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COHOMOLOGY AND DEFORMATIONS OF HOM-LIE-YAMAGUTI COLOR ALGEBRAS

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Abstract. Hom-Lie-Yamaguti color algebras are defined and their representation and cohomology theory is considered. The $(2,3)$-cocycles of a given Hom-Lie-Yamaguti color algebra T are shown to be very useful in a study of its deformations. In particular, it is shown that any $(2,3)$-cocycle of T gives rise to a Hom-Lie-Yamaguti color structure on $T \oplus V$, where V is a T-module, and that a one-parameter infinitesimal deformation of T is equivalent to that a $(2,3)$-cocycle of T (with coefficients in the adjoint representation) defines a Hom-Lie-Yamaguti color algebra of deformation type.

1. Introduction

Lie-Yamaguti algebras (first called "generalized Lie triple systems") were introduced in [25] as an algebraic treatment of characteristic properties of the torsion and curvature of reductive homogeneous spaces. Later on, these algebras were called "Lie triple algebras" in [14] and the recent terminology was introduced in [15]. Cohomology groups of Lie-Yamaguti algebras were defined in [26] while their deformations and extensions were considered in [30]. In the framework of superalgebras, a $\mathbb{Z}_2$-graded generalization of Lie-Yamaguti algebras was studied in [19] and their color generalization was recently considered in [12] (one may refer to [20–22] for basics on color algebras).

The general theory of Hom-algebras started with the introduction of Hom-Lie algebras (see [10, 16]) while Hom-type algebras, other than Hom-Lie, were defined and discussed in [18]. A homological study of Hom-algebras could be found in [27]. During the last years, various Hom-algebraic structures have been widely investigated. An important compartment of the theory of Hom-algebras is the study of their representations, cohomologies and deformations (first papers on the subject are, e.g., [3,4,23]).

As a Hom-type generalization of Lie-Yamaguti algebras, Hom-Lie-Yamaguti algebras were introduced in [7] and their representations and deformations were considered in [17, 31]. Next, a $\mathbb{Z}_2$-graded generalization of Hom-Lie-Yamaguti algebras was considered in [6] and, recently, their representations and deformations were investigated in [9].

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The important role in physics played by color algebras (especially Lie color algebras) is well-known. This motivates their intrinsic algebraic study (see, e.g., [1, 2, 5, 20–22, 24, 28, 29] and references therein). Lie color triple systems seem to be first considered in [13]. The present paper aims to study representations and deformations of a Hom-type generalization of Lie-Yamaguti color algebras, called Hom-Lie-Yamaguti color algebras, as well as $(2, 3)$-cohomology groups associated to a representation of these algebras.

A description of the paper's content is as follows. In Section 2 we remind useful basic notions and next define a Hom-Lie-Yamaguti color algebra. Some construction results of Hom-Lie-Yamaguti color algebras are mentioned and some nontrivial examples are given. In Section 3 we define a representation of a Hom-Lie-Yamaguti color algebra as well as the coboundary operator on the cochain complex of such an algebra with coefficients in the representation space (as usual, this produces the cohomology group). While the general $(2n, 2n + 1)$-cohomology group seems to be not so fruitful, the special case of a $(2, 3)$-cohomology group is found useful in the construction of a Hom-Lie-Yamaguti color structure on the space $T \oplus V$, where T is a Hom-Lie-Yamaguti color algebra and V its representation space. The $(2, 3)$-cohomology group is also applied in Section 4 to study infinitesimal deformations and one-parameter formal deformations of Hom-Lie-Yamaguti color algebras.

All vector spaces and algebras are considered over a fixed algebraically closed field $\mathbb{K}$ of characteristic not 2 or 3 and Γ denotes an abelian group.

2. Definitions, constructions, and examples

A vector space V is said to be Γ-*graded* if V is a direct sum of a family of subspaces $\{V_\gamma, \gamma \in \Gamma\}$, $V := \bigoplus_{\gamma\in\Gamma} V_\gamma$. An element $x \in V$ is said to be *homogeneous of degree* $\gamma \in \Gamma$ whenever $x \in V_\gamma$, in which case we denote $\gamma := x$ since confusion rarely occurs. We assume that all considered elements in V are homogeneous and the set of all homogeneous elements of V will be denoted by $\mathcal{H}(V)$.

Let $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ and $W = \bigoplus_{\gamma\in\Gamma} W_\gamma$ be two Γ-graded vector spaces. A linear map $f : V \to W$ is said to be *homogeneous of degree* $\alpha \in \Gamma$ if $f(V_\gamma) \subseteq W_{\gamma+\alpha}$, $\forall \gamma \in \Gamma$. The map f is said to be *even* if f is homogeneous of degree zero (i.e. $f(V_\gamma) \subseteq W_\gamma$).

A Γ-*graded (binary) algebra* is a Γ-graded vector space $A = \bigoplus_\gamma A_\gamma$ together with a binary operation "$\cdot$" such that $A_\alpha \cdot A_\beta \subseteq A_{\alpha+\beta}$, $\forall \alpha, \beta \in \Gamma$. Likewise are defined a ternary algebra and a binary-ternary algebra.

A map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K}^*$ is called a *bicharacter* on Γ if the following identities hold for all α, β, γ in Γ:

$\varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma)$,
$\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)$,
$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = 1$.

Throughout this paper we assume that ε is a fixed bicharacter on Γ.

A *color algebra* is a triple $(A, \cdot, \varepsilon)$ where $(A, \cdot)$ is a Γ-graded algebra and ε is the fixed bicharacter on Γ. The most studied variety of color algebras is the one of Lie color algebras (one may refer to [20–22] for the introduction and first studies of Lie color algebras). A Hom-type generalization of Lie color algebras was introduced in [28].

Definition 2.1. [28] A *Hom-Lie color algebra* is a quadruple $(A, [,], \varepsilon, \alpha)$ consisting of a Γ-graded binary algebra $(A, [,], \varepsilon)$ and an even linear self-map α of A such that, for all $x, y, z \in \mathcal{H}(A)$, the following identities hold:
$[x, y] = -\varepsilon(x, y)[y, x]$ (ε-skew symmetry),
$[[x, y], \alpha(z)] + \varepsilon(x, y + z)[[y, z], \alpha(x)] + \varepsilon(x + y, z)[[z, x], \alpha(y)] = 0$ (ε-Hom-Jacobi identity).
The Hom-Lie color algebra $(A, [,], \varepsilon, \alpha)$ is said to be *multiplicative* if α is an even endomorphism of $(A, [,])$ i.e. $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for all $x, y \in \mathcal{H}(A)$.

One observes that for $\alpha = id$ in the definition above, we get a Lie color algebra $(A, [,], \varepsilon)$.

The non-ε-skew symmetric counterpart of a Hom-Lie color algebra is the one of a Hom-Leibniz color algebra.

Definition 2.2. A *(left) Hom-Leibniz color algebra* is a quadruple $(A, [,], \varepsilon, \alpha)$ consisting of a Γ-graded binary algebra $(A, [,], \varepsilon)$ and an even linear self-map α of A such that, for all $x, y, z \in \mathcal{H}(A)$, the following identity holds:
$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + \varepsilon(x, y)[\alpha(y), [x, z]]$.

Clearly, in case of ε-skew-symmetry, a Hom-Leibniz color algebra is a Hom-Lie color algebra. For $\alpha = id$ in Definition 2.2 we get a Leibniz color algebra that is first defined in [5].

Besides binary color algebras, ternary or binary-ternary color algebras could also be considered. Thus, as a color generalization of Lie triple systems, Lie color triple systems were considered in [13] while Lie-Yamaguti (LY) color algebras were introduced in [12]. As a Hom-type generalization of LY color algebras, we now give the definition of the basic object of this paper.

Definition 2.3. A *Hom-Lie-Yamaguti color algebra* ($HLYCA$ for short) is a tuple $(T, *, [, ,], \varepsilon, \alpha)$ consisting of a binary-ternary color algebra $(T, *, [, ,], \varepsilon)$, where "$*$" is a binary operation and "$[, ,]$" a ternary one on $T := \bigoplus_{\gamma\in\Gamma} T_\gamma$ such that $T_i * T_j \subseteq T_{i+j}$, $[T_i, T_j, T_k] \subseteq T_{i+j+k}$ $(i, j, k \in \Gamma)$, and a linear self-map α of T such that, for all $x_i, y_j \in \mathcal{H}(T)$,

- (HLY01) $\alpha(x_1 * x_2) = \alpha(x_1) * \alpha(x_2)$;
- (HLY02) $\alpha([x_1, x_2, x_3]) = [\alpha(x_1), \alpha(x_2), \alpha(x_3)]$;
- (HLY1) $x_1 * x_2 = -\varepsilon(x_1, x_2)x_2 * x_1$;
- (HLY2) $[x_1, x_2, x_3] = -\varepsilon(x_1, x_2)[x_2, x_1, x_3]$;
- (HLY3) $[x_1, x_2, x_3] + x_1x_2 * \alpha(x_3) + \varepsilon(x_1, x_2 + x_3)([x_2, x_3, x_1] + x_2x_3 * \alpha(x_1)) + \varepsilon(x_1 + x_2, x_3)([x_3, x_1, x_2] + x_3x_1 * \alpha(x_2)) = 0$;
- (HLY4) $[x_1 * x_2, \alpha(x_3), \alpha(y_1)] + \varepsilon(x_1, x_2 + x_3)[x_2 * x_3, \alpha(x_1), \alpha(y_1)] + \varepsilon(x_1 + x_2, x_3)[x_3 * x_1, \alpha(x_2), \alpha(y_1)] = 0$;
- (HLY5) $[\alpha(x_1), \alpha(x_2), y_1 * y_2] = [x_1, x_2, y_1] * \alpha^2(y_2) + \varepsilon(x_1 + x_2, y_1)\alpha^2(y_1) * [x_1, x_2, y_2]$;
- (HLY6) $[\alpha^2(x_1), \alpha^2(x_2), [y_1, y_2, y_3]] = [[x_1, x_2, y_1], \alpha^2(y_2), \alpha^2(y_3)] + \varepsilon(x_1 + x_2, y_1)[\alpha^2(y_1), [x_1, x_2, y_2], \alpha^2(y_3)] + \varepsilon(x_1 + x_2, y_1 + y_2)[\alpha^2(y_1), \alpha^2(y_2), [x_1, x_2, y_3]]$.

In the sequel $(T, *, [, ,], \varepsilon, \alpha)$ will simply be denoted by T unless otherwise stated.

Observe that, by (HLY01) and (HLY02), a $HLYCA$ is always multiplicative. If $\Gamma = \{0\}$ and $\varepsilon(0, 0) = 1$, a $HLYCA$ is a Hom-Lie-Yamaguti algebra ($HLYA$; see [7]) while for $\Gamma = \mathbb{Z}_2$ and $\varepsilon(i,j) = (-1)^{ij}$ $(i,j \in \mathbb{Z}_2)$ one gets a Hom-Lie-Yamaguti superalgebra ([6]). Clearly, for $\alpha = id$, one gets a *LY* color algebra ([12]). If $[u,v,w] = 0$, $\forall u,v,w \in \mathcal{H}(T)$ then T is a multiplicative Hom-Lie color algebra ([28]) while $u*v = 0$ defines a multiplicative *Hom-Lie color triple system* structure on T by the identities

- (HT0) $\alpha([x_1,x_2,x_3]) = [\alpha(x_1),\alpha(x_2),\alpha(x_3)]$;
- (HT1) $[x_1,x_2,x_3] = -\varepsilon(x_1,x_2)[x_2,x_1,x_3]$;
- (HT2) $[x_1,x_2,x_3] + \varepsilon(x_1,x_2+x_3)([x_2,x_3,x_1] + \varepsilon(x_1+x_2,x_3)([x_3,x_1,x_2] = 0$;
- (HT3) $[\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]] = [[x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3)]$
$+\varepsilon(x_1+x_2,y_1)[\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3)]$
$+\varepsilon(x_1+x_2,y_1+y_2)[\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3]]$.

Definition 2.4. A *HLYCA* $(T,*,[,,],\varepsilon,\alpha)$ is said to be *regular* if α is an automorphism of $(T,*,[,,])$.

From the result below it is clearly seen that any regular *HLYCA* induces a *LY* color algebra.

Proposition 2.5. *Let $(T,*,[,,],\varepsilon,\alpha)$ be a regular HLYCA. Then $(T,*_{\alpha^{-1}},[,,]_{\alpha^{-1}},\varepsilon)$ is a LY color algebra, where $x *_{\alpha^{-1}} y := \alpha^{-1}(x*y)$ and $[x,y,z]_{\alpha^{-1}} := \alpha^{-2}([x,y,z])$ for all $x,y,z \in \mathcal{H}(T)$.*

Proof. The ε-skew symmetry of "$*_{\alpha^{-1}}$" and "$[,,]_{\alpha^{-1}}$" is obvious. Next we have

$$[x_1,x_2,x_3]_{\alpha^{-1}} + x_1x_2 *_{\alpha^{-1}} x_3 + \varepsilon(x_1,x_2+x_3)([x_2,x_3,x_1]_{\alpha^{-1}} + x_2x_3 *_{\alpha^{-1}} x_1)$$
$$+\varepsilon(x_1+x_2,x_3)([x_3,x_1,x_2]_{\alpha^{-1}} + x_3x_1 *_{\alpha^{-1}} x_2)$$
$$= \alpha^{-2}\Big([x_1,x_2,x_3] + x_1x_2 * \alpha(x_3) + \varepsilon(x_1,x_2+x_3)([x_2,x_3,x_1] + x_2x_3 * \alpha(x_1))$$
$$+\varepsilon(x_1+x_2,x_3)([x_3,x_1,x_2] + x_3x_1 * \alpha(x_2))\Big)$$
$$= 0 \text{ (by (HLY3))}.$$

Likewise, using (HLY4), (HLY5) and (HLY6) respectively, one checks that

$$[x_1 *_{\alpha^{-1}} x_2,x_3,y_1]_{\alpha^{-1}} + \varepsilon(x_1,x_2+x_3)[x_2 *_{\alpha^{-1}} x_3,x_1,y_1]_{\alpha^{-1}}$$
$$+\varepsilon(x_1+x_2,x_3)[x_3 *_{\alpha^{-1}} x_1,x_2,y_1]_{\alpha^{-1}} = 0,$$
$$[x_1,x_2,y_1 *_{\alpha^{-1}} y_2]_{\alpha^{-1}} = [x_1,x_2,y_1]_{\alpha^{-1}} *_{\alpha^{-1}} y_2 + \varepsilon(x_1+x_2,y_1)y_1 *_{\alpha^{-1}} [x_1,x_2,y_2]_{\alpha^{-1}},$$
$$[x_1,x_2,[y_1,y_2,y_3]_{\alpha^{-1}}]_{\alpha^{-1}} = [[x_1,x_2,y_1]_{\alpha^{-1}},y_2,y_3]_{\alpha^{-1}}$$
$$+\varepsilon(x_1+x_2,y_1)[y_1,[x_1,x_2,y_2]_{\alpha^{-1}},y_3]_{\alpha^{-1}}$$
$$+\varepsilon(x_1+x_2,y_1+y_2)[y_1,y_2,[x_1,x_2,y_3]_{\alpha^{-1}}]_{\alpha^{-1}}$$

and so $(T,*_{\alpha^{-1}},[,,]_{\alpha^{-1}},\varepsilon)$ is a *LY* color algebra. □

We now consider some construction results for *HLYCA*. These results generalize the ones from [7, 8, 11] and their proofs are similar to that of [8, Theorem 3.4 and Corollary 3.5] so we omit them.

Theorem 2.6. *Let $(T,*,[,,],\varepsilon,\alpha)$ be a HLYCA, $\beta : T \to T$ be an even self-morphism of T (i.e. $\beta(T_\gamma) \subseteq T_\gamma$, $\gamma \in \Gamma$, $\beta(x*y) = \beta(x)*\beta(y)$, $\beta([x,y,z]) = [\beta(x),\beta(y),\beta(z)]$ for all $x,y,z \in \mathcal{H}(T)$ and $\beta\circ\alpha = \alpha\circ\beta$). Then $(T,*_{\beta^n},[,,]_{\beta^n},\varepsilon,$ $\beta^n\circ\alpha)$ is a HLYCA, where $x *_{\beta^n} y := \beta^n(x*y)$, $[x,y,z]_{\beta^n} := \beta^{2n}([x,y,z])$, $n \geq 1$, and $\beta^n = \beta\circ\beta^{n-1}$.*

Corollary 2.7. *Let $(T,*,[,,],\varepsilon)$ be a LY color algebra and β an even self-morphism of T. Define on T a binary operation "$*_\beta$" and a ternary operation "$[,,]_\beta$" by $x*_\beta y := \beta(x*y)$ and $[x,y,z]_\beta = \beta^2([x,y,z])$. Then $(T,*_\beta,[,,]_\beta,\varepsilon,\beta)$ is a HLYCA.*

Observe that Corollary 2.7 above provides examples of $HLYCA$ (either regular or not) starting from any LY color algebra with any self-morphism. Other examples are given below.

EXAMPLE 2.8. Let $(L, [,], \varepsilon, \alpha)$ any multiplicative Hom-Lie color algebra. If define on L a ternary operation by $\{x, y, z\}_\alpha := [[x, y], \alpha(z)]$, $\forall x, y, z \in \mathcal{H}(L)$, then $(L, [,], \{,,\}_\alpha, \varepsilon, \alpha)$ is a $HLYCA$.

As a specific example, consider $\Gamma = \mathbb{Z}_2^3$ and suppose that the vector space $sl(2, \mathbb{C})$ with basis $\{e_1, e_2, e_3\}$ is Γ-graded by letting $sl(2, \mathbb{C}) = sl(2, \mathbb{C})_{(1,1,0)} \oplus sl(2, \mathbb{C})_{(1,0,1)} \oplus sl(2, \mathbb{C})_{(0,1,1)}$, where $sl(2, \mathbb{C})_{(1,1,0)} = \langle e_1 \rangle$, $sl(2, \mathbb{C})_{(1,0,1)} = \langle e_2 \rangle$, $sl(2, \mathbb{C})_{(0,1,1)} = \langle e_3 \rangle$ and the homogeneous subspaces of $sl(2, \mathbb{C})$ graded by the elements of $\mathbb{Z}_2^3$ different from $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ are all zero and so are omitted. Now order the elements $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ by numbers $1, 2, 3$ respectively and define the commutation factor $\varepsilon : \Gamma \times \Gamma \to \mathbb{C}$ by the matrix

$$[\varepsilon(i, j)] = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

with $i, j = 1, 2, 3$. Then the binary operation "$[,]$" on $sl(2, \mathbb{C})$ given by

$[e_1, e_1] = 0$, $[e_2, e_2] = 0$, $[e_3, e_3] = 0$,

$[e_1, e_2] = e_3$, $[e_1, e_3] = e_2$, $[e_2, e_3] = e_1$

makes $sl(2, \mathbb{C})$ into a 3-dimensional Lie color algebra ([24, Example 1]). Next, if define an even linear self-map α of $sl(2, \mathbb{C})$ by $\alpha(e_1) = -e_1$, $\alpha(e_2) = -e_2$, $\alpha(e_3) = e_3$ then, by [28, Theorem 3.14], $sl(2, \mathbb{C})$ turns out to be a multiplicative Hom-Lie color algebra whose multiplication table is given by

$[e_1, e_2]_\alpha = e_3$, $[e_1, e_3]_\alpha = -e_2$, $[e_2, e_3]_\alpha = -e_1$.

Now the ternary operation "$\{,,\}_\alpha$" on the Hom-Lie color algebra $sl(2, \mathbb{C})$ is defined by the table

$\{e_1, e_2, e_1\}_\alpha := [[e_1, e_2]_\alpha, \alpha(e_1)]_\alpha = e_2$, $\{e_1, e_2, e_2\}_\alpha = e_1$, $\{e_1, e_2, e_3\}_\alpha = 0$

$\{e_1, e_3, e_1\}_\alpha = e_3$, $\{e_1, e_3, e_2\}_\alpha = 0$, $\{e_1, e_3, e_3\}_\alpha = e_1$,

$\{e_2, e_3, e_1\}_\alpha = 0$, $\{e_2, e_3, e_2\}_\alpha = e_3$, $\{e_2, e_3, e_3\}_\alpha = e_2$.

So $(sl(2, \mathbb{C}), [,]_\alpha, \{,,\}_\alpha, \varepsilon, \alpha)$ is a 3-dimensional $HLYCA$.

EXAMPLE 2.9. In [12, Section 3] it was shown that any (left) Leibniz color algebra bears a natural LY structure. Suppose that $(L, *, \varepsilon, \alpha)$ is a multiplicative left Hom-Leibniz color algebra. Then considering the Hom-type generalization of results from [12], one gets that $(L, [,], \{,,\}_\alpha, \varepsilon, \alpha)$ is a $HLYCA$ with respect to $[x, y] := x * y - \varepsilon(x, y) y * x$ and $\{x, y, z\}_\alpha := -\frac{1}{2}[x, y] * \alpha(z)$ for all $x, y, z \in \mathcal{H}(L)$.

3. Representation and cohomology

A *Hom-vector color space* is a triple (V, ε, α), where V is a Γ-graded vector space and α an even linear self-map of V. For simplicity we shall often write V for (V, ε, α). A morphism between two Hom-vector color spaces $(V_1, \varepsilon, \alpha_1)$ and $(V_2, \varepsilon, \alpha_2)$ is an even linear map $f : V_1 \to V_2$ such that $f \circ \alpha_1 = \alpha_2 \circ f$. The set of all morphisms between $(V_1, \varepsilon, \alpha_1)$ and $(V_2, \varepsilon, \alpha_2)$ is denoted by $Hom(V_1, V_2)$ and $End(V) := Hom(V, V)$ denotes the set of all self-morphisms of (V, ε, α).

3.1. Representation and semidirect product.

Definition 3.1. Let $(T,*,[,,],\varepsilon,\alpha)$ be a *HLYCA* and (V,ε,β) a Hom-vector color space. A *representation* of T on (V,ε,β) consists of an even linear map $\rho: T\to End(V)$ and even bilinear maps $D,\theta: T\times T\to End(V)$ such that the following conditions hold:

- (CHR01) $\rho(\alpha(x_1))\circ\beta=\beta\circ\rho(x_1)$;
- (CHR02) $D(\alpha(x_1),\alpha(x_2))\circ\beta=\beta\circ D(x_1,x_2)$;
- (CHR03) $\theta(\alpha(x_1),\alpha(x_2))\circ\beta=\beta\circ\theta(x_1,x_2)$;
- (CHR3) $D(x_1,x_2)-\varepsilon(x_1,x_2)\theta(x_2,x_1)+\theta(x_1,x_2)$
$+\rho(x_1*x_2)\circ\beta-\rho(\alpha(x_1))\circ\rho(x_2)+\varepsilon(x_1,x_2)\rho(\alpha(x_2))\circ\rho(x_1)=0$;
- (CHR41) $D(x_1*x_2,\alpha(x_3))+\varepsilon(x_1,x_2+x_3)D(x_2*x_3,\alpha(x_1))$
$+\varepsilon(x_1+x_2,x_3)D(x_3*x_1,\alpha(x_2))=0$;
- (CHR42) $\theta(x_1*x_2,\alpha(y_1))\circ\beta=\varepsilon(x_2,y_1)\theta(\alpha(x_1),\alpha(y_1))\circ\rho(x_2)$
$-\varepsilon(x_1,x_2+y_1)\theta(\alpha(x_2),\alpha(y_1))\circ\rho(x_1)$;
- (CHR51) $D(\alpha(x_1),\alpha(x_2))\circ\rho(y_2)=\varepsilon(x_1+x_2,y_2)\rho(\alpha^2(y_2))\circ D(x_1,x_2)$
$+\rho([x_1,x_2,y_2])\circ\beta^2$;
- (CHR52) $\theta(\alpha(x_1),y_1*y_2)\circ\beta=\varepsilon(x_1,y_1)\rho(\alpha^2(y_1))\circ\theta(x_1,y_2)$
$-\varepsilon(x_1+y_1,y_2)\rho(\alpha^2(y_2))\circ\theta(x_1,y_1)$;
- (CHR61) $D(\alpha^2(x_1),\alpha^2(x_2))\circ\theta(y_1,y_2)$
$=\varepsilon(x_1+x_2,y_1+y_2)\theta(\alpha^2(y_1),\alpha^2(y_2))\circ D(x_1,x_2)$
$+\theta([x_1,x_2,y_1],\alpha^2(y_2))\circ\beta^2$
$+\varepsilon(x_1+x_2,y_1)\theta(\alpha^2(y_1),[x_1,x_2,y_2])\circ\beta^2$;
- (CHR62) $\theta(\alpha^2(x_1),[y_1,y_2,y_3])\circ\beta^2$
$=\varepsilon(x_1+y_1,y_2+y_3)\theta(\alpha^2(y_2),\alpha^2(y_3))\circ\theta(x_1,y_1)$
$-\varepsilon(x_1,y_1+y_3)\varepsilon(y_2,y_3)\theta(\alpha^2(y_1),\alpha^2(y_3))\circ\theta(x_1,y_2)$
$+\varepsilon(x_1,y_1+y_2)D(\alpha^2(y_1),\alpha^2(y_2))\circ\theta(x_1,y_3)$ for all $x_i,y_i\in\mathcal{H}(T)$.

The Hom-vector space (V,ε,β) with the conditions above is called a T-*module* (or the *representation space* of T) and the triple (ρ,D,θ) is also called a representation of T.

Example 3.2. For a given *HLYCA* T there is a natural representation of T on itself (called the *adjoint representation*) given by

$\rho(x_1)(x_2):=x_1*x_2$,
$D(x_1,x_2)(x_3):=[x_1,x_2,x_3]$,
$\theta(x_1,x_2)(x_3):=\varepsilon(x_1+x_2,x_3)[x_3,x_1,x_2]$.

Remark 3.3. For $\Gamma=\mathbb{Z}_2$ and $\varepsilon(i,j):=(-1)^{ij}$, $i,j\in\mathbb{Z}_2$, one gets the representation of Hom-Lie-Yamaguti superalgebras ([9]). For $\Gamma=\{0\}$ and $\varepsilon(0,0)=1$ one gets the representation of Hom-Lie-Yamaguti algebras ([31]) and if, moreover, the twisting map is the identity map then we obtain the representation of Lie-Yamaguti algebras ([26]). The case when $\alpha=id$ in T and $\beta=id$ defines a *representation of the Lie-Yamaguti color algebra* $(T,*,[,,],\varepsilon)$ *on the color space* (V,ε). Obviously if $[x,y,z]=0$ in a *HLYCA* and D and θ are zero maps then we get the representation of Hom-Lie color algebras ([1]).

The following observation will be useful.

Lemma 3.4. *Let (ρ,D,θ) be a representation of a HLYCA T. Then D is ε-skew-symmetric.*

Proof. From (CHR62) we have

(CHR62') $\varepsilon(y_1,y_2)\theta(\alpha^2(x_1),[y_2,y_1,y_3])\circ\beta^2$
$= \varepsilon(x_1,y_1+y_3)\varepsilon(y_2,y_3)\theta(\alpha^2(y_1),\alpha^2(y_3))\circ\theta(x_1,y_2)$
$-\varepsilon(x_1+y_1,y_2+y_3)\theta(\alpha^2(y_2),\alpha^2(y_3))\circ\theta(x_1,y_1)$
$+\varepsilon(y_1,y_2)\varepsilon(x_1,y_2+y_1)D(\alpha^2(y_2),\alpha^2(y_1))\circ\theta(x_1,y_3).$

Adding memberwise (CHR62) and (CHR62') and using the ε-skew-symmetry of the operation "[, ,]", we get

$$0=\theta(\alpha^2(x_1),[y_1,y_2,y_3])\circ\beta^2+\varepsilon(y_1,y_2)\theta(\alpha^2(x_1),[y_2,y_1,y_3])\circ\beta^2$$
$$=\varepsilon(x_1,y_1+y_2)\Big(D(\alpha^2(y_1),\alpha^2(y_2))+\varepsilon(y_1,y_2)D(\alpha^2(y_2),\alpha^2(y_1))\Big)\circ\theta(x_1,y_3)$$

and so

$$D(\alpha^2(y_1),\alpha^2(y_2))=-\varepsilon(y_1,y_2)D(\alpha^2(y_2),\alpha^2(y_1)).$$

Therefore we have

$$D(\alpha^2(y_1),\alpha^2(y_2))\circ\beta^2=-\varepsilon(y_1,y_2)D(\alpha^2(y_2),\alpha^2(y_1))\circ\beta^2$$

i.e., by (CHR02),

$$\beta^2\circ D(y_1,y_2)=-\varepsilon(y_1,y_2)\beta^2\circ D(y_2,y_1)$$

which implies the ε-skew-symmetry of D. □

PROPOSITION 3.5. *Let $(T,*,[,,],\varepsilon,\alpha)$ be a HLYCA and (V,ε,β) a Hom-vector color space. Let $\rho: T\to End(V)$ be a linear map and $D,\theta: T\times T\to End(V)$ bilinear maps. Then (ρ,D,θ) is a representation of T on V if and only if $(T\oplus V,\odot,\{,,\},\varepsilon,\alpha+\beta)$ is a HLYCA, where*

$$(\alpha+\beta)(x_1+u_1):=\alpha(x_1)+\beta(u_1),$$
$$(x_1+u_1)\odot(x_2+u_2):=x_1*x_2+\rho(x_1)(u_2)-\varepsilon(x_1,x_2)\rho(x_2)(u_1),$$
$$\{x_1+u_1,x_2+u_2,x_3+u_3\}:=[x_1,x_2,x_3]+D(x_1,x_2)(u_3)-\varepsilon(x_2,x_3)\theta(x_1,x_3)(u_2)$$
$$+\varepsilon(x_1,x_2+x_3)\theta(x_2,x_3)(u_1)$$

for all $x_i\in\mathcal{H}(T)$ and $u_i\in\mathcal{H}(V)$.

Proof. The axioms (HLY01)-(HLY2) are obvious for $(T\oplus V,\odot,\{,,\},\varepsilon,\alpha+\beta)$ due to (CHR01), (CHR02), and the ε-skew-symmetry of "$*$", "[, ,]" and D (see Lemma 3.4 above). We now check (HLY3) for $(T\oplus V,\odot,\{,,\},\varepsilon,\alpha+\beta)$. We have

$$\{x_1+u_1,x_2+u_2,x_3+u_3\}+(x_1+u_1)(x_2+u_2)\odot(\alpha+\beta)(x_3+u_3)$$
$$+\varepsilon(x_1,x_2+x_3)(\{x_2+u_2,x_3+u_3,x_1+u_1\}+(x_2+u_2)(x_3+u_3)\odot(\alpha+\beta)(x_1+u_1))$$
$$+\varepsilon(x_1+x_2,x_3)(\{x_3+u_3,x_1+u_1,x_2+u_2\}+(x_3+u_3)(x_1+u_1)\odot(\alpha+\beta)(x_2+u_2))$$
$$=[x_1,x_2,x_3]+D(x_1,x_2)(u_3)-\varepsilon(x_2,x_3)\theta(x_1,x_3)(u_2)+\varepsilon(x_1,x_2+x_3)\theta(x_2,x_3)(u_1)$$
$$+(x_1*x_2+\rho(x_1)(u_2)-\varepsilon(x_1,x_2)\rho(x_2)(u_1))\odot(\alpha(x_3)+\beta(u_3))$$
$$+\varepsilon(x_1,x_2+x_3)([x_2,x_3,x_1]+D(x_2,x_3)(u_1)-\varepsilon(x_3,x_1)\theta(x_2,x_1)(u_3)$$
$$+\varepsilon(x_2,x_3+x_1)\theta(x_3,x_1)(u_2)$$
$$+(x_2*x_3+\rho(x_2)(u_3)-\varepsilon(x_2,x_3)\rho(x_3)(u_2))\odot(\alpha(x_1)+\beta(u_1)))$$
$$+\varepsilon(x_1+x_2,x_3)([x_3,x_1,x_2]+D(x_3,x_1)(u_2)-\varepsilon(x_1,x_2)\theta(x_3,x_2)(u_1)$$
$$+\varepsilon(x_3,x_1+x_2)\theta(x_1,x_2)(u_3)$$
$$+(x_3*x_1+\rho(x_3)(u_1)-\varepsilon(x_3,x_1)\rho(x_1)(u_3))\odot(\alpha(x_2)+\beta(u_2)))$$
$$=[x_1,x_2,x_3]+D(x_1,x_2)(u_3)-\varepsilon(x_2,x_3)\theta(x_1,x_3)(u_2)+\varepsilon(x_1,x_2+x_3)\theta(x_2,x_3)(u_1)$$
$$+\varepsilon(x_1,x_2+x_3)\Big([x_2,x_3,x_1]+D(x_2,x_3)(u_1)-\varepsilon(x_3,x_1)\theta(x_2,x_1)(u_3)$$
$$+\varepsilon(x_2,x_3+x_1)\theta(x_3,x_1)(u_2)\Big)$$
$$+\varepsilon(x_1+x_2,x_3)\Big([x_3,x_1,x_2]+D(x_3,x_1)(u_2)-\varepsilon(x_1,x_2)\theta(x_3,x_2)(u_1)$$
$$+\varepsilon(x_3,x_1+x_2)\theta(x_1,x_2)(u_3)\Big)$$

$$
\begin{aligned}
&+(x_1 * x_2) * \alpha(x_3) + \rho(x_1 * x_2)(\beta(u_3))\\
&-\varepsilon(x_1+x_2, x_3)\rho(\alpha(x_3))(\rho(x_1)(u_2) - \varepsilon(x_1,x_2)\rho(x_2)(u_1))\\
&+\varepsilon(x_1, x_2+x_3)\Big((x_2 * x_3) * \alpha(x_1) + \rho(x_2 * x_3)(\beta(u_1))\\
&\qquad -\varepsilon(x_2+x_3, x_1)\rho(\alpha(x_1))(\rho(x_2)(u_3) - \varepsilon(x_2,x_3)\rho(x_3)(u_2))\Big)\\
&+\varepsilon(x_1+x_2, x_3)\Big((x_3 * x_1) * \alpha(x_2) + \rho(x_3 * x_1)(\beta(u_2))\\
&\qquad -\varepsilon(x_3+x_1, x_2)\rho(\alpha(x_2))(\rho(x_3)(u_1) - \varepsilon(x_3,x_1)\rho(x_1)(u_3))\Big)\\
= &[x_1,x_2,x_3] + (x_1 * x_2) * \alpha(x_3) + \varepsilon(x_1, x_2+x_3)([x_2,x_3,x_1] + (x_2 * x_3) * \alpha(x_1))\\
&+\varepsilon(x_1+x_2, x_3)([x_3,x_1,x_2] + (x_3 * x_1) * \alpha(x_2))\\
&+D(x_1,x_2)(u_3) - \varepsilon(x_1,x_2)\theta(x_2,x_1)(u_3) + \theta(x_1,x_2)(u_3)\\
&+\rho(x_1 * x_2)(\beta(u_3)) - \rho(\alpha(x_1))(\rho(x_2)(u_3)) + \varepsilon(x_1,x_2)\rho(\alpha(x_2))(\rho(x_1)(u_3))\\
&+\varepsilon(x_1, x_2+x_3)\Big(D(x_2,x_3)(u_1) - \varepsilon(x_2,x_3)\theta(x_3,x_2)(u_1) + \theta(x_2,x_3)(u_1)\\
&+\rho(x_2 * x_3)(\beta(u_1)) - \rho(\alpha(x_2))(\rho(x_3)(u_1)) + \varepsilon(x_2,x_3)\rho(\alpha(x_3))(\rho(x_2)(u_1))\Big)\\
&+\varepsilon(x_1+x_2, x_3)\Big(D(x_3,x_1)(u_2) - \varepsilon(x_3,x_1)\theta(x_1,x_3)(u_2) + \theta(x_3,x_1)(u_2)\\
&+\rho(x_3 * x_1)(\beta(u_2)) - \rho(\alpha(x_3))(\rho(x_1)(u_2)) + \varepsilon(x_3,x_1)\rho(\alpha(x_1))(\rho(x_3)(u_2))\Big)
\end{aligned}
$$

and so (HLY3) holds for $(T \oplus V, \odot, \{,,\}, \varepsilon, \alpha+\beta)$ if and only if (CHR3) holds.

For (HLY4) we proceed as follows. We compute

$$
\begin{aligned}
&\bullet\ \{(x_1+u_1)\odot(x_2+u_2), (\alpha+\beta)(x_3+u_3), (\alpha+\beta)(y+v)\}\\
&= \{x_1 * x_2 + \rho(x_1)(u_2) - \varepsilon(x_1,x_2)\rho(x_2)(u_1), \alpha(x_3)+\beta(u_3), \alpha(y)+\beta(v)\}\\
&= [x_1 * x_2, \alpha(x_3), \alpha(y)] + D(x_1 * x_2, \alpha(x_3))(\beta(v)) - \varepsilon(x_3,y)\theta(x_1 * x_2, \alpha(y))(\beta(u_3))\\
&\quad +\varepsilon(x_1+x_2, x_3+y)\theta(\alpha(x_3),\alpha(y))(\rho(x_1)(u_2) - \varepsilon(x_1,x_2)\rho(x_2)(u_1)).
\end{aligned}
$$

Likewise we get

$$
\begin{aligned}
&\bullet\ \{(x_2+u_2)\odot(x_3+u_3), (\alpha+\beta)(x_1+u_1), (\alpha+\beta)(y+v)\}\\
&= [x_2 * x_3, \alpha(x_1), \alpha(y)] + D(x_2 * x_3, \alpha(x_1))(\beta(v)) - \varepsilon(x_1,y)\theta(x_2 * x_3, \alpha(y))(\beta(u_1))\\
&\quad +\varepsilon(x_2+x_3, x_1+y)\theta(\alpha(x_1),\alpha(y))(\rho(x_2)(u_3) - \varepsilon(x_2,x_3)\rho(x_3)(u_2))
\end{aligned}
$$

and

$$
\begin{aligned}
&\bullet\ \{(x_3+u_3)\odot(x_1+u_1), (\alpha+\beta)(x_2+u_2), (\alpha+\beta)(y+v)\}\\
&= [x_3 * x_1, \alpha(x_2), \alpha(y)] + D(x_3 * x_1, \alpha(x_2))(\beta(v)) - \varepsilon(x_2,y)\theta(x_3 * x_1, \alpha(y))(\beta(u_2))\\
&\quad +\varepsilon(x_3+x_1, x_2+y)\theta(\alpha(x_2),\alpha(y))(\rho(x_3)(u_1) - \varepsilon(x_3,x_1)\rho(x_1)(u_3)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\{(x_1+u_1)\odot(x_2+u_2), (\alpha+\beta)(x_3+u_3), (\alpha+\beta)(y+v)\}\\
&+\varepsilon(x_1, x_2+x_3)\{(x_2+u_2)\odot(x_3+u_3), (\alpha+\beta)(x_1+u_1), (\alpha+\beta)(y+v)\}\\
&+\varepsilon(x_1+x_2, x_3)\{(x_3+u_3)\odot(x_1+u_1), (\alpha+\beta)(x_2+u_2), (\alpha+\beta)(y+v)\}\\
&= [x_1 * x_2, \alpha(x_3), \alpha(y)] + \varepsilon(x_1, x_2+x_3)[x_2 * x_3, \alpha(x_1), \alpha(y)]\\
&\qquad +\varepsilon(x_1+x_2, x_3)[x_3 * x_1, \alpha(x_2), \alpha(y)]\\
&+D(x_1 * x_2, \alpha(x_3))(\beta(v)) + \varepsilon(x_1, x_2+x_3)D(x_2 * x_3, \alpha(x_1))(\beta(v))\\
&\qquad +\varepsilon(x_1+x_2, x_3)D(x_3 * x_1, \alpha(x_2))(\beta(v))\\
&+\varepsilon(x_3, y)\Big(-\theta(x_1 * x_2, \alpha(y))(\beta(u_3)) + \varepsilon(x_2,y)\theta(\alpha(x_1),\alpha(y))(\rho(x_2)(u_3))\\
&\qquad -\varepsilon(x_1, x_2+y)\theta(\alpha(x_2),\alpha(y))(\rho(x_1)(u_3))\Big)\\
&+\varepsilon(x_1+x_2, x_3)\varepsilon(x_2, y)\Big(-\theta(x_3 * x_1, \alpha(y))(\beta(u_2)) + \varepsilon(x_1,y)\theta(\alpha(x_3),\alpha(y))(\rho(x_1)(u_2))\\
&\qquad -\varepsilon(x_3, x_1+y)\theta(\alpha(x_1),\alpha(y))(\rho(x_3)(u_2))\Big)
\end{aligned}
$$

$$+\varepsilon(x_1, x_2+x_3)\varepsilon(x_1, y)\Big(-\theta(x_2*x_3, \alpha(y))(\beta(u_1))+\varepsilon(x_3, y)\theta(\alpha(x_2), \alpha(y))(\rho(x_3)(u_1)) -\varepsilon(x_2, x_3+y)\theta(\alpha(x_3), \alpha(y))(\rho(x_2)(u_1))\Big)$$

and so (HLY4) holds for $(T\oplus V, \odot, \{,,\}, \varepsilon, \alpha+\beta)$ if and only if (CHR41) and (CHR42) hold.

For (HLY5) we compute

- $\{(\alpha+\beta)(x_1+u_1), (\alpha+\beta)(x_2+u_2), (y_1+v_1)\odot(y_2+v_2)\}$
$= [\alpha(x_1), \alpha(x_2), y_1*y_2] + D(\alpha(x_1), \alpha(x_2))(\rho(y_1)(v_2) - \varepsilon(y_1, y_2)\rho(y_2)(v_1))$
$-\varepsilon(x_2, y_1+y_2)\theta(\alpha(x_1), y_1*y_2)(\beta(u_2)) + \varepsilon(x_1, x_2+y_1+y_2)\theta(\alpha(x_2), y_1*y_2)(\beta(u_1))$,
- $\{x_1+u_1, x_2+u_2, y_1+v_1\}\odot(\alpha+\beta)^2(y_2+v_2)$
$= [x_1, x_2, y_1]*\alpha^2(y_2) + \rho([x_1, x_2, y_1])(\beta^2(v_2))$
$-\varepsilon(x_1+x_2+y_1, y_2)\rho(\alpha^2(y_2))\Big(D(x_1, x_2)(v_1) - \varepsilon(x_2, y_1)\theta(x_1, y_1)(u_2) +\varepsilon(x_1, x_2+y_1)\theta(x_2, y_1)(u_1)\Big)$

and

- $\varepsilon(x_1+x_2, y_1)(\alpha+\beta)^2(y_1+v_1)\odot\{x_1+u_1, x_2+u_2, y_2+v_2\}$
$= \varepsilon(x_1+x_2, y_1)\alpha^2(y_1)*[x_1, x_2, y_2] + \varepsilon(x_1+x_2, y_1)\rho(\alpha^2(y_1))(D(x_1, x_2)(v_2))$
$-\varepsilon(x_1+x_2, y_1)\varepsilon(x_2, y_2)\rho(\alpha^2(y_1))(\theta(x_1, y_2)(u_2))$
$+\varepsilon(x_1+x_2, y_1)\varepsilon(x_1, x_2+y_2)\rho(\alpha^2(y_1))(\theta(x_2, y_2)(u_1))$
$-\varepsilon(y_1, y_2)\rho([x_1, x_2, y_2])(\beta^2(v_1))$.

Therefore (HLY5) holds for $(T\oplus V, \odot, \{,,\}, \varepsilon, \alpha+\beta)$ if and only if

$$\underbrace{D(\alpha(x_1), \alpha(x_2))(\rho(y_1)(v_2))}_{(a)} -\varepsilon(y_1, y_2)\underbrace{D(\alpha(x_1), \alpha(x_2))(\rho(y_2)(v_1))}_{(a')}$$
$$-\varepsilon(x_2, y_1+y_2)\underbrace{\theta(\alpha(x_1), y_1*y_2)(\beta(u_2))}_{(b)} +\varepsilon(x_1, x_2+y_1+y_2)\underbrace{\theta(\alpha(x_2), y_1*y_2)(\beta(u_1))}_{(b')}$$
$$= \underbrace{\rho([x_1, x_2, y_1])(\beta^2(v_2))}_{(a)} -\varepsilon(x_1+x_2+y_1, y_2)\underbrace{\rho(\alpha^2(y_2))(D(x_1, x_2)(v_1))}_{(a')}$$
$$+\varepsilon(x_1+x_2+y_1, y_2)\varepsilon(x_2, y_1)\underbrace{\rho(\alpha^2(y_2))(\theta(x_1, y_1)(u_2))}_{(b)}$$
$$-\varepsilon(x_1+x_2+y_1, y_2)\varepsilon(x_1, x_2+y_1)\underbrace{\rho(\alpha^2(y_2))(\theta(x_2, y_1)(u_1))}_{(b')}$$
$$+\varepsilon(x_1+x_2, y_1)\underbrace{\rho(\alpha^2(y_1))(D(x_1, x_2)(v_2))}_{(a)}$$
$$-\varepsilon(x_1+x_2, y_1)\varepsilon(x_2, y_2)\underbrace{\rho(\alpha^2(y_1))(\theta(x_1, y_2)(u_2))}_{(b)}$$
$$+\varepsilon(x_1+x_2, y_1)\varepsilon(x_1, x_2+y_2)\underbrace{\rho(\alpha^2(y_1))(\theta(x_2, y_2)(u_1))}_{(b')}$$
$$-\varepsilon(y_1, y_2)\underbrace{\rho([x_1, x_2, y_2])(\beta^2(v_1))}_{(a')}$$

i.e. if and only if (CH51) and (CH52) hold.

Finally, proceeding as for (HLY5) above, one checks that (HLY6) holds for $(T\oplus V, \odot, \{,,\}, \varepsilon, \alpha+\beta)$ if and only if (CHR61) and (CHR62) hold. This completes the proof. □

Definition 3.6. The *HLYCA* $T \oplus V$ as constructed in Proposition 3.5 is called the *semidirect product* of T and V.

3.2. Cohomology. For *HLYCAs* we now define cohomology complexes, as a generalization of the ones of Lie-Yamaguti algebras ([26]), Hom-Lie-Yamaguti algebras ([31]) or Hom-Lie-Yamaguti superalgebras ([9]).

Definition 3.7. Let $(T, *, [,,], \varepsilon, \alpha)$ be a *HLYCA* and (V, ε, β) a Hom-vector color space. The set $C^n(T, V)$ of n-cochains on T with values in V is the set of n-linear maps $f : T^n \to V$ satisfying

$$f(x_1, \cdots, x_i, x_{i+1}, \cdots, x_n) = -\varepsilon(x_i, x_{i+1}) f(x_1, \cdots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \cdots, x_n), \tag{3.1}$$

$$f(\alpha(x_1), \cdots, \alpha(x_n)) = \beta(f(x_1, \cdots, x_n)), \tag{3.2}$$

where $1 \leq i \leq n-1$, $C^0(T, V) = V$ and $(x_1, \cdots x_n) \in \mathcal{H}(T)^n$.

For integers $n < 0$ we let $C^n(T, V) = 0$. A map f is said to be *even* (resp. *of degree* r) when $f(x_1, \cdots, x_n) \in V_0$ (resp. $f(x_1, \cdots, x_n) \in V_r$) for all $(x_1, \cdots x_n) \in \mathcal{H}(T)^n$. Therefore $C^n(T, V)$, $\forall n$, are Γ-graded vector spaces.

Next the coboundary operator δ is defined as follows.

Definition 3.8. For any $(f, g) \in C^{2n}(T, V) \times C^{2n+1}(T, V)$, the *coboundary operator* is the map $\delta : C^{2n}(T, V) \times C^{2n+1}(T, V) \to C^{2n+2}(T, V) \times C^{2n+3}(T, V)$, $(f, g) \mapsto (\delta_I f, \delta_{II} g)$, where $\delta_I f$ and $\delta_{II} g$ are given by

$$\begin{aligned}
&(\delta_I f)(x_1, \cdots, x_{2n+2})\\
&= \varepsilon(g + x_1 + \cdots x_{2n}, x_{2n+2}) \rho(\alpha^{2n}(x_{2n+1}))(g(x_1, \cdots, x_{2n}, x_{2n+1}))\\
&-\varepsilon(g + x_1 + \cdots x_{2n+1}, x_{2n+2}) \rho(\alpha^{2n}(x_{2n+2}))(g(x_1, \cdots x_{2n}, x_{2n+1}))\\
&-g(\alpha(x_1), \cdots, \alpha(x_{2n}), x_{2n+1} * x_{2n+2})\\
&+\textstyle\sum_{k=1}^{n} (-1)^{n+k+1} \varepsilon(f + x_1 + \cdots + x_{2k-2} + x_{2k+1} + \cdots + x_{2n+2}, x_{2k-1} + x_{2k})\\
&\qquad D(\alpha^{2n-1}(x_{2k-1}), \alpha^{2n-1}(x_{2k}))(f(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+2}))\\
&+\textstyle\sum_{k=1}^{n} \sum_{j=2k+1}^{2n+2} (-1)^{n+k} \varepsilon(x_{2k-1} + x_{2k}, x_{2k+1} + x_{2k+2} + \cdots + x_{j-1})\\
&f(\alpha^2(x_1), \cdots, \alpha^2(x_{2k-2}), \hat{x}_{2k-1}, \hat{x}_{2k}, [x_{2k-1}, x_{2k}, x_j], \alpha^2(x_{j+1}), \cdots, \alpha^2(x_{2n+2})),
\end{aligned}$$

$$\begin{aligned}
&(\delta_{II} g)(x_1, \cdots, x_{2n+3})\\
&= \varepsilon(g + x_1 + \cdots + x_{2n+1}, x_{2n+2} + x_{2n+3}) \theta(\alpha^{2n}(x_{2n+2}), \alpha^{2n}(x_{2n+3}))\\
&\qquad (g(x_1, \cdots, x_{2n}, x_{2n+1}))\\
&-\varepsilon(g + x_1 + \cdots + x_{2n}, x_{2n+1} + x_{2n+3}) \varepsilon(x_{2n+2}, x_{2n+3})\\
&\theta(\alpha^{2n}(x_{2n+1}), \alpha^{2n}(x_{2n+3}))(g(x_1, \cdots, x_{2n}, x_{2n+2}))\\
&+\textstyle\sum_{k=1}^{n+1} (-1)^{n+k+1} \varepsilon(g + x_1 + \cdots + x_{2k-2}, x_{2k-1} + x_{2k})\\
&\qquad D(\alpha^{2n}(x_{2k-1}), \alpha^{2n}(x_{2k}))(g(x_1, \cdots, \hat{x}_{2k-1}, x_{2k}, \cdots, x_{2n+3}))\\
&+\textstyle\sum_{k=1}^{n+1} \sum_{j=2k+1}^{2n+3} (-1)^{n+k} \varepsilon(x_{2k-1} + x_{2k}, x_{2k+1} + x_{2k+2} + \cdots + x_{j-1})\\
&g(\alpha^2(x_1), \cdots, \alpha^2(x_{2k-2}), \hat{x}_{2k-1}, \hat{x}_{2k}, [x_{2k-1}, x_{2k}, x_j], \alpha^2(x_{j+1}), \cdots, \alpha^2(x_{2n+3})),
\end{aligned}$$

where the symbol $\hat{z}$ indicates that the letter z is omitted and the "h" in $\varepsilon(h + x_1 + \cdots)$ denotes the degree of the map $h \in C^m(T, V)$ with $m = 2n$ or $2n+1$.

Remark 3.9. In case when $\Gamma = \mathbb{Z}_2$ and $\varepsilon(i, j) = (-1)^{ij}$, $i, j \in \mathbb{Z}_2$, then the coboundary operator δ as defined above is the one defined in [9] for Hom-Lie-Yamaguti superalgebras. If $\Gamma = \{0\}$ and $\varepsilon(0, 0) = 1$ then δ is the one defined in [31] for Hom-Lie-Yamaguti algebras and if, moreover, the twisting map is the identity map then one recovers the Yamaguti's coboundary operator defined for Lie-Yamaguti algebras in [26].

Proposition 3.10. *With the notations as above, for any $(f,g) \in C^{2n}(T,V) \times C^{2n+1}(T,V)$, the following equalities hold:*

$$(\delta_I f)(\alpha(x_1), \cdots, \alpha(x_{2n+2})) = (\beta \circ \delta_I f)(x_1, \cdots, x_{2n+2}) \tag{3.3}$$

$$(\delta_{II} g)(\alpha(x_1), \cdots, \alpha(x_{2n+3})) = (\beta \circ \delta_{II} g)(x_1, \cdots, x_{2n+3}). \tag{3.4}$$

Thus the map $\delta : (f,g) \mapsto (\delta_I f, \delta_{II} g)$ is well-defined.

Proof. By Definition 3.7 and applying (3.2), (CHR01) or (CHR02) in suitable places, we have

$$\begin{aligned}
&(\delta_I f)(\alpha(x_1), \cdots, \alpha(x_{2n+2}))\\
&= \varepsilon(g + x_1 + \cdots + x_{2n}, x_{2n+1})\rho(\alpha^{2n+1}(x_{2n+1}))(g(\alpha(x_1), \cdots, \alpha(x_{2n}), \alpha(x_{2n+2})))\\
&-\varepsilon(g + x_1 + \cdots + x_{2n+1}, x_{2n+2})\rho(\alpha^{2n+1}(x_{2n+2}))(g(\alpha(x_1), \cdots, \alpha(x_{2n}), \alpha(x_{2n+1})))\\
&-g(\alpha^2(x_1), \cdots, \alpha^2(x_{2n}), \alpha(x_{2n+1} * x_{2n+2}))\\
&+\textstyle\sum_{k=1}^{n}(-1)^{n+k+1}\varepsilon(f + x_1 + \cdots + x_{2k-2} + x_{2k-1} + \cdots + x_{2n+2}, x_{2k-1} + x_{2k})\\
&\qquad D(\alpha^{2n}(x_{2k-1}), \alpha^{2n}(x_{2k}))(f(\alpha(x_1), \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots,\\
&\qquad\qquad \alpha(x_{2n+2})))\\
&+\textstyle\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^{n+k}\varepsilon(x_{2k-1} + x_{2k}, x_{2k+1} + x_{2k+2} + \cdots + x_{j-1})\\
&\quad f(\alpha^3(x_1), \cdots, \alpha^3(x_{2k-2}), \hat{x}_{2k-1}, \hat{x}_{2k}, \alpha([x_{2k-1}, x_{2k}, x_j]), \alpha^3(x_{j+1}), \cdots,\\
&\qquad\qquad \alpha^3(x_{2n+2}))\\
&= \varepsilon(g + x_1 + \cdots + x_{2n}, x_{2n+1})\rho(\alpha^{2n+1}(x_{2n+1}))(\beta(g(x_1, \cdots, x_{2n}, x_{2n+2})))\\
&-\varepsilon(g + x_1 + \cdots + x_{2n+1}, x_{2n+2})\rho(\alpha^{2n+1}(x_{2n+2}))(\beta(g(x_1, \cdots, x_{2n}, x_{2n+1})))\\
&-\beta(g(\alpha(x_1), \cdots, \alpha(x_{2n}), x_{2n+1} * x_{2n+2}))\\
&+\textstyle\sum_{k=1}^{n}(-1)^{n+k+1}\varepsilon(f + x_1 + \cdots + x_{2k-2} + x_{2k-1} + \cdots + x_{2n+2}, x_{2k-1} + x_{2k})\\
&\qquad D(\alpha^{2n}(x_{2k-1}), \alpha^{2n}(x_{2k}))(\beta(f(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+2})))\\
&+\textstyle\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^{n+k}\varepsilon(x_{2k-1} + x_{2k}, x_{2k+1} + x_{2k+2} + \cdots + x_{j-1})\\
&\quad \beta(f(\alpha^2(x_1), \cdots, \alpha^2(x_{2k-2}), \hat{x}_{2k-1}, \hat{x}_{2k}, [x_{2k-1}, x_{2k}, x_j], \alpha^2(x_{j+1}), \cdots,\\
&\qquad\qquad \alpha^2(x_{2n+2})))
\end{aligned}$$

(by (3.2))

$$\begin{aligned}
&= \varepsilon(g + x_1 + \cdots + x_{2n}, x_{2n+1})\beta(\rho(\alpha^{2n}(x_{2n+1}))(g(x_1, \cdots, x_{2n}, x_{2n+2})))\\
&-\varepsilon(g + x_1 + \cdots + x_{2n+1}, x_{2n+2})\beta(\rho(\alpha^{2n}(x_{2n+2}))(g(x_1, \cdots, x_{2n}, x_{2n+1})))\\
&-\beta(g(\alpha(x_1), \cdots, \alpha(x_{2n}), x_{2n+1} * x_{2n+2}))\\
&+\textstyle\sum_{k=1}^{n}(-1)^{n+k+1}\varepsilon(f + x_1 + \cdots + x_{2k-2} + x_{2k-1} + \cdots + x_{2n+2}, x_{2k-1} + x_{2k})\\
&\qquad \beta(D(\alpha^{2n-1}(x_{2k-1}), \alpha^{2n-1}(x_{2k}))(f(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots,\\
&\qquad\qquad x_{2n+2})))\\
&+\textstyle\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^{n+k}\varepsilon(x_{2k-1} + x_{2k}, x_{2k+1} + x_{2k+2} + \cdots + x_{j-1})\\
&\quad \beta(f(\alpha^2(x_1), \cdots, \alpha^2(x_{2k-2}), \hat{x}_{2k-1}, \hat{x}_{2k}, [x_{2k-1}, x_{2k}, x_j], \alpha^2(x_{j+1}), \cdots,\\
&\qquad\qquad \alpha^2(x_{2n+2})))
\end{aligned}$$

(by (CHR01) and (CHR02))

$$= (\beta \circ \delta_I f)(x_1, \cdots, x_{2n+2})$$

which proves (3.3). One also proves (3.4) in a similar way. □

By some tedious and lenghty computation, one could check that $\delta \circ \delta = 0$ (i.e $\delta_I \circ \delta_I = 0$ and $\delta_{II} \circ \delta_{II} = 0$). The subspace $Z^{2n}(T,V) \times Z^{2n+1}(T,V)$ of $C^{2n}(T,V) \times C^{2n+1}(T,V)$ spanned by the (f,g)'s such that $\delta(f,g) = 0$ is called *the space of cocycles* while the subspace $B^{2n}(T,V) \times B^{2n+1}(T,V) := \delta(C^{2n}(T,V) \times C^{2n+1}(T,V))$ is called *the space of coboundaries.*

For $n \geq 2$, the $(2n, 2n+1)$-*cohomology group of* T *with coefficients in* V is defined to be the quotient-space $H^{2n}(T,V)\times H^{2n+1}(T,V) := (Z^{2n}(T,V)\times Z^{2n+1}(T,V))/(B^{2n}(T,V)\times B^{2n+1}(T,V))$.

For some constructions that are of interest in our setting (see below), we will rather consider the $(2,3)$-cohomology group of a $HLYCA$ T with coefficients in (V, ε, β).

Denote by $C^2(T,V)$ the space of bilinear maps $\nu : T\times T \to V$ such that $\nu(x_1, x_2) = -\varepsilon(x_1, x_2)\nu(x_2, x_1)$ and

- (CC01) $\nu(\alpha(x_1), \alpha(x_2)) = (\beta \circ \nu)(x_1, x_2)$,

and by $C^3(T,V)$ the space of trilinear maps $\omega : T\times T\times T \to V$ such that $\omega(x_1, x_2, x_3) = -\varepsilon(x_1, x_2)\omega(x_2, x_1, x_3)$ and

- (CC02) $\omega(\alpha(x_1), \alpha(x_2), \alpha(x_3)) = (\beta \circ \omega)(x_1, x_2, x_3)$.

Definition 3.11. Let $(T, *, [,,], \varepsilon, \alpha)$ be a $HLYCA$, (V, ε, β) a Hom-vector color space and (ρ, D, θ) a representation of T on (V, ε, β). Then $(\nu, \omega) \in C^2(T,V) \times C^3(T,V)$ is called a $(2,3)$-*cocycle* if, for all $x_i, y_i \in \mathcal{H}(T)$, the following conditions hold:

- (CC1) $$\omega(x_1, x_2, x_3) - \rho(\alpha(x_1))\nu(x_2, x_3) + \nu(x_1 * x_2, \alpha(x_3)) +\varepsilon(x_1, x_2 + x_3)\Big(\omega(x_2, x_3, x_1) - \rho(\alpha(x_2))\nu(x_3, x_1) + \nu(x_2 * x_3, \alpha(x_1))\Big) +\varepsilon(x_1 + x_2, x_3)\Big(\omega(x_3, x_1, x_2) - \rho(\alpha(x_3))\nu(x_1, x_2) + \nu(x_3 * x_1, \alpha(x_2))\Big) = 0;$$
- (CC2) $$\varepsilon(x_2 + x_3, y_1)\theta(\alpha(x_1), \alpha(y_1))\nu(x_2, x_3) +\varepsilon(x_1, x_2 + x_3)\varepsilon(x_1 + x_3, y_1)\theta(\alpha(x_2), \alpha(y_1))\nu(x_3, x_1) +\varepsilon(x_1 + x_2, x_3)\varepsilon(x_1 + x_2, y_1)\theta(\alpha(x_3), \alpha(y_1))\nu(x_1, x_2) +\omega(x_1 * x_2, \alpha(x_3), \alpha(y_1)) +\varepsilon(x_1, x_2 + x_3)\omega(x_2 * x_3, \alpha(x_1), \alpha(y_1)) +\varepsilon(x_1 + x_2, x_3)\omega(x_3 * x_1, \alpha(x_2), \alpha(y_1)) = 0;$$
- (CC3) $$\omega(\alpha(x_1), \alpha(x_2), y_1 * y_2) + D(\alpha(x_1), \alpha(x_2))\nu(y_1, y_2) = \varepsilon(x_1 + x_2, y_1)\rho(\alpha^2(y_1))\omega(x_1, x_2, y_2) -\varepsilon(x_1 + x_2 + y_1, y_2)\rho(\alpha^2(y_2))\omega(x_1, x_2, y_1) +\nu([x_1, x_2, y_1], \alpha^2(y_2)) + \varepsilon(x_1 + x_2, y_1)\nu(\alpha^2(y_1), [x_1, x_2, y_2]);$$
- (CC4) $$\omega(\alpha^2(x_1), \alpha^2(x_2), [y_1, y_2, y_3]) + D(\alpha^2(x_1), \alpha^2(x_2))\omega(y_1, y_2, y_3) = \varepsilon(x_1 + x_2 + y_1, y_2 + y_3)\theta(\alpha^2(y_2), \alpha^2(y_3))\omega(x_1, x_2, y_1) -\varepsilon(x_1 + x_2, y_1 + y_3)\varepsilon(y_2, y_3)\theta(\alpha^2(y_1), \alpha^2(y_3))\omega(x_1, x_2, y_2) +\varepsilon(x_1 + x_2, y_1 + y_2)D(\alpha^2(y_1), \alpha^2(y_2))\omega(x_1, x_2, y_3) +\omega([x_1, x_2, y_1], \alpha^2(y_2), \alpha^2(y_3)) +\varepsilon(x_1 + x_2, y_1)\omega(\alpha^2(y_1), [x_1, x_2, y_2], \alpha^2(y_3)) +\varepsilon(x_1 + x_2, y_1 + y_2)\omega(\alpha^2(y_1), \alpha^2(y_2), [x_1, x_2, y_3]).$$

Observe that (CC3) and (CC4) mean $\delta_I \nu = 0$ and $\delta_{II}\omega = 0$ respectively.

The space of $(2,3)$-cocycles is denoted by $Z^2(T,V) \times Z^3(T,V)$.

Let $f : T \to V$ be a linear map. The map f is called a *derivation of* T *into* V if

$$f(x_1 * x_2) = \varepsilon(f, x_1)\rho(x_1)f(x_2) - \varepsilon(x_1 + f, x_2)\rho(x_2)f(x_1), \tag{3.5}$$

$$f([x_1, x_2, x_3]) = \varepsilon(x_1 + f, x_2 + x_3)\theta(x_2, x_3)f(x_1) - \varepsilon(x_2 + f, x_3)\varepsilon(f, x_1)\theta(x_1, x_3)f(x_2) + \varepsilon(f, x_1 + x_2)D(x_1, x_2)f(x_3). \tag{3.6}$$

Observe that if (ρ, D, θ) is the adjoint representation of the LY color algebra $(T, *, [,,], \varepsilon)$ (see Example 3.2), then f is a derivation of degree f of $(T, *, [,,], \varepsilon)$ [12].

We can now define a $(2,3)$-coboundary.

Definition 3.12. Let $(T,*,[,,],\varepsilon,\alpha)$ be a $HLYCA$, (V,ε,β) a Hom-vector color space and (ρ,D,θ) a representation of $(T,*,[,,],\varepsilon,\alpha)$ on (V,ε,β). Then $(\nu,\omega)\in C^2(T,V)\times C^3(T,V)$ is called a $(2,3)$-*coboundary* if there exists a map $f:T\to V$ such that

- (BB01) $f\circ\alpha=\beta\circ f$;
- (BB1) $\nu(x_1,x_2)=\varepsilon(f,x_1)\rho(x_1)f(x_2)-\varepsilon(x_1+f,x_2)\rho(x_2)f(x_1)-f(x_1*x_2)$;
- (BB2) $\omega(x_1,x_2,x_3)=\varepsilon(x_1+f,x_2+x_3)\theta(x_2,x_3)f(x_1)$ $-\varepsilon(x_2+f,x_3)\varepsilon(f,x_1)\theta(x_1,x_3)f(x_2)$ $+\varepsilon(f,x_1+x_2)D(x_1,x_2)f(x_3)-f([x_1,x_2,x_3])$

for all $x_i\in\mathcal{H}(T)$.

The space of $(2,3)$-coboundaries is denoted by $B^2(T,V)\times B^3(T,V)$.

Proposition 3.13. *The space of* $(2,3)$*-coboundaries is contained in the space of* $(2,3)$*-cocycles.*

Proof. We shall verify that any $(2,3)$-coboundary (ν,ω) satisfies the conditions (CC01), (CC02) and (CC1)-(CC4). With ν and ω given by (BB1) and (BB2) respectively, we obviously have $\nu(x_1,x_2)=-\varepsilon(x_1,x_2)\nu(x_2,x_1)$ and $\omega(x_1,x_2,x_3)=-\varepsilon(x_1,x_2)\omega(x_2,x_1,x_3)$ by Lemma 3.4 and the ε-skew-symmetry of the operations "$*$" and "$[,,]$". Now we have

$$\begin{aligned}
&\nu(\alpha(x_1),\alpha(x_2))-(\beta\circ\nu)(x_1,x_2)=\varepsilon(f,x_1)\rho(\alpha(x_1))f(\alpha(x_2))\\
&\quad-\varepsilon(x_1+f,x_2)\rho(\alpha(x_2))f(\alpha(x_1))-f(\alpha(x_1)*\alpha(x_2))\\
&\quad-\beta\Big(\varepsilon(f,x_1)\rho(x_1)f(x_2)-\varepsilon(x_1+f,x_2)\rho(x_2)f(x_1)-f(x_1*x_2)\Big)\ \text{(by (BB1))}\\
&=\varepsilon(f,x_1)\Big(\rho(\alpha(x_1))\circ\beta\circ f-\beta\circ\rho(x_1)\circ f\Big)(x_2)\\
&\quad-\varepsilon(x_1+f,x_2)\Big(\rho(\alpha(x_2))\circ\beta\circ f-\beta\circ\rho(x_2)\circ f\Big)(x_1)-(f\circ\alpha-\beta\circ f)(x_1*x_2)\\
&=0\ \text{(by (CHR01) and (BB01))}
\end{aligned}$$

so (CC01) holds for ν. Likewise, using successively (BB2), (CHR02), (CHR03) and (BB01), we have

$$\omega(\alpha(x_1),\alpha(x_2),\alpha(x_3))-(\beta\circ\omega)(x_1,x_2,x_3)=0$$

and so (CC02) holds for ω. Next we have

$$\begin{aligned}
&\omega(x_1,x_2,x_3)-\rho(\alpha(x_1))\nu(x_2,x_3)+\nu(x_1*x_2,\alpha(x_3))\\
&+\varepsilon(x_1,x_2+x_3)\Big(\omega(x_2,x_3,x_1)-\rho(\alpha(x_2))\nu(x_3,x_1)+\nu(x_2*x_3,\alpha(x_1))\Big)\\
&+\varepsilon(x_1+x_2,x_3)\Big(\omega(x_3,x_1,x_2)-\rho(\alpha(x_3))\nu(x_1,x_2)+\nu(x_3*x_1,\alpha(x_2))\Big)\\
&=\varepsilon(f,x_1+x_2)\Big(D(x_1,x_2)-\varepsilon(x_1,x_2)\theta(x_2,x_1)+\theta(x_1,x_2)+\rho(x_1*x_2)\circ\beta\\
&\qquad-\rho(\alpha(x_1))\circ\rho(x_2)+\varepsilon(x_1,x_2)\rho(\alpha(x_2))\circ\rho(x_1)\Big)f(x_3)\\
&\varepsilon(f+x_1,x_2+x_3)\Big(D(x_2,x_3)-\varepsilon(x_2,x_3)\theta(x_3,x_2)+\theta(x_2,x_3)+\rho(x_2*x_3)\circ\beta\\
&\qquad-\rho(\alpha(x_2))\circ\rho(x_3)+\varepsilon(x_2,x_3)\rho(\alpha(x_3))\circ\rho(x_2)\Big)f(x_1)\\
&\varepsilon(x_1+x_2,x_3)\varepsilon(f,x_3+x_1)\Big(D(x_3,x_1)-\varepsilon(x_3,x_1)\theta(x_1,x_3)+\theta(x_3,x_1)\\
&\qquad+\rho(x_3*x_1)\circ\beta\\
&\qquad-\rho(\alpha(x_3))\circ\rho(x_1)+\varepsilon(x_3,x_1)\rho(\alpha(x_1))\circ\rho(x_3)\Big)f(x_2)
\end{aligned}$$

$$-f\Big([x_1,x_2,x_3]+x_1x_2*\alpha(x_3)+\varepsilon(x_1,x_2+x_3)([x_2,x_3,x_1]+x_2x_3*\alpha(x_1))$$
$$+\varepsilon(x_1+x_2,x_3)([x_3,x_1,x_2]+x_3x_1*\alpha(x_2))\Big)$$
$= 0$ (by (CHR3) and (HLY3))
and so (CC1) holds for (ν,ω).

Likewise, as for (CC1) above, one checks that (CC2) holds by (CHR41), (CHR42) and (HLY4), (CC3) holds by (CHR51), (CHR52) and (HLY5) and finally (CC4) holds for (ν,ω) by (CHR61), (CHR62) and (HLY6).

Therefore we conclude that the space of $(2,3)$-coboundaries is contained in the space of $(2,3)$-cocycles. This completes the proof. □

Consistent with Proposition 3.13, we obtain the $(2,3)$-cohomology group of a $HLYCA$.

Definition 3.14. The $(2,3)$-cohomology group of a $HLYCA$ T with coefficients in the corresponding T-module V is the quotient-space

$$H^2(T,V)\times H^3(T,V) := (Z^2(T,V)\times Z^3(T,V))/(B^2(T,V)\times B^3(T,V)).$$

As described below, a representation of a $HLYCA$ T with T-module V and $(2,3)$-cocycle give rise to a $HLYCA$ structure on the space $T\oplus V$.

Proposition 3.15. *Let $(T,*,[,,],\varepsilon,\alpha)$ be a $HLYCA$, (ρ,D,θ) a representation of T on the space (V,ε,β), and (ν,ω) a $(2,3)$-cocycle of T with coefficients in V. If define on $T\oplus V$ the map $\alpha+\beta$ and operations "$*_\nu$" and "$[,,]_\omega$" by*

$(\alpha+\beta)(x+u) := \alpha(x)+\beta(u)$,
$(x_1+u_1)*_\nu(x_2+u_2) := x_1*x_2+\rho(x_1)(u_2)-\varepsilon(x_1,x_2)\rho(x_2)(u_1)+\nu(x_1,x_2)$,
$[x_1+u_1,x_2+u_2,x_3+u_3]_\omega := [x_1,x_2,x_3]+D(x_1,x_2)(u_3)$
$\qquad -\varepsilon(x_2,x_3)\theta(x_1,x_3)(u_2)+\varepsilon(x_1,x_2+x_3)\theta(x_2,x_3)(u_1)$
$\qquad +\omega(x_1,x_2,x_3)$,

then $T\oplus_{(\nu,\omega)}V := (T\oplus V,*_\nu,[,,]_\omega,\varepsilon,\alpha+\beta)$ is a $HLYCA$.

Proof. Upon the conditions (CC01), (CC02), (CC1)-(CC4) on (ν,ω), the proof repeats the one of Proposition 3.5 above (see also [31] for the case of Hom-Lie-Yamaguti algebras). □

4. Deformations

In this section the $(2,3)$-cohomology group is applied to study deformations of Hom-Lie-Yamaguti color algebras.

4.1. Infinitesimal deformations. In [31] infinitesimal deformations of Hom-Lie-Yamaguti algebras were considered. Here we extend the study from [31] to the color setting.

Let $(T,*,[,,],\varepsilon,\alpha)$ be a $HLYCA$. Let $\nu: T\times T\to T$ and $\omega: T\times T\times T\to T$ be bilinear and trilinear maps respectively.

Definition 4.1. The pair (ν,ω) is said to define a *Hom-Lie-Yamaguti color structure of deformation type* on the space T if (ν,ω) satisfies (HLY01)-(HLY6) except that (HLY3) is replaced with the condition

(HLY3') $\nu(\nu(x_1,x_2),\alpha(x_3))+\varepsilon(x_1,x_2+x_3)\nu(\nu(x_2,x_3),\alpha(x_1))$
$\qquad +\varepsilon(x_1+x_2,x_3)\nu(\nu(x_3,x_1),\alpha(x_2))=0.$

Consider now a t-parametrized family of binary operations "$*_t$" and ternary operations "$[,,]_t$" on T defined by

$x_1 *_t x_2 := x_1 * x_2 + t\nu(x_1, x_2)$,

$[x_1, x_2, x_3]_t := [x_1, x_2, x_3] + t\omega(x_1, x_2, x_3)$

for all $x_i \in \mathcal{H}(T)$.

Definition 4.2. The pair (ν, ω) is said to *generate a t-parameter infinitesimal deformation* of T if $T_t := (T, *_t, [,,]_t, \varepsilon, \alpha)$ is a $HLYCA$.

We have the following characterization of t-parameter infinitesimal deformations.

Theorem 4.3. *The pair (ν, ω) generates a t-parameter infinitesimal deformation of a $HLYCA$ T if and only if*

(i) (ν, ω) defines a Hom-Lie-Yamaguti color structure of deformation type on the space T,

(ii) (ν, ω) is a $(2,3)$-cocycle of T with coefficients in the adjoint representation (ρ, D, θ).

Proof. If T_t is a $HLYCA$ then the operations "$*_t$" and "$[,,]_t$" must verify the system (HLY01)-(HLY6).

From (HLY01) we have

$$\begin{aligned}0 &= \alpha(x_1 *_t x_2) - \alpha(x_1) *_t \alpha(x_2)\\ &= \alpha(x_1 * x_2) - \alpha(x_1) * \alpha(x_2) + t(\alpha(\nu(x_1, x_2)) - \nu(\alpha(x_1), \alpha(x_2)))\end{aligned}$$

and so (HLY01) holds for T_t if and only if ν satisfies (CC01) and (ρ, D, θ) is the adjoint representation.

Likewise (HLY02) holds for T_t if and only if ω satisfies (CC02) and (ρ, D, θ) is the adjoint representation.

From (HLY3) we have

$$\begin{aligned}0 &= [x_1, x_2, x_3]_t + x_1x_2 *_t \alpha(x_3) + \varepsilon(x_1, x_2 + x_3)([x_2, x_3, x_1]_t + x_2x_3 *_t \alpha(x_1))\\ &\qquad +\varepsilon(x_1 + x_2, x_3)([x_3, x_1, x_2]_t + x_3x_1 *_t \alpha(x_2))\\ &= [x_1, x_2, x_3] + x_1x_2 * \alpha(x_3) + \varepsilon(x_1, x_2 + x_3)([x_2, x_3, x_1] + x_2x_3 * \alpha(x_1))\\ &\qquad +\varepsilon(x_1 + x_2, x_3)([x_3, x_1, x_2] + x_3x_1 * \alpha(x_2))\\ &+t\Big(\omega(x_1, x_2, x_3) + \nu(x_1 * x_2, \alpha(x_3))\\ &\quad +\varepsilon(x_1, x_2 + x_3)(\omega(x_2, x_3, x_1) + \nu(x_2 * x_3, \alpha(x_1)))\\ &\quad +\varepsilon(x_1 + x_2, x_3)(\omega(x_3, x_1, x_2) + \nu(x_3 * x_1, \alpha(x_2)))\Big)\\ &+t\Big(\nu(x_1, x_2) * \alpha(x_3) + \varepsilon(x_1, x_2 + x_3)\nu(x_2, x_3) * \alpha(x_1)\\ &\quad +\varepsilon(x_1 + x_2, x_3)\nu(x_3, x_1) * \alpha(x_2)\Big)\\ &+t^2\Big(\nu(\nu(x_1, x_2), \alpha(x_3)) + \varepsilon(x_1, x_2 + x_3)\nu(\nu(x_2, x_3), \alpha(x_1))\\ &\quad +\varepsilon(x_1 + x_2, x_3)\nu(\nu(x_3, x_1), \alpha(x_2))\Big)\end{aligned}$$

and so (HLY3) holds for T_t if and only if (ν, ω) satisfies (CC1), (ρ, D, θ) is the adjoint representation, and (HLY3') holds.

From (HLY4) we have

$$\begin{aligned}0 &= [x_1 *_t x_2, \alpha(x_3), \alpha(y_1)]_t + \varepsilon(x_1, x_2 + x_3)[x_2 *_t x_3, \alpha(x_1), \alpha(y_1)]_t\\ &\qquad +\varepsilon(x_1 + x_2, x_3)[x_3 *_t x_1, \alpha(x_2), \alpha(y_1)]_t\\ &+t\Big(\omega(x_1 * x_2, \alpha(x_3), \alpha(y_1)) + [\nu(x_1, x_2), \alpha(x_3), \alpha(y_1)]\\ &\qquad +\varepsilon(x_1, x_2 + x_3)(\omega(x_2 * x_3, \alpha(x_1), \alpha(y_1)) + [\nu(x_2, x_3), \alpha(x_1), \alpha(y_1)])\end{aligned}$$

$$+\varepsilon(x_1+x_2,x_3)(\omega(x_3*x_1,\alpha(x_2),\alpha(y_1))+[\nu(x_3,x_1),\alpha(x_2),\alpha(y_1)])\Big)$$
$$+t^2\Big(\omega(\nu(x_1,x_2),\alpha(x_3),\alpha(y_1))+\varepsilon(x_1,x_2+x_3)\omega(\nu(x_2,x_3),\alpha(x_1),\alpha(y_1))$$
$$+\varepsilon(x_1+x_2,x_3)\omega(\nu(x_3,x_1),\alpha(x_2),\alpha(y_1))\Big)$$

and so (HLY4) holds for T_t if and only if (ν,ω) satisfies (CC2) and (HLY4), and (ρ, D, θ) is the adjoint representation.

The left-hand side of (HLY5) is

$$[\alpha(x_1),\alpha(x_2),y_1 *_t y_2]_t = [\alpha(x_1),\alpha(x_2),y_1*y_2]$$
$$+t(\omega(\alpha(x_1),\alpha(x_2),y_1*y_2)+[\alpha(x_1),\alpha(x_2),\nu(y_1,y_2)])$$
$$+t^2\omega(\alpha(x_1),\alpha(x_2),\nu(y_1,y_2))$$

and its right-hand side is

$$[x_1,x_2,y_1]_t *_t \alpha^2(y_2)+\varepsilon(x_1+x_2,y_1)\alpha^2(y_1) *_t [x_1,x_2,y_2]_t$$
$$= t\Big(\omega(x_1,x_2,y_1)*\alpha^2(y_2)+\varepsilon(x_1+x_2,y_1)\alpha^2(y_1)*\omega(x_1,x_2,y_2)$$
$$+\nu([x_1,x_2,y_1],\alpha^2(y_2))+\varepsilon(x_1+x_2,y_1)\nu(\alpha^2(y_1),[x_1,x_2,y_2])\Big)$$
$$+t^2\Big(\nu(\omega(x_1,x_2,y_1),\alpha^2(y_2))+\varepsilon(x_1+x_2,y_1)\nu(\alpha^2(y_1),\omega(x_1,x_2,y_2))\Big).$$

Therefore (HLY5) holds for T_t if and only if (ν,ω) satisfies (CC3) and (HLY5), and (ρ, D, θ) is the adjoint representation.

The left-hand side of (HLY6) is

$$[\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]_t]_t$$
$$= [\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]]$$
$$+t\Big(\omega(\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3])+[\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3)]\Big)$$
$$+t^2\omega(\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3))$$

while its right-hand side is

$$[[x_1,x_2,y_1]_t,\alpha^2(y_2),\alpha^2(y_3)]_t+\varepsilon(x_1+x_2,y_1)[\alpha^2(y_1),[x_1,x_2,y_2]_t,\alpha^2(y_3)]_t$$
$$+\varepsilon(x_1+x_2,y_1+y_2)[\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3]_t]_t$$
$$= [[x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3)]+\varepsilon(x_1+x_2,y_1)[\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3)]$$
$$+\varepsilon(x_1+x_2,y_1+y_2)[\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3]]$$
$$+t\Big(\omega([x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3))+[\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3)]$$
$$+\varepsilon(x_1+x_2,y_1)\omega(\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3))$$
$$+\varepsilon(x_1+x_2,y_1)[\alpha^2(y_1),\omega(x_1,x_2,y_2),\alpha^2(y_3)]$$
$$+\varepsilon(x_1+x_2,y_1+y_2)\omega(\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3])$$
$$\varepsilon(x_1+x_2,y_1+y_2)[\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3)]\Big)$$
$$+t^2\Big(\omega(\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3))$$
$$+\varepsilon(x_1+x_2,y_1)\omega(\alpha^2(y_2),\omega(x_1,x_2,y_2),\alpha^2(y_3))$$
$$+\varepsilon(x_1+x_2,y_1+y_2)\omega(\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3))\Big).$$

Therefore (HLY6) holds for T_t if and only if ω satisfies (CC4) and (HLY6), and (ρ, D, θ) is the adjoint representation. This completes the proof. □

4.2. 1-parameter formal deformations. Here we consider 1-parameter formal deformations of Hom-Lie-Yamaguti color algebras. A $\mathbb{Z}_2$-graded version of the study below was given in [9] and, for the ungraded version, one refers to [17].

Let $(T, *, [,,], \varepsilon, \alpha)$ be a $HLYCA$ and $\mathbb{K}[[t]]$ the ring of power series in one variable t with coefficients in $\mathbb{K}$. Denote by $T[[t]]$ the set of power series where coefficients are homogeneous elements in the vector color space T.

DEFINITION 4.4. Let $(T, *, [,,], \varepsilon, \alpha)$ be a $HLYCA$. A 1-*parameter formal deformation* of T is given by a pair (f_t, g_t), where f_t is a $\mathbb{K}[[t]]$-bilinear map $f_t : T[[t]] \times T[[t]] \to T[[t]]$, $f_t := \sum_{i\geq 0} f_i t^i$, and g_t a $\mathbb{K}[[t]]$-trilinear map $g_t : T[[t]] \times T[[t]] \times T[[t]] \to T[[t]]$, $g_t := \sum_{i\geq 0} g_i t^i$, where $f_0 := *$, $g_0 := [,,]$, and each f_i is an even $\mathbb{K}$-bilinear map $T \times T \to T$ (extended to be even $\mathbb{K}[[t]]$-bilinear), each g_i is an even $\mathbb{K}$-trilinear map $T \times T \times T \to T$ (extended to be even $\mathbb{K}[[t]]$-trilinear), such that $(T[[t]], f_t, g_t, \varepsilon, \alpha)$ is a $HLYCA$ over $\mathbb{K}[[t]]$. A deformation is said to be *of order k* whenever $f_t := \sum_{i=0}^{k} f_i t^i$, $g_t := \sum_{i=0}^{k} g_i t^i$.

Observe that the ε-skew-symmetry of f_t and g_t is equivalent to the ε-skew-symmetry of f_i and g_i respectively for each positive integer i.

That $(T[[t]], f_t, g_t, \varepsilon, \alpha)$ is a $HLYCA$ is equivalent to the following set of axioms (called the *deformation equations of a $HLYCA$*):

$$(\alpha \circ f_t)(x_1, x_2) = f_t(\alpha(x_1), \alpha(x_2)), \tag{4.1}$$

$$(\alpha \circ g_t)(x_1, x_2, x_3)) = g_t(\alpha(x_1), \alpha(x_2), \alpha(x_3)), \tag{4.2}$$

$$f_t(x_1, x_2) = -\varepsilon(x_1, x_2) f_t(x_2, x_1), \tag{4.3}$$

$$g_t(x_1, x_2, x_3) = -\varepsilon(x_1, x_2) g_t(x_2, x_1, x_3), \tag{4.4}$$

$$\begin{aligned} &g_t(x_1, x_2, x_3) + f_t(f_t(x_1, x_2), \alpha(x_3)) \\ &+ \varepsilon(x_1, x_2 + x_3)(g_t(x_2, x_3, x_1) + f_t(f_t(x_2, x_3), \alpha(x_1))) \\ &+ \varepsilon(x_1 + x_2, x_3)(g_t(x_3, x_1, x_2) + f_t(f_t(x_3, x_1), \alpha(x_2))) = 0, \end{aligned} \tag{4.5}$$

$$\begin{aligned} g_t(f_t(x_1, x_2), \alpha(x_3), \alpha(y_1)) + \varepsilon(x_1, x_2 + x_3) g_t(f_t(x_2, x_3), \alpha(x_1), \alpha(y_1)) \\ + \varepsilon(x_1 + x_2, x_3) g_t(f_t(x_3, x_1), \alpha(x_2), \alpha(y_1)) = 0, \end{aligned} \tag{4.6}$$

$$\begin{aligned} g_t(\alpha(x_1), \alpha(x_2), f_t(y_1, y_2)) =& f_t(g_t(x_1, x_2, y_1), \alpha^2(y_2)) \\ &+ \varepsilon(x_1 + x_2, y_1) f_t(\alpha^2(y_1), g_t(x_1, x_2, y_2)), \end{aligned} \tag{4.7}$$

$$\begin{aligned} &g_t(\alpha^2(x_1), \alpha^2(x_2), g_t(y_1, y_2, y_3)) \\ &= g_t(g_t(x_1, x_2, y_1), \alpha^2(y_2), \alpha^2(y_3)) \\ &+ \varepsilon(x_1 + x_2, y_1) g_t(\alpha^2(y_1), g_t(x_1, x_2, y_2), \alpha^2(y_3)) \\ &+ \varepsilon(x_1 + x_2, y_1 + y_2) g_t(\alpha^2(y_1), \alpha^2(y_2), g_t(x_1, x_2, y_3)) \end{aligned} \tag{4.8}$$

for all $x_i, y_i \in \mathcal{H}(T[[t]])$.

One observes that the conditions (4.1)-(4.4) mean that $(f_t, g_t) \in C^2(T,T)\times C^3(T,T)$. Also for a formal deformation of order k, the conditions (4.5)-(4.8) are written respectively as

$$\begin{aligned}
&g_k(x_1,x_2,x_3) + \sum\nolimits_{i=0}^{k} f_i(f_{k-i}(x_1,x_2),\alpha(x_3))\\
&+\varepsilon(x_1,x_2+x_3)\Big(g_k(x_2,x_3,x_1) + \sum\nolimits_{i=0}^{k} f_i(f_{k-i}(x_2,x_3),\alpha(x_1))\Big)\\
&+\varepsilon(x_1+x_2,x_3)\Big(g_k(x_3,x_1,x_2) + \sum\nolimits_{i=0}^{k} f_i(f_{k-i}(x_3,x_1),\alpha(x_2))\Big) = 0,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\sum\nolimits_{i=0}^{k}\Big(g_i(f_{k-i}(x_1,x_2),\alpha(x_3),\alpha(y_1))\\
&+\varepsilon(x_1,x_2+x_3)g_i(f_{k-i}(x_2,x_3),\alpha(x_1),\alpha(y_1))\\
&+\varepsilon(x_1+x_2,x_3)g_i(f_{k-i}(x_3,x_1),\alpha(x_2),\alpha(y_1))\Big) = 0,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&\sum\nolimits_{i=0}^{k} g_i(\alpha(x_1),\alpha(x_2),f_{k-i}(y_1,y_2))\\
&= \sum\nolimits_{i=0}^{k}\Big(f_i(g_{k-i}(x_1,x_2,y_1),\alpha^2(y_2)) + \varepsilon(x_1+x_2,y_1)f_i(\alpha^2(y_1),g_{k-i}(x_1,x_2,y_2))\Big),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&\sum\nolimits_{i=0}^{k} g_i(\alpha^2(x_1),\alpha^2(x_2),g_{k-i}(y_1,y_2,y_3))\\
&= \sum\nolimits_{i=0}^{k}\Big(g_i(g_{k-i}(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3))\\
&+\varepsilon(x_1+x_2,y_1)g_i(\alpha^2(y_1),g_{k-i}(x_1,x_2,y_2),\alpha^2(y_3))\\
&+\varepsilon(x_1+x_2,y_1+y_2)g_i(\alpha^2(y_1),\alpha^2(y_2),g_{k-i}(x_1,x_2,y_3))\Big).
\end{aligned} \tag{4.12}$$

The case $k = 1$ is of particular interest and then (4.9)-(4.12) read as

$$\begin{aligned}
&g_1(x_1,x_2,x_3) + f_1(x_1,x_2)*\alpha(x_3) + f_1(x_1*x_2,\alpha(x_3))\\
&+\varepsilon(x_1,x_2+x_3)\Big(g_1(x_2,x_3,x_1) + f_1(x_2,x_3)*\alpha(x_1) + f_1(x_2*x_3,\alpha(x_1))\Big)\\
&+\varepsilon(x_1+x_2,x_3)\Big(g_1(x_3,x_1,x_2) + f_1(x_3,x_1)*\alpha(x_2) + f_1(x_3*x_1,\alpha(x_2))\Big) = 0,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&[f_1(x_1,x_2),\alpha(x_3),\alpha(y_1)] + g_1(x_1*x_2,\alpha(x_3),\alpha(y_1))\\
&+\varepsilon(x_1,x_2+x_3)\Big([f_1(x_2,x_3),\alpha(x_1),\alpha(y_1)] + g_1(x_2*x_3,\alpha(x_1),\alpha(y_1))\Big)\\
&+\varepsilon(x_1+x_2,x_3)\Big([f_1(x_3,x_1),\alpha(x_2),\alpha(y_1)] + g_1(x_3*x_1,\alpha(x_2),\alpha(y_1))\Big) = 0,
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&[\alpha(x_1),\alpha(x_2),f_1(y_1,y_2)] + g(\alpha(x_1),\alpha(x_2),y_1*y_2)\\
&= g_1(x_1,x_2,y_1)*\alpha^2(y_2) + f_1([x_1,x_2,y_1],\alpha^2(y_2))\\
&+\varepsilon(x_1+x_2,y_1)\Big(\alpha^2(y_1)*g_1(x_1,x_2,y_2) + f_1(\alpha^2(y_1),[x_1,x_2,y_2])\Big),
\end{aligned} \tag{4.15}$$

(4.16)

$$\begin{aligned}&[\alpha^2(x_1),\alpha^2(x_2),g_1(y_1,y_2,y_3)]+g_1(\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3])\\&=[g_1(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3)]+g_1([x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3))\\&+\varepsilon(x_1+x_2,y_1)\Big([\alpha^2(y_1),g_1(x_1,x_2,y_2),\alpha^2(y_3)]+g_1(\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3))\Big)\\&+\varepsilon(x_1+x_2,y_1+y_2)\Big([\alpha^2(y_1),\alpha^2(y_2),g_1(x_1,x_2,y_3)]\\&+g_1(\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3])\Big).\end{aligned}$$

Comparing (4.13)-(4.16) with (CC1)-(CC4), it is easy to see that (f_1,g_1) is a $(2,3)$-cocycle with respect to the adjoint representation and $\delta_I f_1=0$, $\delta_{II} g_1=0$. Therefore, Theorem 4.3 implies that (f_1,g_1) generates a t-parameter infinitesimal deformation of $(T,*,[,,],\varepsilon,\alpha)$. In this case (f_1,g_1) is also called an infinitesimal deformation.

Definition 4.5. Let T be a $HLYCA$. Two 1-parameter formal deformations (f_t,g_t) and (f'_t,g'_t) of T are said to be *equivalent* (denote it by $(f_t,g_t)\sim(f'_t,g'_t)$) if there exists a formal isomorphism of $\mathbb{K}[[t]]$-modules $\phi_t(x)=\sum_{i\geq 0}\phi_i(x)t^i:(T[[t]],f_t,g_t,\varepsilon,\alpha)\to(T[[t]],f'_t,g'_t,\varepsilon,\alpha)$, where ϕ_i are even $\mathbb{K}$-linear self-maps of T (extended to be even $\mathbb{K}[[t]]$-linear) such that

$\phi_0=id_T$,

$\phi_t\circ\alpha=\alpha\circ\phi_t$,

$(\phi_t\circ f_t)(x,y)=f'_t(\phi_t(x),\phi_t(y))$,

$(\phi_t\circ g_t)(x,y,z)=g'_t(\phi_t(x),\phi_t(y),\phi_t(z))$

for all $x,y,z\in\mathcal{H}(T[[t]])$. If $(f_i,g_i)=(0,0)$, $\forall i\geq 1$, then the deformation (f_t,g_t) given by (f_0,g_0) is called the *null deformation* (which is also denoted by (f_0,g_0)). If $(f_t,g_t)\sim(f_0,g_0)$ as deformations, then (f_t,g_t) is called the *trivial deformation.* If every 1-parameter formal deformation of T is trivial, then T is called an *analytically rigid HLYCA.*

The following results are proved in the same way as in [9, Theorem 28 and Theorem 29] in case of Hom-Lie-Yamaguti superalgebras.

Theorem 4.6. *Let (f_t,g_t) and (f'_t,g'_t) be two equivalent* 1*-parameter formal deformations of a $HLYCA$ T. Then the infinitesimal deformations (f_1,g_1) and (f'_1,g'_1) belong to the same cohomology class in $H^2(T,T)\times H^3(T,T)$.*

Theorem 4.7. *Let T be a $HLYCA$ such that $H^2(T,T)\times H^3(T,T)=0$. Then T is analytically rigid.*

The relationships between $(2,3)$-cohomology groups of Hom-Lie-Yamaguti color algebras and their abelian extensions could be of interest for a further investigation.

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