpness of the functions comes from examining the sharpness of the functions for Theorem 2.4. Let $\varphi(z) = -ie^{i\theta}\psi(e^{-i\alpha}z)$, and wisely choose the analytic dilatation $\omega(z)$ and α. □

The mapping properties of these functions are shown in Figures 1 and 2. The figures illustrate the images of concentric circles and equally spaced rays.

Theorem 3.2. *Let* $f = h + \overline{g} \in \mathcal{S}^0_{\mathcal{H}}(k,\theta)$. *For* $|z| \leq r$, *we have*

$$|f(z)| \leq \frac{2k}{(1-k)^2}\ln\left(\frac{1-r}{1-kr}\right) + \frac{(1+k)r}{2(1-k)(1-r)}. \tag{3.3}$$

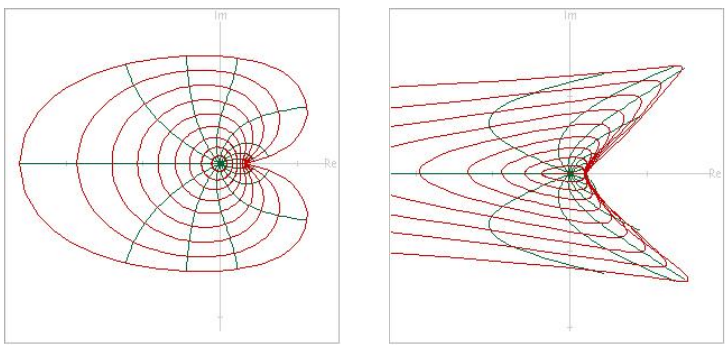
Proof. Since $f(z) = h(z) + \overline{g(z)}$, we have the following equalities:



(a) Image of $\mathbb{D}$ under mapping $\varphi(z) = (z/(1+iz))$

(b) shear of $\varphi(z) = (z/(1+iz))$

Figure 1: The shear of $\varphi(z) = (z/(1+iz))$ with $k = 1, \theta = 0$.



(a) Image of $\mathbb{D}$ under mapping $\varphi(z) = 1/2\log((1+z)^2/(1+z^2))$

(b) shear of $\varphi(z) = 1/2\log((1+z)^2/(1+z^2))$

Figure 2: The shear of $\varphi(z) = 1/2\log((1+z)^2/(1+z^2))$ with $k = 1, \theta = 0$.

$$
\begin{aligned}
f(z) &= h(z) + \overline{g(z)} \\
&= \int_0^r h'(\rho e^{i\gamma}) e^{i\gamma} d\rho + \overline{\int_0^r g'(\rho e^{i\gamma}) e^{i\gamma} d\rho} \\
&= \int_0^r h'(\rho e^{i\gamma}) e^{i\gamma} d\rho + \int_0^r \overline{g'(\rho e^{i\gamma})} e^{-i\gamma} d\rho \\
&= \int_0^r f_z(\rho e^{i\gamma}) e^{i\gamma} d\rho + \int_0^r f_{\overline{z}}(\rho e^{i\gamma}) e^{-i\gamma} d\rho.
\end{aligned}
\tag{3.4}
$$

Thus

$$|f(z)| = |h(z) + g(z)| \le |h(z)| + |g(z)| \\ \le \int_0^r |f_z(\rho e^{i\gamma})| d\rho + \int_0^r |f_{\overline{z}}(\rho e^{i\gamma})| d\rho. \tag{3.5}$$

Applying inequalities in Theorems 3.1 and 3.2 to (3.4) and (3.5) yields

$$|f(z)| \le \int_0^r \frac{1}{(1-k\rho)(1-\rho)^2} d\rho + \int_0^r \frac{k\rho}{(1-k\rho)(1-\rho)^2} d\rho \\ = \frac{2k}{(1-k)^2} \ln\left(\frac{1-r}{1-kr}\right) + \frac{(1+k)r}{2(1-k)(1-r)}.$$

□

Corollary 3.3. *In Thorem 3.2, if* $|f_{\bar{z}}| \le k|f_z|$ *then*

$$|f(z)| \le \frac{2k(1+k)}{(1-k)^2} \ln(\frac{1-r}{1-kr}) \quad |z| \le 1.$$

It is easy to check that if f *is conformal, i.e.* $k = 0$ *then the estimate is sharp because the estimate of* f_z *is sharp.*

Sheil-Small [11] proved that if $f \in \mathcal{S}^0_{\mathcal{H}}$ and $f(\mathbb{D})$ is convex in one direction, then the following bounds hold for the coefficients:

$$|a_n| \le \frac{(n+1)(2n+1)}{6}, \quad |b_n| \le \frac{(n-1)(2n-1)}{6},$$

where $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k + \overline{\sum\limits_{k=1}^{\infty} b_k z^k}$.

In Theorems 3.1 and 3.2, we described how the geometry of the related analytic function $\varphi(z)$ affects bounds of a harmonic function and its derivatives. The following theorem shows how the geometry of $\varphi(z)$ affects the coefficients $|a_n|$ and $|b_n|$. We begin by looking at Hengartner and Schober's result in [3].

Theorem 3.4.([3]) *If* $\psi(z) = a_0 + (\alpha + i\beta)z + \sum\limits_{k=2}^{\infty} a_k z^k$ *is analytic in* $\mathbb{D}$ *and satisfies* $\mathbf{Re}\left\{(1-z^2)\psi'(z)\right\} \ge 0$*, then*

$$|a_n| \le \alpha \quad \text{for} \quad n = 2, 4, 6, \dots \tag{3.6}$$

and

$$|a_n| \le \left(1 - \frac{1}{n}\right)\alpha + \frac{1}{n}|\alpha + i\beta| \quad \text{for} \quad n = 1, 3, 5, \dots \tag{3.7}$$

Consequently

$$|a_n| \le |\psi'(0)| \quad \text{for} \quad n = 1, 2, 3, 4, \dots \tag{3.8}$$

Equality is obtained in all three inequalities by $\psi(z) = 1/(1-z)$. Furthermore, among bounds which depend on both α and β, (3.6) *is sharp for the function*

$$\psi(z) = \frac{\alpha}{1-z} + \frac{\beta i}{2}\log\frac{1+z}{1-z}, \quad \alpha > 0.$$

Again, suppose that $\varphi(z)$ is a function that is analytic and convex in the direction of $\theta(0 \le \theta < 1)$. Furthermore, let $\varphi(z)$ be normalized by (2.3). By setting

$$\psi(z) := ie^{-i\theta}\varphi(e^{-i\alpha}z) \tag{3.9}$$

the function ψ satisfies (2.1). Thus ψ is univalent and convex in the direction of the imaginary axis in $\mathbb{D}$. Making use of Theorem 2.2 for ψ, we have

$$\mathbf{Re}\{(1-z^2)\psi'(z)\} = \mathbf{Re}\{ie^{-i\theta}(1-z^2)\varphi'(z)\} \ge 0.$$

Therefore, we can apply Theorem 2.4 to $\varphi(z)$, obtain that the result still holds, with $\psi(z)$ replaced by $\varphi(z)$.

Theorem 3.5. *Let $f = h + \overline{g} \in \mathcal{S}^0_{\mathcal{H}}(k,\theta)$. Then for $|z| \le r$, we have*

$$|a_n| \le \frac{n+1}{2}, \quad |b_n| \le \frac{n-1}{2}, \quad \textit{for} \quad n \ge 2.$$

Proof. Initiating with the integral representations

$$h(z) = \int_0^z \frac{\varphi'(\zeta)}{1-e^{2i\theta}\omega(\zeta)}d\zeta, \quad g(z) = \int_0^z \frac{\omega(\zeta)\varphi'(\zeta)}{1-e^{2i\theta}\omega(\zeta)}d\zeta,$$

where $\omega(z) = (g'(z)/h'(z))$. Let

$$\varphi(z) = \sum_{n=1}^{\infty}\phi_n z^n,$$

and

$$\frac{\omega(z)}{1-e^{2i\theta}\omega(z)} = \sum_{n=1}^{\infty} w_n z^n.$$

For

$$\begin{aligned} g(z) &= \int_0^z \left[\phi_1 + 2\phi_2\zeta + 3\phi_3\zeta^3 + \dots\right]\left[w_1\zeta + w_2\zeta^2 + w_3\zeta^3 + \dots\right] d\zeta \\ &= \int_0^z \left[\phi_1 w_1\zeta + (\phi_1 w_2 + 2\phi_2 w_1)\zeta^2 + (\phi_1 w_3 + 2\phi_2 w_2 + 3\phi_3 w_1)\zeta^3 + \dots\right] d\zeta \\ &= \frac{1}{2}(\phi_1 w_1)z^2 + \frac{1}{3}(\phi_1 w_2 + 2\phi_2 w_1)z^3 + \frac{1}{4}(\phi_1 w_3 + \phi_2 w_2 + 3\phi_3 w_1)z^4 + \dots. \end{aligned}$$

We have

$$\begin{aligned} b_1 &= 0, \\ b_2 &= \frac{1}{2}(\phi_1 w_1), \\ b_3 &= \frac{1}{3}(\phi_1 w_2 + 2\phi_2 w_1), \\ &\vdots \\ b_n &= \frac{1}{n}\sum_{k=1}^{n-1} k\phi_k w_{n-k} \quad \text{for} \quad n \geq 2. \end{aligned}$$

Now for $h(z)$,

$$\begin{aligned} h(z) &= \int_0^z \varphi'(\zeta)\frac{1}{1-e^{2i\theta}\omega(\zeta)}d\zeta \\ &= \int_0^z \varphi'(\zeta)e^{2i\theta}\left(\frac{\omega(\zeta)}{1-e^{2i\theta}\omega(\zeta)} + \frac{1}{e^{2i\theta}}\right)d\zeta \\ &= e^{2i\theta}\left(\int_0^z \varphi'(\zeta)\frac{\omega(\zeta)}{1-e^{2i\theta}\omega(\zeta)}d\zeta\right) + \int_0^z \varphi'(\zeta)d\zeta. \end{aligned}$$

Therefore

$$a_n = e^{2i\theta}b_n + \phi_n = \phi_n + e^{2i\theta}\left(\frac{1}{n}\sum_{k=1}^{n-1} k\phi_k w_{n-k}\right).$$

Since $\omega(z)/(1-e^{2i\theta}\omega(z))$ is subordinated by $z/(1-e^{2i\theta}z)$, we have $|w_n| \leq 1$ for all natural n by [7, p.238]. Also, the descriptions following theorem **??** concludes that $|\phi_k| \leq |\phi'(0)| = 1$. In consequence:

$$|b_n| = |\frac{1}{n}\sum_{k=1}^{n-1} k\phi_k w_{n-k}| \leq \frac{1}{n}\sum_{k=1}^{n-1} k|\phi_k||w_{n-k}| \leq \frac{1}{n}\sum_{k=1}^{n-1} k = \frac{n-1}{2}.$$

Similarly, we have

$$|a_n| \leq |\phi_n| + |e^{2i\theta}||\frac{1}{n}\sum_{k=1}^{n-1} k\phi_k w_{n-k}| = \frac{1}{n}\sum_{k=1}^{n} k = \frac{n+1}{2}. \qquad \square$$

Acknowledgements. The authors thanks to the referee, especially for his comment on Corollary 3.3. A part of this research was carried out while the third author was visiting the university of Alberta. The author is grateful to his colleagues in the department of mathematical and statistical siecnces for their kind hosting.

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