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Synchronization of Non-integer Chaotic Systems with Uncertainties, Disturbances and Input Non-linearities

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Abstract. In this paper, we examine and analyze the concept of different non-integer chaotic systems with external disturbances, uncertainties, and input non-linearities. We consider both drive and response systems with external bounded disturbances and uncertainties. We also consider non-linear control inputs. For synchronization, we introduce the adaptive sliding mode technique, in which we establish the stability of the controlled system by a control which estimates uncertainties and disturbances, and then applies a suitable sliding surface to control them. We use computer simulations to established the efficacy and adeptness of the prospective scheme.

1. Introduction

In recent times, non-linear dynamical systems have become a hot topic among researchers. Discovered by Henri Poincaré [26], chaos is a complex phenomenon found in most non-linear dynamical systems, describing the sensitive dependence of the evolution of the system on the initial conditions. Poincaré observed that two neighboring points in state space can very quickly become isolated. The phenomenon of chaos has deep applications in viscoelasticity [14], dielectric polarization, electromagnetic waves [9], diffusion, signal processing, mathematical biology and, of course, chaotic systems. Different procedures have been used to investigate the behavior of the chaotic non-linear systems that surround us. Among these procedures are plotting phase portraits, poincaré sections, or bifurcation diagrams, or finding Lyapunov exponents.

To understand the behaviour of non-linear systems and to stabilize their control, Pecora and Carroll established the idea of synchronization. Under synchronization,

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trajectories of coupled systems evolve together in a usual pattern. Based on different control techniques such as adaptive backstepping [27], linear and nonlinear feedback synchronization [3], active control [25], sliding mode control [6], adaptive sliding mode technique [12], and time delay feedback [5], researchers have developed many synchronization schemes– schemes such as complete and anti synchronization [10, 21], phase and anti-phase synchronization [20], projective and hybrid function projective synchronization [13, 22], generalised synchronization [28].

To combine the concept of integer order differentiation and n-fold integration, Leibnitz and L'Hospital in 1675 gave the theory of integrals and derivatives of arbitrary order. Systems represented by non-integer differential equations [23] have been studied extensively in recent years. These studies focus on real-life systems and have many multidisciplinary applications. Specifically, it has been seen that non-integer systems, which generalize many well-known integer order systems, have chaotic and hyper-chaotic behavior. Some such systems are Lorenz systems [8], Chen systems, Rössler systems [15], Liu-systems [6], Genesio-Tesi systems [7], Chua systems [29], complex t-system [19] and complex Lu-systems [24].

In this manuscript, we synchronize two different non-integer chaotic systems. We treat a non-integer chaotic Liu-system [6] as drive a system, and a chaotic Genesio-Tesi system [7] as response a system. We do so with model uncertainties, and external bounded disturbances, but also with non-linear input. The consideration of these elements together seems to be novel in the literature. As uncertainties and disturbances introduce a dreadful change in chaotic systems, dynamics, and synchronization reduce this. Researchers have introduced various schemes [4, 12] to examine the synchronization of chaotic systems with various disturbances and uncertainties. Generally the sliding mode control technique is an efficient approach for dealing with uncertainties and disturbances.

In practice when we encounter a controller in a real-life systems, physical limitations cause some non-linearity in the control inputs. It has been shown that non-linear input can cause a severe decay in the system performance. If the controller is poorly designed, then system failure becomes worse. Therefore, non-linear input effects must be taken into consideration when evaluating and implementing a control scheme for chaotic systems. Researchers have designed various techniques to synchronize integer-order chaotic systems with non-linear inputs [2, 16]. It seems, however, synchronization among non-integer chaotic systems in the presence of external disturbances, model uncertainties and non-linear input has not been discovered. In our paper, we investigate the synchronization under these perturbations. We introduce an adaptive sliding mode control scheme to synchronize the considered systems. We estimated the disturbances and uncertainties through a adaptive control rule and we chose a suitable sliding surface to counter their effect. We design the appropriate controllers using known control techniques and Lyapunov stability theory.

As motivated above, we summarize here the main aspects of the this paper.

1. We propose a novel synchronization scheme for non-integer chaotic systems.

2. We design non-linear control inputs for non-integer chaotic systems.
3. We compare our proposed methodology with the previously published literature. We consider disturbances, uncertainties, and non-linear input. Even with this, our methodology yields better results than the previously publishedliterature.
4. We ilustrate an application in secure communication in Section 5.
5. We use numerical simulations to validate and visualize our results.

2. Preliminaries and Problem Formulation

The fractional order derivative can be defined in various forms [23] such as Riemann-Lioville's derivative, Grünwald Letnikov's derivative, Caputo's derivative. Here we have taken Caputo's derivative defined as

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \qquad t > t_0 $$

where $\alpha \in \mathbb{R}^+$ and $\Gamma(.)$ is the Gamma function.

Consider the drive system of n dimensions with model uncertainties and external disturbances

$$ D^{\alpha} u_i = f_i(u_1, u_2....u_n) + \Delta f_i(u_1, u_2....u_n, t) + d_i(t) \tag{2.1} $$

for $i = 1, \dots, n$, where $u(t) = [u_1, u_2, ..., u_n]^T \in \mathbb{R}^n$ are state variables of the system (2.1), $f_i(u) : \mathbb{R}^{n\times 1} \to \mathbb{R}$, are continuous functions, $\Delta f_i(u)$ are model uncertainties and $d_i(t)$ are external disturbances for $i = 1, 2, 3, ..., n$.

Consider the response system of n dimensions with model uncertainties, external disturbances, and non-linear control inputs as

$$ D^{\alpha} v_i = g_i(v_1, v_2....v_n) + \Delta g_i(v_1, v_2....v_n, t) + d_i'(t) + \Phi_i(U_i) \tag{2.2} $$

for $i = 1, \dots, n$, where $v(t) = [v_1, v_2, ..., v_n]^T \in \mathbb{R}^n$ are state variables of the system (2.2), $g_i(v) : \mathbb{R}^{n\times 1} \to \mathbb{R}$ are continuous functions, $\Delta g_i(v)$ are model uncertainties, $d_i'(t)$ are external disturbances, and $\Phi_i(U_i)$ are the non-linear control inputs for controller U_i $i = 1, 2, 3, ..., n$.

Assumption 1. The trajectories of non-integer chaotic systems are bounded so here we have assumed that the model uncertainties $\Delta f_i(u)$ and $\Delta g_i(v)$ are bounded. This implies that there exist constants $\vartheta_i^m > 0$ and $\vartheta_i^s > 0$ such that

$$ |\Delta f_i(u)| < \vartheta_i^m \qquad \text{and} \qquad |\Delta g_i(v)| < \vartheta_i^s. $$

Consequently we have

$$ |\Delta f_i(u) - \Delta g_i(v)| < \vartheta_i $$

for $i = 1, 2, ..., n$.

Assumption 2. It is assumed that if external disturbances $d_i(t)$ and $d_i'(t)$ are norm bounded then there exist constants $\nu_i^m > 0$ and $\nu_i^s > 0$ such that

$$|d_i(t)| < \nu_i^m \qquad \text{and} \qquad |g_i(v)| < \nu_i^s.$$

Thus for $i = 1, 2, ..., n$, we get

$$|\Delta f_i(u) - \Delta g_i(v)| < \nu_i.$$

Assumption 3. The control inputs $\Phi_i(U_i)$ are continuous non-linear functions and satisfy

$$\omega_i U_i^2 \leq U_i \Phi_i(U_i) \leq \eta_i U_i^2$$

where $i = 1, 2, ..., n$ and $\omega_i, \eta > 0$ are constant parameters.

In order to achieve synchronization, here we define synchronization error as $e_i = u_i - v_i$, for $i = 1, 2, ..., n$. The error dynamics is attained as

$$\begin{aligned} D^\alpha e_i &= g_i(v_1, v_2....v_n) + \Delta g_i(v_1, v_2....v_n, t) + d_i'(t) \\ &\quad - f_i(u_1, u_2....u_n) - \Delta f_i(u_1, u_2....u_n, t) - d_i(t) + \Phi_i(U_i) \end{aligned} \tag{2.3}$$

for $i = 1, \ldots, n$.

To achieve synchronization, we have to establish that the error system (2.3) is stable. For that our aim is to design control laws for any two non-integer chaotic systems with model uncertainties, external disturbances, and non-linear input to established that it is stable asymptotically: $\lim_{t\to\infty} \parallel e_i(t) \parallel = 0, i = 1, 2, ..., n$.

To minimize the error, we choose the suitable sliding surface which is as follows:

$$s_i(t) = \mu_i D^{\alpha-1} e_i(t) + \int_0^t e_i(\xi) d\xi \tag{2.4}$$

where $s(t) \in \mathbb{R}, s(t) = [s_1, s_2, ..., s_n]^T$ and the sliding surface parameters $\mu_i, i = 1, 2, .., n$ are chosen in such a manner that they are positive.

To discuss the error system (2.3) at the chosen sliding surface (2.4), it is necessary that it should satisfy the following condition for $i = 1, 2, .., n$

$$s_i(t) = 0, \dot{s_i}(t) = 0 \tag{2.5}$$

The derivative of (2.4) yields the following equation

$$\dot{s_i}(t) = \mu_i D^\alpha e_i(t) + e_i(t) \tag{2.6}$$

Then, by considering the necessary condition $\dot{s_i}(t) = 0$, we obtain

$$D^\alpha e_i(t) = -\frac{1}{\mu_i} e_i(t) \tag{2.7}$$

Hence, the system (2.3) is asymptotically stable which shows that the slave system (2.2) can be tackled by the master system (2.1) by constructing the appropriate control inputs.

Our next step is to design the appropriate control inputs in order to stabilize the error system and attain synchronization on the chosen sliding surface $s(t) = 0$. The control inputs are designed as follows

$$U_i = \left[\frac{1}{\omega_i}\left(\frac{1}{\mu_i}|e_i| + |g_i - f_i| + \hat{\vartheta}_i + \hat{\nu}_i + \lambda_i\right)\right] sign(s_i) = \zeta_i sign(s_i) \tag{2.8}$$

where $\hat{\vartheta}_i$ and $\hat{\nu_i}$ are estimates of ϑ_i and ν_i respectively and λ_i are switching gain.

The adaptive laws are chosen as

$$\dot{\hat{\vartheta}}_i = \dot{\hat{\nu}}_i = \mu_i|s_i|, i = 1, 2, .., n. \tag{2.9}$$

Theorem 2.1. *For the error system (2.3) with control laws (2.8) and adaptive laws (2.9), if the following condition is fulfilled:*

$$(\mu_i\eta_i - \lambda_i) < 0, \tag{2.10}$$

then the synchronization error converges to $s_i = 0$. Thereby the synchronization between (2.1) and (2.2) can be achieved.

Proof. Consider the Lyapunov function given as

$$V_i = \frac{1}{2}\sum_{i=1}^{N}[s_i^2 + (\hat{\vartheta}_i - \vartheta_i)^2 + (\hat{\nu}_i - \nu_i)^2]$$

The derivative of V_i is

$$\dot{V}_i = \sum_{i=1}^{N}[s_i\dot{s}_i + (\hat{\vartheta}_i - \vartheta_i)\dot{\hat{\vartheta}}_i + (\hat{\nu}_i - \nu_i)\dot{\hat{\nu}}_i]$$

By substituting the value of $\dot{s}_i$,

$$\dot{V}_i = \sum_{i=1}^{N}[s_i(\mu_i D^\alpha e_i(t) + e_i(t)) + (\hat{\vartheta}_i - \vartheta_i)\dot{\hat{\vartheta}}_i + (\hat{\nu}_i - \nu_i)\dot{\hat{\nu}}_i]$$

Using adaptive laws (2.9) and substituting the values of $D^\alpha e_i(t)$, we obtain

$$\begin{aligned}\dot{V}_i = &\sum_{i=1}^{N}[s_i(\mu_i(g_i + \Delta g_i + d_i'(t) - f_i - \Delta f_i - d_i(t) + \Phi_i(U_i)) + e_i(t)) \\ &+ (\hat{\vartheta}_i - \vartheta_i)(\mu_i|s_i|) + (\hat{\nu}_i - \nu_i)(\mu_i|s_i|)]\end{aligned}$$

Using Assumption (2) $\Phi_i(U_i) \leq \eta_i U_i$ and $-s_i\varphi_i(U_i) \leq -\omega\zeta_i|s_i|$, where $U_i = \zeta_i sign(s_i)$ and (2.3), we have

$$\begin{aligned}\dot{V}_i \leq & \sum_{i=1}^{N}[s_i(\mu_i(g_i + \Delta g_i + d_i'(t) - f_i - \Delta f_i - d_i(t) + \Phi_i(U_i)) + \mu_i\eta_i + e_i(t)) \\ & + (\hat{\vartheta}_i - \vartheta_i)(\mu_i|s_i|) + (\hat{\nu}_i - \nu_i)(\mu_i|s_i|)] \\ & \leq \sum_{i=1}^{N}[|s_i||e_i| + |s_i|\mu_i\eta_i + |s_i|\mu_i|g_i - f_i| - s_i\mu_i\Phi_i(U_i) \\ & + \hat{\vartheta}_i\mu_i|s_i| + \hat{\nu}_i\mu_i|s_i|] \\ & \leq \sum_{i=1}^{N}[|s_i||e_i| + |s_i|\mu_i\eta_i + |s_i|\mu_i|g_i - f_i| - \mu_i\omega\zeta_i|s_i| \\ & + \hat{\vartheta}_i\mu_i|s_i| + \hat{\nu}_i\mu_i|s_i|]\end{aligned}$$

Substituting $\zeta_i = \frac{1}{\omega_i}\left(\frac{1}{\mu_i}|e_i| + |g_i - f_i| + \hat{\vartheta}_i + \hat{\nu}_i + \lambda_i\right)$ into the above inequality:

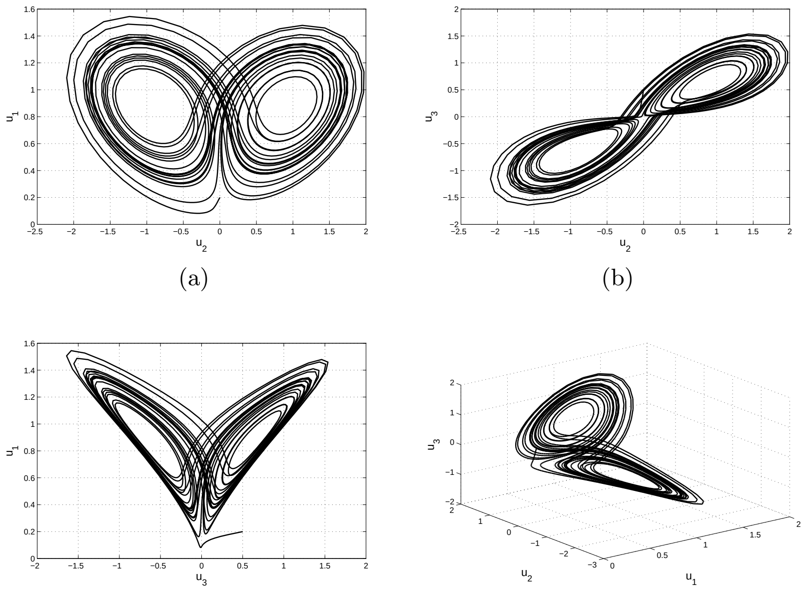
$$\begin{aligned}\dot{V}_i \leq & \sum_{i=1}^{N}[|s_i||e_i| + |s_i|\mu_i\eta_i + |s_i|\mu_i|g_i - f_i| \\ & - \mu_i\omega\frac{1}{\omega_i}\left(\frac{1}{\mu_i}|e_i| + |g_i - f_i| + \hat{\vartheta}_i + \hat{\nu}_i + \lambda_i\right)|s_i| + \hat{\vartheta}_i\mu_i|s_i| \\ & + \hat{\nu}_i\mu_i|s_i|]\end{aligned}$$

Using equations (2.8), (2.9) and (2.10) and simplifying, we get the following inequality:

$$\begin{aligned}\dot{V}_i & \leq \sum_{i=1}^{N}[(\mu_i\eta_i - \lambda_i)|s_i|] \\ & = \sum_{i=1}^{N}[-(\lambda_i - \mu_i\eta_i)|s_i|] \\ & = \sum_{i=1}^{N}[-\Theta_i|s_i|] \\ & = -\Theta_i|s_i| = \Omega_i(\xi) \leq 0.\end{aligned}$$

Integrating the above equation from 0 to t yields

$$V_i(0) \geq V_i(t) + \int_0^t \Omega_i(\xi)d\xi.$$



(a) (b)

(c) (d)

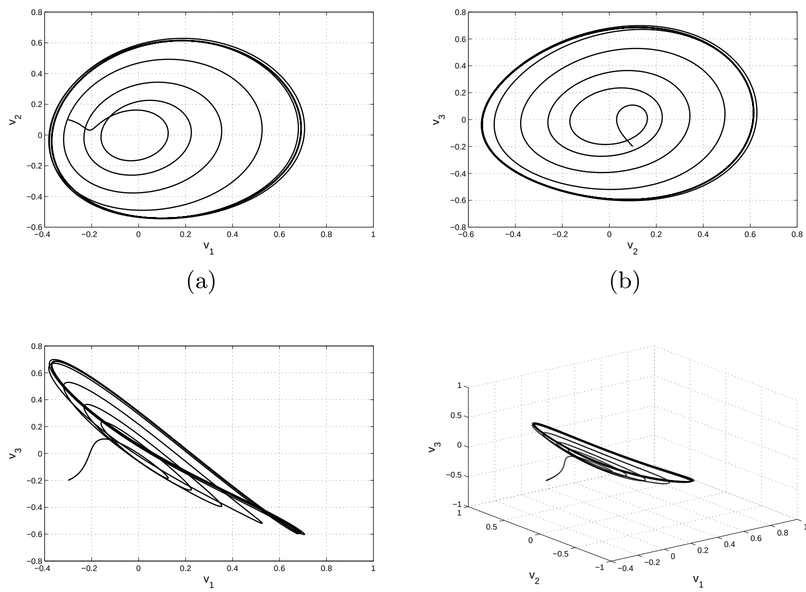
Fig.1: Phase Portraits of fractional order Liu chaotic system for $\alpha = 0.95$
(a)$u_2 - u_1$ axis (b)$u_2 - u_3$ axis (c)$u_3 - u_1$ axis (d)$u_1 - u_2 - u_3$ axis.

Since $\dot{V}_i(t) < 0, V_i(0) - V_i(t) \geq 0$ is positive and finite, the limit $\lim_{t\to\infty} \Omega_i(\xi)$ exists and is finite (i.e. $\lim_{t\to\infty} \Omega_i(\xi) = V_i(0) - V_i(t) \geq 0$). Using the Barbalat Lemma ([11, lemma 8.2]), $\lim_{t\to\infty} \int_0^t \Omega_i(\xi) d\xi = 0$, which implies $|s_i| = 0$. Thus the error dynamical system is asymptotically stable. Hence, synchronization is achieved between non-integer chaotic systems on the considered stationary surface. This completes the proof. □

Remark 2.2.([1, 17]) The signum function behaves as a rigid switcher in the prospective control law and it can cause chattering. Therefore, we modify the controller to prevent chattering:

$$U_i = \left[\frac{1}{\omega_i}\left(\frac{1}{\mu_i}|e_i| + |g_i - f_i| + \hat{\vartheta}_i + \hat{\nu}_i + \lambda_i\right)\right] tanh(\delta_i s_i) \tag{2.11}$$

where $\delta_i > 0$ is a constant.



(a) (b) (c) (d)

Fig.2: Phase Portraits of fractional order Genesio-Tesi chaotic system for $\alpha = 0.95$ (a) $v_1 - v_2$ axis (b) $v_2 - v_3$ axis (c)$v_1 - v_3$ axis (d)$v_1 - v_2 - v_3$ axis.

Remark 2.3. After substituting the controller (2.11) into $\dot{V}_i$ for error dynamical system (2.3), we have

$$\dot{V}_i \leq \sum_{i=1}^{N} [(\mu_i \eta_i - \lambda_i) tanh(\delta_i s_i)].$$

Using condition (2.10) and $(\mu_i \eta_i - \lambda_i)|tanh(\delta_i s_i)| \leq 0$, we have

$$\dot{V}_i \leq \sum_{i=1}^{N} [(\mu_i \eta_i - \lambda_i)|tanh(\delta_i s_i)||s_i|] \leq 0.$$

Using Theorem 2.1 together with the Barbalat Lemma, we obtain $|s_i| = 0$ subsequently the error system is stable with controllers (2.11) and adaptive laws (2.9).

3. Illustration Example

To check the applicability and efficacy of the proposed control scheme, we consider the following two different non-integer chaotic systems:

Drive system [6]

$$
\begin{aligned}
D^{\alpha}u_1 &= -au_1 - eu_2^2 + 0.1\cos\pi u_1 + 0.5\sin t \\
D^{\alpha}u_2 &= bu_2 - ku_1u_3 + 0.1\cos 2\pi u_2 + 0.5\sin t \\
D^{\alpha}u_3 &= -cu_3 + mu_1u_2 + 0.1\cos 3\pi u_3 + 0.5\sin t
\end{aligned} \tag{3.1}
$$

where u_1, u_2, u_3 are state variables. The parameters a, b, c, d, and e are non-negative constants. For the parameter values $a = 1, m = 4, b = 2.5, c = 5, e = 1, k = 4, m = 4$, initial conditions $(u_1(0), u_2(0), u_3(0)) = (0.2, 0, 0.5)$., and fractional order $\alpha = 0.95$ the system (3.1) exhibits chaotic behaviour, as seen in Fig.1.

The uncertainties and disturbances for the drive system are taken as

$$\Delta f_i = 0.1\cos(i\pi u_i) \text{ and } d_i = 0.5\sin t$$

for $i = 1, 2, 3$.

Response system [7]

$$
\begin{aligned}
D^{\alpha}v_1 &= v_2 - 0.1\cos\pi v_1 - 0.5\sin t + \Phi_1(U_1) \\
D^{\alpha}v_2 &= v_3 - 0.1\cos 2\pi v_2 - 0.5\sin t + \Phi_2(U_2) \\
D^{\alpha}v_3 &= -fv_1 - gv_2 - hv_3 + iv_1^2 - 0.1\cos 3\pi v_3 - 0.5\sin t + \Phi_3(U_3)
\end{aligned} \tag{3.2}
$$

where v_1, v_2, v_3 are state variables. Parameters f, g, h, and i are non-negative constants. For the parameter values $f = 1, g = 1.1, h = 0.4, i = 1$ and initial conditions $(v_1(0), v_2(0), v_3(0)) = (-0.3, 0.1, -0.2)$. and for fractional order $\alpha = 0.95$ the system (3.2) shows the chaotic behaviour in Fig.2.

The uncertainties and disturbances for response system are taken as

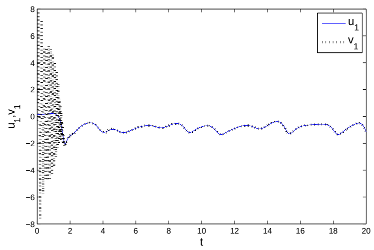
$$\Delta f_i = -0.1\cos(i\pi v_i) \text{ and } d_i = -0.5\sin t$$

for $i = 1, 2, 3$.

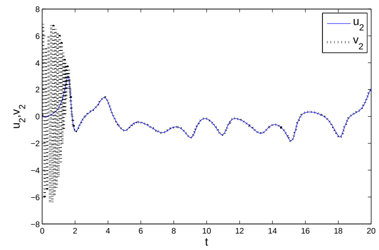
The non-linear control inputs are taken as $\Phi_i(U_i) = [5 + 3\sin t]U_i$. Also, it is assumed that $\omega_i = 1, \eta_i = 4$.

The error dynamical system can be written as

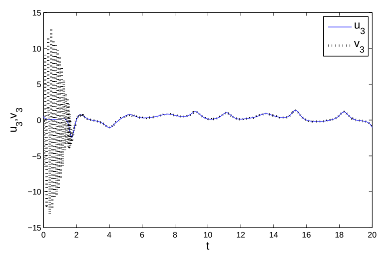
$$
\begin{aligned}
D^{\alpha}e_1(t) =& -au_1 - eu_2^2 + 0.1\cos\pi u_1 + 0.5\sin t - v_2 + 0.1\cos\pi v_1 \\
&+ 0.5\sin t - \Phi_1(U_1) \\
D^{\alpha}e_2(t) =& \; bu_2 - ku_1u_3 + 0.1\cos 2\pi u_2 + 0.5\sin t - v_3 + 0.1\cos 2\pi v_2 \\
&+ 0.5\sin t - \Phi_2(U_2) \\
D^{\alpha}e_3(t) =& -cu_3 + mu_1u_2 + 0.1\cos 3\pi u_3 + 0.5\sin t + fv_1 + gv_2 \\
&+ hv_3 - iv_1^2 + 0.1\cos 3\pi v_3 + 0.5\sin t - \Phi_3(U_3)
\end{aligned} \tag{3.3}
$$



(a)



(b)



(c)

Fig.3: Synchronized state trajectories which are synchronized at time t=2 unit(approx.)(a)$u_1 - v_1$ (b)$u_2 - v_2$ (c)$u_3 - v_3$

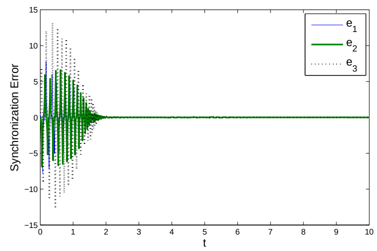
Choosing a suitable sliding surface (2.4) using control law (2.11) and adaptive law (2.9), we take $\mu_i = 0.1$, $\lambda_i = 0.5$ and $\delta_i = 200$. This yields $(\mu_i \eta_i - \lambda_i) = 0.1 * 4 - 0.5 = -0.1 < 0$.

Using Theorem 2.1 and (2.11), we get the following control laws.

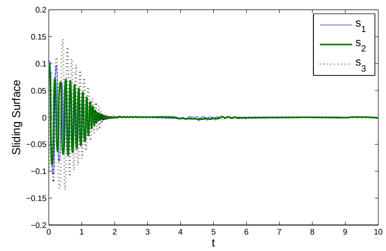
$$
\begin{aligned}
U_1 &= [10|e_1| + |-au_1 - eu_2^2 - v_2| + \hat{\vartheta}_1 + \hat{\nu}_1 + 0.5] \tan h(200 s_1) \\
U_2 &= [10|e_2| + |bu_2 - ku_1u_3 - v_3| + \hat{\vartheta}_2 + \hat{\nu}_2 + 0.5] \tan h(200 s_2) \\
U_3 &= [10|e_3| + |-cu_3 + mu_1u_2 + fv_1 + gv_2 + hv_3 - iv_1^2| \\
&\quad + \hat{\vartheta}_3 + \hat{\nu}_3 + 0.5] \tan h(200 s_3)
\end{aligned} \tag{3.4}
$$

For the systems (3.1) and (3.2) take $(u_1(0), u_2(0), u_3(0)) = (0.2, 0, 0.5)$ and $(v_1(0), v_2(0), v_3(0)) = (-0.3, 0.1, -0.2)$. Also take $\hat{\vartheta}_1(0) = 0.1, \hat{\vartheta}_2(0) = 0.1, \hat{\vartheta}_3(0) = 0.1$ and $\hat{\nu}_1(0) = 0.1, \hat{\nu}_2(0) = 0.1, \hat{\nu}_3(0) = 0.1$.

Figures 1 and 2 show the phase portraits of system (3.1) and (3.2). Figure 3 shows the synchronized state trajectories of system (3.1) and (3.2). Figure 4 shows the synchronized error and that the sliding surface converges to zero at approximately time $t = 2$ seconds. Figure 5 shows the estimated values of uncertainties

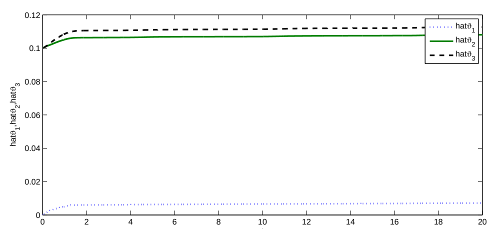


(a)

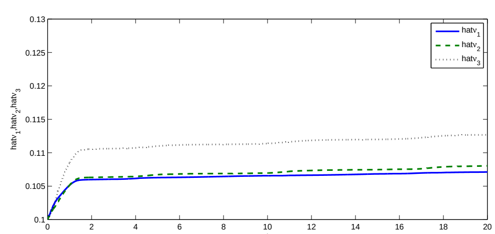


(b)

Fig.4: (a) Synchronization error (b) sliding surface converging to zero at time t=2 unit(approx.).



(a)



(b)

Fig.5: Estimated values of (a)uncertainties bounds (b)disturbances bounds.

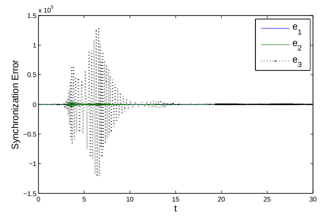


Fig.6: Synchronization Error for integer order Liu system and Genesio-Tesi system.

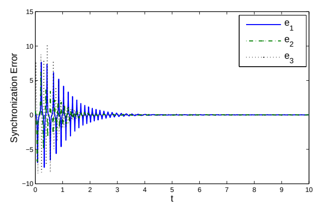


Fig.7: Synchronization Error for two identical non-integer Genesio-Tesi system.

and disturbance bounds.

4. Comparison of the Proposed Scheme with the Previous Published Literature

1. First of all, we compared our synchronization result to the integer-order Liu system and Genesio-Tesi system for the same set of parameter values and initial conditions. For $\alpha = 1$, synchronization was achieved at approximately $t = 15$ seconds. as seen in Fig.6. Therefore, from Fig.4(a) and Fig.6, we see that our scheme gives better results for non-integer chaotic systems.

2. In [6], the author adopted the active control and sliding mode control methods to analyse the complete synchronization between two identical non-integer Genesio Tesi systems with parameter values $a = 6, b = 2.92, c = 1.2$ & $d = 1$, initial conditions $(x_1(0), x_2(0), x_3(0)) = (0.3, 0.7, 1.2)$, $(y_1(0), y_2(0), y_3(0)) = (0.1, -0.3, -0.7)$ and $\alpha = 0.97$. The author achieved synchronization with the active control method at approximately $t = 20$, and with the sliding mode method at time $t = 15$. When we implemented our proposed methodology for the same systems with same set of parameter values and initial conditions in the presence of a set of disturbances, uncertainties, and non-linear input, we achieve synchronization at time $t = 4$, as shown in Fig.7.

3. In [18], the complete synchronization between two identical non-integer Genesio-Tesi systems with fifth order non-linearity based on the adaptive control method

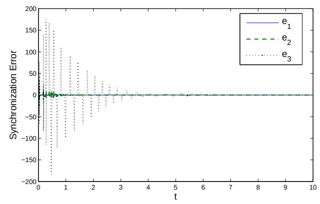


Fig.8: Synchronization Error for two identical non-integer Genesio-Tesi system with fifth order non-linearity.

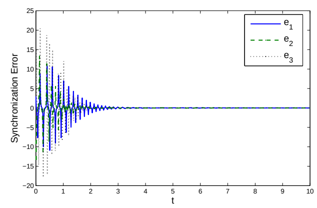


Fig.9: Synchronization Error for two identical non-integer Genesio-Tesi system.

was studied with parameter values $\beta_1 = -2, \beta_2 = 3.5, \beta_3 = 0.3$ & $\beta_4 = -1$, initial conditions $(x_1(0), x_2(0), x_3(0)) = (-0.2, 0.5, 0.2)$, $(y_1(0), y_2(0), y_3(0)) = (0.5, 1, -1)$ and $\alpha = 0.95$. They achieved synchronization at time $t = 75$. Our scheme for the same systems with same parameter values and initial conditions in the presence of disturbances, uncertainties, and input non-linearities, achieves synchronization at time $t = 4$, as shown in Fig.8.

4. In Section 4 of [25], the author anti-synchronized two identical non-integer Genesio-Tesi systems using the active control method with parameter values $a = 6, b = 2.92, c = 1.2$ & $m = 1$, initial conditions $(x_1(0), x_2(0), x_3(0)) = (2, -3, 4)$, $(y_1(0), y_2(0), y_3(0)) = (-1, 6, -6)$ and $\alpha = 0.95$. Anti-syncronization occured at time $t = 6$. When we applied our proposed scheme for the systems in [26] with same parameter values and initial conditions in the presence of disturbances, uncertainties and input non-linearities, we achieves synchronization at time $t = 3.5$, as shown in Fig.9.

5. In Section 5 of [25], the author anti-synchronized a non-integer Genesio-Tesi system and a Qi system using the active control method with parameter values $a = 6, b = 2.92, c = 1.2, m = 1, p = 35, q = 8/3$, & $r = 80$ initial conditions $(x_1(0), x_2(0), x_3(0)) = (-2, 3, 5)$, $(y_1(0), y_2(0), y_3(0)) = (-1, -1, -2)$ and $\alpha = 0.96$. They acheived anti-synchronization at time $t = 8$. When we adopted our scheme for the same with same parameter values and initial conditions in the presence of disturbances, uncertainties, and input non-linearities, we achieved synchronization

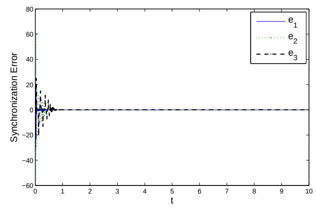


Fig.10: synchronization Error for non-integer Genesio-Tesi system and Qi-system.

at time $t = 1$, as shown in Fig.10.

Our methodology significantly beats published liturature in these settings.

5. Application to Secure Communication

Non-integer, chaotic systems have applications in many fields, such as physics, chemical science, and secure communication. In this manuscript, we show an application in secure communication.

Here we take a simple additive encryption masking scheme, to validate our proposed scheme.

The information input signal is selected as $I_S = \sin t + \cos t$ and u_3 is a chaotic carrier. The chaotic encrypted signal $C_S = I_S + u_3$. The original input information signal is regained by our proposed methodology– the decrypted signal is $D_S = C_S - v_3$. The results are shown in Fig. 11.

6. Conclusion

In this paper, a robust adaptive sliding mode technique has been used to achieve synchronization between two different fractional-order chaotic systems with model uncertainties, external disturbances, and non-linear inputs. Synchronization of non-integer chaotic systems in the presence of uncertainties, disturbances and non-linear control inputs has not been examined in the prior literature. We synchronized non-integer chaotic Liu-system and Genesio-Tesi systems. We chose a suitable sliding surface and estimated the bounded uncertainties and disturbances using update laws to achieve the desired synchronization and reduce the consequence of external uncertainties and disturbances and non-linear input. Then, using the considered control scheme and Lyapunov stability theory, we designed appropriate controllers. Although we have taken non-integer chaotic systems with uncertainties, disturbances, and non-linearities, we get better synchronization results. This scheme should perform a significant role to enhance security in communications. Computational methods were used to evaluate the efficiency of the considered scheme.

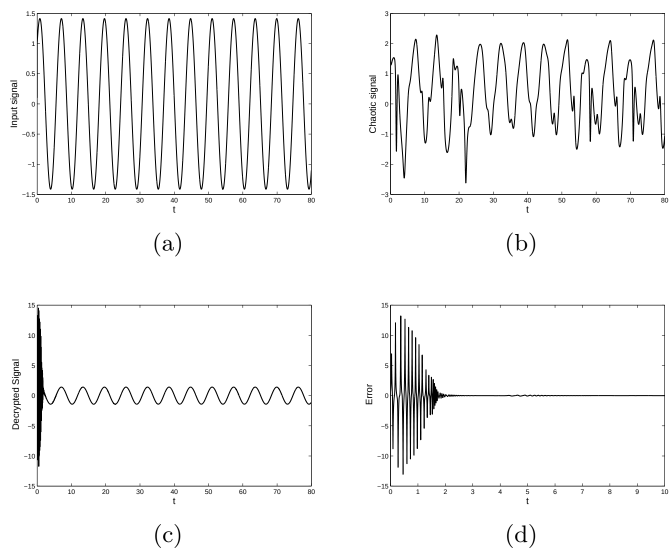


Fig.11: Additive encryption masking scheme (a) Information input signal(I_S) (b) Chaotic encrypted signal(C_S) (c) Decrypted signal(D_S) (d)Error between original signal and decrypted signal.

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