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BESSEL-WRIGHT TRANSFORM IN THE SETTING OF QUANTUM CALCULUS

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Abstract. This work is devoted to the study of a q-harmonic analysis related to the q-analog of the Bessel-Wright integral transform [6]. We establish some important properties of this transform and we focalise our attention in studying the associated transmutation operator.

1. Introduction

Recently, a new discipline was created and interested a lot of researchers : the q-harmonic analysis [9] who find application in the q-deformed mechanics. This theory was first elaborated by Koornwinder and R.F. Swarttouw [12] and then by Fitouhi et al. [3, 5, 7] who investigated in the q-Bessel function and the q-Fourier transform and related q-harmonic analysis, q-orthogonality relation, q-Plancherel formulae, q-Hardy Theorem [3].

Motivated by these gigantic growth and taking into account the work of the classical Bessel-Wright operator [6, 11], it is quite legitimate to try to extend these results to the quantum calculus case. We define the analog of the Bessel-Wright operator [2, 6, 11] and we try to give a q-integral representation of the q-Bessel-Wright function introduced via the q-hypergeometric functions.

To make this work self containing, we shall try to generalize our results starting by presenting some results associated with the q-Bessel transform, deeply studied on [3–5, 7], which will be useful in the sequel. Next, we present the basic tool in our work: the q-transmutation operator. This q-transmutation operator associated to the q-Bessel-Wright function is deeply studied in section 3 and the most important results are proved as well as the q-Bessel-Wright operator.

Finally, in Section 4, we introduce the q-Bessel-Wright transform. We prove many results that will be a strong step in studying a large class of q-integral transform.

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2. Preliminaries on q-harmonic analysis

Throughout this paper, we will assume that $0 < q < 1, \alpha > -1$ and $\beta > 0$. We refer to [9] for the definitions, notations and properties of the q-shifted factorials, the Jackson's q-derivative and the Jackson's q-integrals.

The q-shifted factorial are defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

and

$$\mathbb{R}_q^+ = \{q^n \ : \ n \in \mathbb{Z}\}.$$

The q-derivative of a function f is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad \text{if} \quad x \neq 0.$$

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by [9]

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n,$$

$$\int_0^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n,$$

provided the sums converge absolutely.

The space $\mathcal{L}_{q,p,\alpha}$, $1 \leq p < \infty$ denote the set of functions on $\mathbb{R}_q^+$ for which

$$\|f\|_{q,p,\alpha} = \left[\int_{-\infty}^{+\infty} |f(x)|^p \, |x|^{2\alpha+1} \, d_qx\right]^{1/p} < \infty,$$

and $\widetilde{\mathcal{L}}_{q,p,(\alpha,\beta)}\left(\mathbb{R}_q^+\right)$ the set of functions on $\mathbb{R}_q^+$ for which

$$\left[\int_0^{+\infty} |x^{2\beta} f(x)|^p \, |x|^{2\alpha+1} \, d_qx\right]^{1/p} < \infty.$$

We define the q-inner product $\langle , \rangle_\alpha$ in the Hilbert space $\mathcal{L}_{q,2,\alpha}$ as follow

$$f, g \in \mathcal{L}_{q,2,\alpha} \Rightarrow \langle f, g\rangle_\alpha = \int_{-\infty}^{+\infty} f(x)g(x) \, |x|^{2\alpha+1} \, d_qx.$$

Similarly $\mathcal{C}_{q,0}$ is the space of continuous functions at 0 for which

$$\|f\|_{q,\infty} = \sup_{x\in\mathbb{R}_q^+} |f(x)| < \infty$$

and $\mathcal{C}_{q,b}$ the space of continuous functions at 0 and bounded on $\mathbb{R}_q^+$.

The space $\mathcal{S}(\mathbb{R}_q^+)$ denote the set of the q-analogue of the Schwartz space f defined on $\mathbb{R}_q^+$ such

that for all $n \in \mathbb{N}$ $D_q^n f$ is continuous at 0, and verifying

$$\lim_{x\longrightarrow\infty} x^k \left|D_q^n f(x)\right| = 0, \quad \forall n, k \in \mathbb{N},$$

and $\mathcal{D}_{q,a}\left(\mathbb{R}_q^+\right)$ denote the subspace of $\mathcal{S}(\mathbb{R}_q^+)$ with compact support in $[-a, a]$.

The q-Paley-Wiener space is the sets of functions on $\mathbb{R}_q^+$ defined as follow

$$PW_{q,a}^{\alpha}=\left\{f\left(x\right)=\int_0^a u\left(t\right)j_\alpha\left(xt;q^2\right)d_qt,\quad u\in\mathcal{D}_{q,a}\right\}.$$

The normalized q-Bessel function is defined by

$$j_\alpha(x,q^2)=\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+1)}}{(q^{2\alpha+2},q^2)_n(q^2,q^2)_n}x^{2n}.$$

The q-Bessel function has the following q-Mehler integral representation [7]

$$j_\alpha(x,q^2)=c_{q,\alpha+1}\left(-\frac{1}{2}\right)\int_0^1 W_{\alpha+1}\left(t;q^2\right)\cos(xt;q^2)d_qt$$

where

(1) $$c_{q,\alpha}\left(\xi\right)=\frac{\left(q^{2\xi+2};q^2\right)_\infty\left(q^{2\alpha};q^2\right)_\infty}{\left(q^2;q^2\right)_\infty\left(q^{2\alpha+2\xi+2};q^2\right)_\infty}$$

and W_α the function defined by formula

(2) $$W_\alpha\left(t;q^2\right)=\frac{1}{(1-q)}\frac{\left(q^2t^2;q^2\right)_\infty}{\left(q^{2\alpha}t^2;q^2\right)_\infty}.$$

And these two formulas (1) and (2) give the following formula

(3) $$c_{q,\alpha}\left(\xi\right)\frac{1}{\left(q^{2\xi+2};q^2\right)_n}\int_0^1 W_\alpha\left(t;q^2\right)t^{2n}t^{2\xi+1}d_qt=\frac{1}{\left(q^{2\alpha+2+2\xi};q^2\right)_n}.$$

The q-Bessel operator as follow

$$\Delta_{q,\alpha}f(x)=\frac{1}{x^2}\left[f(q^{-1}x)-(1+q^{2\alpha})f(x)+q^{2\alpha}f(qx)\right].$$

The function $x\mapsto j_\alpha(\lambda x,q^2)$ is an eigenfunction of $\Delta_{q,\alpha}$ for the eigenvalue $-\lambda^2$.

The q-Bessel Fourier transform $\mathcal{F}_{q,\alpha}$ is defined by

$$\mathcal{F}_{q,\alpha}\left(f\right)\left(x\right)=c_{q,\alpha}\int_0^{+\infty}f\left(t\right)j_\alpha\left(xt;q^2\right)t^{2\alpha+1}d_qt.$$

Theorem 1. *We have*

$$\mathcal{F}_{q,\alpha}:\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right)\to\mathcal{C}_{q,0}\left(\mathbb{R}_q^+\right)$$

and

$$\mathcal{F}_{q,\alpha}:\mathcal{S}(\mathbb{R}_q^+)\to\mathcal{S}(\mathbb{R}_q^+)$$

and

$$\mathcal{F}_{q,\alpha}:\mathcal{D}_{q,a}(\mathbb{R}_q^+)\to PW_{q,a}^{\alpha}.$$

Proof. Using [5, Proposition 3.1], we get

$$\mathcal{F}_{q,\alpha}:\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right)\to\mathcal{C}_{q,0}\left(\mathbb{R}_q^+\right).$$

Via [5, Corollary 3.3], we get

$$\mathcal{F}_{q,\alpha}:\mathcal{S}(\mathbb{R}_q^+)\to\mathcal{S}(\mathbb{R}_q^+)$$

and by [4, Theorem 1], we have

$$\mathcal{F}_{q,\alpha}:\mathcal{D}_{q,a}(\mathbb{R}_q^+)\to PW_{q,a}^{\alpha}.$$

□

3. On the q-Bessel-Wright function

We start this section by defining the q-Bessel-Wright function

$$j_{(\alpha,\beta)}(x,q^2) = {}_2\Phi_2\left(q^2, -; q^{2\alpha+2}, q^{2\beta+2}; q^2; q^2x^2\right) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2},q^2)_n(q^{2\beta+2},q^2)_n} x^{2n}. \tag{4}$$

Proposition 1. *The q-Bessel-Wright function $x \mapsto j_{(\alpha,\beta)}(\lambda x, q^2)$ is an eigenfunction of the q-Bessel-Wright difference operator $\Delta_{(\alpha,\beta)}$ associated to the eigenvalue $-\lambda^2$*

$$\Delta_{(\alpha,\beta)}f(x) = \frac{1}{x^2}\left[f\left(q^{-1}x\right) - \left(q^{2\beta}+q^{2\alpha}\right)f(x) + q^{2(\alpha+\beta)}f(qx) - \left(1-q^{2\alpha}\right)\left(1-q^{2\beta}\right)f(0)\right].$$

Proof. We have

$$\frac{(1-q^{2\alpha+2i})(1-q^{2\beta+2i})}{q^{2i}} = q^{-2i} - \left(q^{2\alpha}+q^{2\beta}\right) + q^{2(\alpha+\beta)+2i}.$$

So we get

$$\begin{aligned}\Delta_{(\alpha,\beta)}j_{(\alpha,\beta)}(x,q^2) &= \sum_{n=1}^{\infty}(-1)^n\prod_{i=1}^{n}\frac{q^{-2n}-\left(q^{2\alpha}+q^{2\beta}\right)+q^{2(\alpha+\beta)+2n}}{q^{-2i}-\left(q^{2\alpha}+q^{2\beta}\right)+q^{2(\alpha+\beta)+2i}}x^{2n-2}\\ &= -\Delta_{(\alpha,\beta)}j_{(\alpha,\beta)}(x,q^2),\end{aligned}$$

which leads to the result. □

Proposition 2. *The q-Bessel-Wright difference operator could be factorized as follow*

$$\Delta_{(\alpha,\beta)} = \partial_{q,\alpha}\partial^*_{q,\beta}$$

where

$$\begin{aligned}\partial_{q,\alpha}f(x) &= \frac{f(x)-q^{2\alpha+1}f(qx)}{x}\\ \partial^*_{q,\beta}f(x) &= \frac{\left[f\left(q^{-1}x\right)-f(0)\right]-q^{2\beta}\left[f(x)-f(0)\right]}{x}.\end{aligned}$$

Proof. A simple calculation leads to the result

$$\begin{aligned}\partial_{q,\alpha}\partial^*_{q,\beta}f(x) &= \frac{\frac{\left[f\left(q^{-1}x\right)-f(0)\right]-q^{2\beta}[f(x)-f(0)]}{x} - q^{2\alpha+1}\frac{[f(x)-f(0)]-q^{2\beta}[f(qx)-f(0)]}{qx}}{x}\\ &= \frac{f\left(q^{-1}x\right)-\left(q^{2\beta}+q^{2\alpha}\right)f(x)-q^{2\beta+2\alpha}f(qx)-(1-q^{2\alpha})\left(1-q^{2\beta}\right)f(0)}{x^2}\\ &= \Delta_{(\alpha,\beta)}f(x).\end{aligned}$$

□

PROPOSITION 3. *We have*

$$\partial_{q,\alpha}j_{\left(\alpha+\frac{1}{2},\beta\right)}(x;q^2) \ = -q^2\left(1-q^{2\beta+4}\right)xj_{\left(\alpha+\frac{1}{2},\beta+1\right)}(qx;q^2)$$

and

$$\partial^*_{q,\beta}j_{(\alpha,\beta)}(qx;q^2) = -qx\left(1-q^{2\alpha+4}\right)j_{(\alpha+1,\beta)}(qx;q^2) - \frac{1-q^{2\beta}}{qx}.$$

Proof. The following calculation

$$\begin{aligned}
&\partial_{q,\alpha}j_{\left(\alpha+\frac{1}{2},\beta\right)}(x;q^2)\\
=\ & \sum_{n=1}^{\infty}(-1)^n\frac{q^{n(n+1)}}{(q^{2\alpha+3};q^2)_n(q^{2\beta+2};q^2)_n}x^{2n-1}\left(1-q^{2\alpha+1+2n}\right)\\
=\ & -\left(1-q^{2\beta+4}\right)q^2x\sum_{n=1}^{\infty}(-1)^{n-1}\frac{q^{n(n-1)}}{(q^{2\alpha+3};q^2)_{n-1}(q^{2\beta+4};q^2)_{n-1}}(qx)^{2(n-1)}\\
=\ & -q^2\left(1-q^{2\beta+4}\right)xj_{\left(\alpha+\frac{1}{2},\beta+1\right)}(qx;q^2),
\end{aligned}$$

and

$$\begin{aligned}
&\partial^*_{q,\beta}j_{(\alpha,\beta)}(qx;q^2)\\
=\ & \frac{\left[j_{(\alpha,\beta)}(x;q^2)-1\right]-q^{2\beta}\left[j_{(\alpha,\beta)}(qx;q^2)-1\right]}{qx}\\
=\ & \frac{\left[j_{(\alpha,\beta)}(x;q^2)-1\right]-q^{2\beta}\left[j_{(\alpha,\beta)}(qx;q^2)-1\right]}{qx}-\frac{1-q^{2\beta}}{qx}\\
=\ & \sum_{n=1}^{\infty}(-1)^n\frac{q^{n(n+1)}}{(q^{2\alpha+2};q^2)_n(q^{2\beta+2};q^2)_n}x^{2n-1}\left(\frac{1-q^{2\beta+2n}}{q}\right)-\frac{1-q^{2\beta}}{qx}\\
=\ & -qx\left(1-q^{2\alpha+4}\right)\sum_{n=1}^{\infty}(-1)^n\frac{q^{n(n-1)}}{(q^{2(\alpha+1)+2};q^2)_{n-1}(q^{2\beta+2};q^2)_{n-1}}(qx)^{2(n-1)}-\frac{1-q^{2\beta}}{qx}\\
=\ & -qx\left(1-q^{2\alpha+4}\right)j_{(\alpha+1,\beta)}(qx;q^2)-\frac{1-q^{2\beta}}{qx}
\end{aligned}$$

leads to the result. □

In this step of our work, we introduce a q-Transmutation operator. This will allow us to stake out the q-Bessel-Wright function properties from the usual q-Bessel function. The bellow proposition will be a strong step in generalizing the q-Bessel properties (integrability, bounds, etc) to be applied to the q-Bessel-Wright function.

DEFINITION 1. The q-Bessel intertwining operator $\mathcal{R}_{q,\beta}$ is defined by

$$\mathcal{R}_{q,\beta}(f)(x) = c_{q,\beta}(0)\int_0^1 W_\beta(t)\,tf(xt)\,d_qt$$

where $c_{q,\beta}$ is defined by the formula (1), and W_β is the function defined by the formula (2).

THEOREM 2. *If*

$$\alpha+1<p$$

then we have

$$\mathcal{R}_{q,\beta}:\mathcal{L}_{q,p,\alpha}\left(\mathbb{R}^+_q\right)\to\mathcal{L}_{q,p,\alpha}\left(\mathbb{R}^+_q\right).$$

Proof. Let $f \in \mathcal{L}_{q,p,\alpha}\left(\mathbb{R}_q^+\right)$. Using the generalized Minkowski integral inequality [10], we obtain

$$\left\|\mathcal{R}_{q,\beta}(f)\right\|_{q,p,\alpha} = \left[\int_0^\infty \left|\int_0^1 W_\beta(t)\, t f(xt)\, d_qt\right|^p x^{2\alpha+1} d_qx\right]^{\frac{1}{p}}$$
$$\leq \int_0^1 W_\beta(t)\, t \left[\int_0^{+\infty} |f(xt)|^p x^{2\alpha+1} dx\right]^{\frac{1}{p}} d_qt$$
$$\leq \int_0^1 W_\beta(t)\, t \left[\int_0^{+\infty} |f(x)|^p x^{2\alpha+1} dx\right]^{\frac{1}{p}} t^{1-\frac{2(\alpha+1)}{p}} d_qt$$
$$\leq \int_0^1 W_\beta(t)\, t^{1-\frac{2(\alpha+1)}{p}} d_qt \times \|f\|_{q,p,\alpha} < \infty.$$

□

Proposition 4. *The operator $\mathcal{R}_{q,\beta}$ has the following inverse operator*

$$\mathcal{R}_{q,\beta}^{-1}(f)(x) = \frac{\left(q^{-2\beta}; q^2\right)_\infty}{\left(q^2; q^2\right)_\infty} \int_0^1 W_{-\beta}(t)\, f(xt)\, t d_qt.$$

Proof. We introduce the following operator

$$H_{q,\beta}(f)(x) = c_{q,\beta}(0)\left(1-q^2\right)^\beta x^{2\beta} \int_0^1 W_\beta(t)\, f(xt)\, t d_qt, \quad \forall x \in \mathbb{R}_q^+.$$

Using [1, lemma 2] we obtain

$$H_{q,\beta}^{-1}(f) = H_{q,-\beta}(f).$$

Note that

$$H_{q,\beta}(f)(x) = \left(1-q^2\right)^\beta x^{2\beta} \mathcal{R}_{q,\beta}(f)(x)$$

which implies

$$H_{q,\beta}^{-1}(f)(x) = \left(1-q^2\right)^{-\beta} x^{-2\beta} \mathcal{R}_{q,\beta}^{-1}(f)(x), \quad \forall x \in \mathbb{R}_q^+.$$

So we get

$$\mathcal{R}_{q,\beta}^{-1}(f)(x) = \frac{\left(q^{-2\beta}; q^2\right)_\infty}{\left(q^2; q^2\right)_\infty} \int_0^1 W_{-\beta}(t)\, f(xt)\, t d_qt$$

□

Proposition 5. *The q-Bessel-Wright function is related to the q-Bessel function via the following formula*

$$j_{(\alpha,\beta)}\left(x; q^2\right) = \mathcal{R}_{q,\beta}\left(j_\alpha\left(x; q^2\right)\right)$$

and we have

(5) $$\left|j_{(\alpha,\beta)}(x, q^2)\right| \leq 1, \quad \forall x \in \mathbb{R}_q^+.$$

Proof. Using formula (4), we get

$$\begin{aligned}
\mathcal{R}_{q,\beta}\left(j_\alpha\left(x;q^2\right)\right) &= \sum_{n=0}^{+\infty}(-1)^n\frac{q^{n(n+1)}}{\left(q^{2\alpha+2};q^2\right)_n}x^{2n}\left[c_{q,\alpha}(0)\frac{1}{\left(q^2;q^2\right)_n}\int_0^1 W_\beta\left(t;q^2\right)t^{2n+1}d_qt\right]\\
&= \sum_{n=0}^{+\infty}(-1)^n\frac{q^{n(n+1)}}{\left(q^{2\alpha+2};q^2\right)_n\left(q^{2\beta+2};q^2\right)_n}x^{2n}\\
&= j_{(\alpha,\beta)}\left(x;q^2\right).
\end{aligned}$$

□

We define the q-Transmutation operator $\mathcal{W}_{q,\alpha,\beta}$ as follow

$$\mathcal{W}_{q,\alpha,\beta}(f)(x) = c_{q,\beta}(0)\,x^{-2\alpha}\int_x^\infty \frac{\left(q^2\frac{x^2}{u^2};q^2\right)_\infty}{(1-q)\left(q^{2\beta}\frac{x^2}{u^2};q^2\right)_\infty}f(u)\,u^{2\alpha-1}d_qu,$$

and which can be also written it in the following form

$$\mathcal{W}_{q,\alpha,\beta}(f)(x) = \frac{c_{q,\beta}(0)}{(1-q)}\int_1^\infty \frac{\left(\frac{q^2}{t^2};q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2};q^2\right)_\infty}f(xt)\,t^{2\alpha-1}d_qt.$$

In the following, we prove that the operator $\mathcal{W}_{q,\alpha,\beta}$ leaves invariant the spaces $\mathcal{S}_q\left(\mathbb{R}_q^+\right)$ and $\mathcal{D}_q\left(\mathbb{R}_q^+\right)$ and that under some conditions, it send the space $\mathcal{L}_{q,p,\alpha}\left(\mathbb{R}_q^+\right)$ to the space $\mathcal{L}_{q,p,\alpha}\left(\mathbb{R}_q^+\right)$.

Theorem 3. *The q-Bessel-Wright Transmutation operator satisfy the following properties*

1. *We have*

$$\mathcal{W}_{q,\alpha,\beta}:\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right)\to\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right) \tag{6}$$

2. *If $\alpha+1<p$ then we have*

$$\mathcal{W}_{q,\alpha,\beta}:\mathcal{L}_{q,p,\alpha}\left(\mathbb{R}_q^+\right)\to\mathcal{L}_{q,p,\alpha}\left(\mathbb{R}_q^+\right)$$

3. *We have*

$$\mathcal{W}_{q,\alpha,\beta}:\mathcal{S}_q\left(\mathbb{R}_q^+\right)\to\mathcal{S}_q\left(\mathbb{R}_q^+\right) \tag{7}$$

and

$$\mathcal{W}_{q,\alpha,\beta}:\mathcal{D}_q\left(\mathbb{R}_q^+\right)\to\mathcal{D}_q\left(\mathbb{R}_q^+\right).$$

Proof. We have

$$f\in\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right)$$

Using the generalized Minkowski integral inequality [10], we obtain

$$\begin{aligned}
&\left\|\mathcal{W}_{q,\alpha,\beta}\left(f\right)\left(x\right)\right\|_{q,\alpha,p}\\
=\;& \left[\int_0^\infty \left|\mathcal{W}_{q,\alpha,\beta}\left(f\right)\left(x\right)\right|^p x^{2\alpha+1}d_qx\right]^{\frac{1}{p}}\\
=\;& \frac{c_{q,\beta}\left(0\right)}{\left(1-q\right)}\left[\int_0^\infty\left|\int_1^\infty \frac{\left(\frac{q^2}{t^2};q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2};q^2\right)_\infty} f\left(xt\right)t^{2\alpha-1}d_qt\right|^p x^{2\alpha+1}d_qx\right]^{\frac{1}{p}}\\
\leq\;& \frac{c_{q,\beta}\left(0\right)}{\left(1-q\right)}\int_1^\infty \frac{\left(\frac{q^2}{t^2};q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2};q^2\right)_\infty}\left[\int_0^{+\infty}\left|f\left(xt\right)\right|^p x^{2\alpha+1}dx\right]^{\frac{1}{p}} t^{2\alpha-1}d_qt\\
\leq\;& \frac{c_{q,\beta}\left(0\right)}{\left(1-q\right)}\int_1^\infty \frac{\left(\frac{q^2}{t^2};q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2};q^2\right)_\infty}\left[\int_0^{+\infty}\left|f\left(x\right)\right|^p x^{2\alpha+1}d_qx\right]^{\frac{1}{p}} t^{2\alpha-1-\frac{2(\alpha+1)}{p}}d_qt\\
\leq\;& \frac{c_{q,\beta}\left(0\right)}{\left(1-q\right)}\int_1^\infty \frac{\left(\frac{q^2}{t^2};q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2};q^2\right)_\infty} t^{2\alpha-1-\frac{2(\alpha+1)}{p}}d_qt\left\|f\right\|_{q,p,\alpha}\\
\leq\;& \frac{c_{q,\beta}\left(0\right)}{\left(1-q\right)}\frac{\left(q^2;q^2\right)_\infty\left(q^{\frac{2(\alpha+1)}{p}-2\alpha};q^2\right)_\infty}{\left(q^{\frac{2(\alpha+1)}{p}-2\alpha};q^2\right)_\infty\left(q^{2\beta};q^2\right)_\infty}\left\|f\right\|_{q,p,\alpha}<\infty, \text{if } 1>\alpha\left(p-1\right).
\end{aligned}$$

Hence, for $p=1$, we have

$$\mathcal{W}_{q,\alpha,\beta}:\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right)\to\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right)\left(\mathbb{R}_q^+\right).$$

To prove the second result, we have

$$\begin{aligned}
\mathcal{W}_{q,\alpha,\beta}\left(f\right)\left(x\right) &= \frac{c_{q,\beta}\left(0\right)}{\left(1-q\right)}\int_1^\infty \frac{\left(\frac{q^2}{t^2};q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2};q^2\right)_\infty} f\left(xt\right)t^{2\alpha-1}d_qt\\
&= \frac{c_{q,\beta}\left(0\right)}{\left(1-q\right)}\Phi\left(x\right).
\end{aligned}$$

Let $(p,n)\in\mathbb{N}^2$,

$$D_q^n\Phi\left(x\right)=\int_1^\infty \frac{\left(\frac{q^2}{t^2};q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2};q^2\right)_\infty}t^nD_q^nf\left(xt\right)t^{2(\alpha)-1}d_qt.$$

Using the Dominated Convergence Theorem, we get

$$\begin{aligned}
&\lim_{x\to+\infty} x^p \left|D_q^n \Phi(x)\right| \\
\leq\ & \lim_{x\to+\infty} x^p \int_1^\infty \frac{\left(\frac{q^2}{t^2};q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2};q^2\right)_\infty} t^n \left|D_q^n f(xt)\right| t^{2\alpha-1} d_q t \\
\leq\ & \int_1^\infty \frac{\left(\frac{q^2}{t^2};q^2\right)_\infty}{\left(\frac{q^{2\beta}}{t^2};q^2\right)_\infty} t^n \lim_{x\to+\infty} \left[(tx)^p \left|D_q^n f(xt)\right|\right] t^{2\alpha-1} d_q t = 0.
\end{aligned}$$

So

$$\mathcal{W}_{q,\alpha,\beta} : \mathcal{S}_q(\mathbb{R}_q) \to \mathcal{S}_q(\mathbb{R}_q).$$

Let $f \in \mathcal{D}_{q,a}(\mathbb{R}_q)$, we have

$$\begin{aligned}
&\mathcal{W}_{q,\alpha,\beta}(f)(x) \\
=\ & c_{q,\beta}(0)\, x^{-2\alpha} \int_x^\infty \frac{\left(q^2\frac{x^2}{u^2};q^2\right)_\infty}{(1-q)\left(q^{2\beta}\frac{x^2}{u^2};q^2\right)_\infty} f(u)\, u^{2\alpha-1} d_q u \\
=\ & \begin{cases} 0, \quad \text{if } x > a \\ c_{q,\beta}(0)\, x^{-2\alpha} \displaystyle\int_x^a \frac{\left(q^2\frac{x^2}{u^2};q^2\right)_\infty}{(1-q)\left(q^{2\beta}\frac{x^2}{u^2};q^2\right)_\infty} f(u)\, u^{2\alpha-1} d_q u, \ \text{if } x < a \end{cases}
\end{aligned}$$

We conclude that $\mathcal{W}_{q,\alpha,\beta}(f) \in \mathcal{D}_{q,a}\left(\mathbb{R}_q^+\right)$. □

Proposition 6. *Let $f, g \in \mathcal{L}_{q,p,\alpha}\left(\mathbb{R}_q^+\right) \cap \mathcal{L}_{q,\overline{p},\alpha}\left(\mathbb{R}_q^+\right)$, such that $\frac{1}{p} + \frac{1}{\overline{p}} = 1$. We get*

$$\langle f, \mathcal{R}_{q,\beta}(g)\rangle_\alpha = \langle \mathcal{W}_{q,\alpha,\beta}(f), g\rangle_\alpha. \tag{8}$$

Proof. Indeed,

$$\begin{aligned}
\frac{(1-q)}{c_{q,\beta}(0)} \langle g, \mathcal{R}_{q,\beta}(f)\rangle_\alpha &= \int_0^\infty g(x) \left[\int_0^1 \frac{(q^2t^2;q^2)_\infty}{(q^{2\beta}t^2;q^2)_\infty} t f(xt)\, d_q t\right] x^{2\alpha+1} d_q x \\
&= \int_0^\infty g(x) \left[\int_0^x \frac{\left(q^2\frac{u^2}{x^2};q^2\right)_\infty}{\left(q^{2\beta}\frac{u^2}{x^2};q^2\right)_\infty} u f(u)\, d_q u\right] x^{2\alpha-1} d_q x \\
&= \int_0^\infty f(u) \left[u^{-2\alpha} \int_u^\infty \frac{\left(q^2\frac{u^2}{x^2};q^2\right)_\infty}{\left(q^{2\beta}\frac{u^2}{x^2};q^2\right)_\infty} g(x)\, x^{2\alpha-1} d_q x\right] u^{2\alpha+1} d_q u \\
&= \frac{(1-q)}{c_{q,\beta}(0)} \langle \mathcal{W}_{q,\alpha,\beta}(g), f\rangle_\alpha.
\end{aligned}$$

The computation are justified by the Fubini's Theorem and Theorem 6 & 7:

$$\begin{aligned}
&\int_0^{+\infty} \int_0^1 \left| g(x) \frac{(q^2t^2;q^2)_\infty}{(q^{2\beta}t^2;q^2)_\infty} t f(xt)\right| x^{2\alpha+1} d_q t\, d_q x \\
\leq\ & \int_0^{+\infty} |g(x)|\, x^{2\alpha+1} d_q x \times \left\|\mathcal{R}_{q,\beta}(f)\right\|_{q,\overline{p},\alpha} \\
\leq\ & \|g\|_{q,p,\alpha} \times \left\|\mathcal{R}_{q,\beta}(f)\right\|_{q,\overline{p},\alpha} < \infty.
\end{aligned}$$

□

4. The q-Bessel-Wright transform

In this section, we define the q-Bessel-Wright transform, and we investigate in the related q-harmonic analysis.

DEFINITION 2. Let $f \in \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q)$, we define the q-Bessel-Wright transform as follow

$$\mathcal{F}_{(\alpha,\beta)}(f)\left(x;q^2\right) = c_{q,(\alpha,\beta)} \int_0^{+\infty} f(t)\, j_{(\alpha,\beta)}\left(xt;q^2\right) t^{2\alpha+1} d_q t$$

where

$$c_{q,(\alpha,\beta)} = \left(\frac{1}{1-q}\right)^2 \frac{\left(q^{2\alpha+2};q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \frac{\left(q^{2\beta+2};q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}.$$

PROPOSITION 7. *Let $f, g \in \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q)$ we have*

$$\int_0^{+\infty} \mathcal{F}_{(\alpha,\beta)}(f)(x)\, g(x)\, x^{2\alpha+1} d_q x = \int_0^{+\infty} f(x)\, \mathcal{F}_{(\alpha,\beta)}(g)(x)\, x^{2\alpha+1} d_q x.$$

Proof. Let $f, g \in \mathcal{L}_{q,1,\alpha}(\mathbb{R}_q)$. Then we have

$$\begin{aligned}
& \int_0^{+\infty} \mathcal{F}_{(\alpha,\beta)}(f)(x)\, g(x)\, x^{2\alpha+1} d_q x \\
= & \int_0^{+\infty} \left[c_{q,(\alpha,\beta)} \int_0^{+\infty} f(t)\, j_{(\alpha,\beta)}\left(xt;q^2\right) t^{2\alpha+1} d_q t \right] g(x)\, x^{2\alpha+1} d_q x \\
= & \int_0^{+\infty} \left[c_{q,(\alpha,\beta)} \int_0^{+\infty} j_{(\alpha,\beta)}\left(xt;q^2\right) g(x)\, x^{2\alpha+1} d_q x \right] f(t)\, t^{2\alpha+1} d_q t \\
= & \int_0^{+\infty} f(t)\, \mathcal{F}_{(\alpha,\beta)}(g)(t)\, t^{2\alpha+1} d_q t
\end{aligned}$$

The computation are justified by the Fubini's Theorem. In fact

$$\begin{aligned}
& \int_0^{+\infty} \left| \int_0^{+\infty} f(t)\, j_{(\alpha,\beta)}\left(xt;q^2\right) g(x) \right| t^{2\alpha+1} x^{2\alpha+1} d_q t\, d_q x \\
\leq & \int_0^{+\infty} \left| \int_0^{+\infty} f(t)\, g(x) \right| t^{2\alpha+1} x^{2\alpha+1} d_q t\, d_q x \\
\leq & \int_0^{+\infty} |f(t)|\, t^{2\alpha+1} d_q t \int_0^{+\infty} |g(x)|\, x^{2\alpha+1} d_q x \\
\leq & \|f\|_{q,1,\alpha} \times \|g\|_{q,1,\alpha}.
\end{aligned}$$

So we get the result. □

COROLLARY 1. *Let $f \in \mathcal{L}_{q,p,\alpha}\left(\mathbb{R}_q^+\right)$ we have*

$$\mathcal{F}_{(\alpha,\beta)}(f) = \mathcal{F}_{q,\alpha} \circ \mathcal{W}_{q,\alpha,\beta}(f)$$

and

$$\mathcal{F}^{-1}_{(\alpha,\beta)}(f)\left(x;q^2\right) = \mathcal{W}^{-1}_{q,\alpha,\beta} \circ \mathcal{F}_{q,\alpha}(f)$$

where $\mathcal{F}_{q,\alpha}$ is the q-Bessel transform [4].

Proof. Using (8), we have

$$\begin{aligned}
&\frac{1}{c_{q,\upsilon}}\mathcal{F}_{(\alpha,\beta)}(f)\left(x;q^2\right)\\
&= \int_0^{+\infty} f(t)\, j_{(\alpha,\beta)}\left(xt;q^2\right)t^{2\alpha+1}d_qt\\
&= \left\langle f,\mathcal{R}_{\alpha,q}\left(j_\alpha\left(x.;q^2\right)\right)\right\rangle_\alpha\\
&= \left\langle \mathcal{W}_{q,\alpha,\beta}(f), j_\alpha\left(x.;q^2\right)\right\rangle_\alpha\\
&= \mathcal{F}_{q,\alpha}\circ\mathcal{W}_{q,\alpha,\beta}(x).
\end{aligned}$$

Using $\mathcal{F}_{q,\alpha}^{-1}=\mathcal{F}_{q,\alpha}$, we get

$$\mathcal{F}_{(\alpha,\beta)}^{-1}(f)\left(x;q^2\right)=\mathcal{W}_{q,\alpha,\beta}^{-1}(f)\circ\mathcal{F}_{q,\alpha}$$

In fact $\mathcal{W}_{q,\alpha,\beta}^{-1}(f)$ exists because $\mathcal{R}_{q,\beta}^{-1}$ exists and we have

$$\left\langle f,\mathcal{R}_{q,\beta}(g)\right\rangle_\alpha=\left\langle \mathcal{W}_{q,\alpha,\beta}(f),g\right\rangle_\alpha.$$

□

THEOREM 4. *Let* $f\in\mathcal{L}_{q,1,\alpha}(\mathbb{R}_q)$ *then we have*

$$\left\|\ \mathcal{F}_{(\alpha,\beta)}f\right\|_{q,\infty}\leq c_{q,(\alpha,\beta)}\|f\|_{q,1,\alpha}.$$

Proof. Using formula (5)

$$\begin{aligned}
\left\|\ \mathcal{F}_{(\alpha,\beta)}f\right\|_{q,\infty} &= \sup_{x\in\mathbb{R}_q^+}\left|\mathcal{F}_{(\alpha,\beta)}f(x)\right|\\
&\leq c_{q,(\alpha,\beta)}\int_0^{+\infty}|f(t)|\sup_{x\in\mathbb{R}_q^+}\left|j_{(\alpha,\beta)}\left(xt;q^2\right)\right|t^{2\alpha+1}d_qt\\
&\leq c_{q,(\alpha,\beta)}\|f\|_{q,1,\alpha},
\end{aligned}$$

so we get the result. □

THEOREM 5. *The q-Bessel-Wright transform verifies the following properties*

:

$$\mathcal{F}_{(\alpha,\beta)}:\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right)\to\mathcal{C}_{q,0}\left(\mathbb{R}_q^+\right).$$

: *If* $\alpha+1<p$ *then we have*

$$\mathcal{F}_{(\alpha,\beta)}:\mathcal{L}_{q,p,\alpha}\left(\mathbb{R}_q^+\right)\to\mathcal{L}_{q,p,\alpha}\left(\mathbb{R}_q^+\right)$$

:

$$\mathcal{F}_{(\alpha,\beta)}:\mathcal{S}(\mathbb{R}_q^+)\to\mathcal{S}(\mathbb{R}_q^+)$$

:

$$\mathcal{F}_{(\alpha,\beta)}:\mathcal{D}_a(\mathbb{R}_q^+)\to PW_{q,a}^\alpha$$

Proof. Using the fact that

$$\mathcal{F}_{q,\alpha}\circ\mathcal{W}_{q,\alpha,\beta},$$

and the fact that the operator $\mathcal{W}_{q,\alpha,\beta}$ send the space $\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right)$ to $\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right)$. We combine this result with Theorem (1), we get

$$\mathcal{F}_{(\alpha,\beta)}:\mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right)\to\mathcal{C}_{q,0}\left(\mathbb{R}_q^+\right)$$

Since the operator $\mathcal{W}_{q,\alpha,\beta}$ leaves $\mathcal{S}(\mathbb{R}_q^+)$ and $\mathcal{D}_a(\mathbb{R}_q^+)$ invariant, using the Theorem (1) we get

$$\mathcal{F}_{(\alpha,\beta)} : \mathcal{S}(\mathbb{R}_q^+) \to \mathcal{S}(\mathbb{R}_q^+)$$

and

$$\mathcal{F}_{(\alpha,\beta)} : \mathcal{D}_a(\mathbb{R}_q^+) \to PW_{q,a}^{\alpha}$$

□

5. Example and Application

To highlight the contribution of our work, we propose to investigate in the $\left(\alpha, \frac{1}{2}\right)$ q-Bessel-Wright transform.

PROPOSITION 8. *Let $\beta = \frac{1}{2}$, the function $j_{\left(\alpha,\frac{1}{2}\right)}$*

$$\begin{aligned} j_{\left(\alpha,\frac{1}{2}\right)}(x;q^2) &= {}_2\Phi_2\left(q^2, \_; q^{2\alpha+2}, q^3; q^2; q^2x^2\right) \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2},q^2)_n(q^3,q^2)_n} x^{2n} \end{aligned}$$

is a q-analog of the Struve function [8].

PROPOSITION 9. *The q-Struve function $x \mapsto j_{\left(\alpha,\frac{1}{2}\right)}(\lambda x, q^2)$ is an eigenfunction of the q-Struve difference operator $\Delta_{\left(\alpha,\frac{1}{2}\right)}$ associated to the eigenvalue $-\lambda^2$where*

$$\Delta_{\left(\alpha,\frac{1}{2}\right)} f(x) = \frac{1}{x^2}\left[f\left(q^{-1}x\right) - \left(q + q^{2\alpha}\right) f(x) + q^{2\alpha+1} f(qx) - \left(1 - q^{2\alpha}\right)(1-q) f(0)\right]$$

DEFINITION 3. Let $f \in \mathcal{L}_{q,1,\alpha+\frac{1}{2}}(\mathbb{R}_q)$, we define the q-Struve transform [8] as follow

$$\mathcal{F}_{\left(\alpha,\frac{1}{2}\right)}(f)\left(x;q^2\right) = c_{q,\left(\alpha,\frac{1}{2}\right)} \int_0^{+\infty} f(t)\, j_{\left(\alpha,\frac{1}{2}\right)}\left(xt;q^2\right) t^{2\alpha+1} d_q t$$

where

$$c_{q,\left(\alpha,\frac{1}{2}\right)} = \left(\frac{1}{1-q}\right)^2 \frac{\left(q^{2\alpha+2};q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \frac{\left(q^3;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}.$$

LEMMA 1. *The q-Struve transform verifies the following properties*

1.

$$\mathcal{F}_{\left(\alpha,\frac{1}{2}\right)} : \mathcal{L}_{q,1,\alpha}\left(\mathbb{R}_q^+\right) \to \mathcal{C}_{q,0}\left(\mathbb{R}_q^+\right)$$

2.

$$\mathcal{F}_{\left(\alpha,\frac{1}{2}\right)} : \mathcal{S}(\mathbb{R}_q^+) \to \mathcal{S}(\mathbb{R}_q^+)$$

3.

$$\mathcal{F}_{\left(\alpha,\frac{1}{2}\right)} : \mathcal{D}_a(\mathbb{R}_q^+) \to PW_{q,a}^{\alpha}$$

4.

$$\left\| \mathcal{F}_{\left(\alpha,\frac{1}{2}\right)} f \right\|_{q,\infty} \le c_{q,\left(\alpha,\frac{1}{2}\right)} \|f\|_{q,1,\alpha+\frac{1}{2}} .$$

Proof. Since we fixed $\beta = \frac{1}{2}$,We satisfy conditions of Theorem 6 and 7, and so we get the result. □

REMARK 1. In the same way, we can define the following q-analog transform

- The Hankel transform when $\alpha = 0$.
- The $\mathcal{Y}$ transform when $\alpha = \frac{1}{2}$.
- The Hartley transform when $\alpha = -\frac{1}{4}$.
- The Hardy transform which was first introduced by G.H. Hardy [13].

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