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ON L-FUZZY SEMI-PRIME IDEALS OF A POSET AND SEPARATION THEOREMS

Derso Abeje Engidaw and Tilahun Bimerew Alemu

Abstract. In this paper, the relations between L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of a poset and L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of the lattice of all ideals of a poset are established. A result analogous to Separation Theorem is obtained using L-fuzzy semi-prime ideals.

1. Introduction

Fuzzy set theory was first introduced by L. A. Zadeh in 1965 as an extension of the classical notion of set theory [38]. He defined a fuzzy subset of a nonempty set S as a function from S to a unit interval $[0, 1]$ of real numbers. J. A. Goguen in [20] introduced the notion of L-fuzzy subsets by replacing the unit interval $[0, 1]$ by a complete lattice L in the definition of fuzzy subsets. Swamy and Swamy [37] initiated that complete lattices satisfying the infinite meet distributive law are the most appropriate candidates to have the truth values of general fuzzy statements.

The study of fuzzy sub-algebras of various algebraic structure has been started after Rosenfeld wrote his seminal paper [34] on fuzzy subgroups. This paper has provided sufficient motivations to researchers to study the fuzzy sub-algebras of different algebraic structures, like rings, modules, vector-spaces, lattices, and more recently in MS-algebras, universal algebras and pseudo-complemented semi-lattice etc. (See [18, 19, 29, 30, 37], [1, 14, 32], [16, 25], [2, 12, 15, 28, 36], [6, 7], [3–5], [13]). B. A Alaba et al. introduced several generalizations of L-fuzzy ideals and filters of a poset whose truth values are in a complete lattice satisfying the infinite meet distributive law.(See [8] and [9]). In addition in [10] and [11], we introduced certain comprehensive results on the notion of L-fuzzy prime ideals and L-fuzzy semiprime ideals of a poset.

Initiated by the above ideas and concepts, in this paper, we establish the relations between the L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of a poset and the L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of lattices of all ideals of a poset and some counter examples are also given. We also extend and prove an analogue of Stone's Theorem for finite posets which has been studied by V. S. Kharat and K. A. Mokbel [26] using L-fuzzy semi-prime ideals.

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2. Preliminaries

For the necessary concepts, terminologies and notations, we refer to [17] and [21]. A pair $(Q, \leq)$ is called a partially ordered set or simply a poset if Q is a non-empty set and " $\leq$ " is a partial order on Q. An element $x \in Q$ is called a lower bound of S if $x \leq s$ for all $s \in S$. An upper bound is defined dually. The set of all lower bounds of S is denoted by S^l and the set of all upper bounds of S by S^u. By the sets S^{ul} and S^{lu} we mean $\{S^u\}^l$ and $\{S^l\}^u$, respectively. For any $a, b \in Q$, the sets $\{a\}^l$ and $\{a, b\}^l$ are denoted by a^l and $(a, b)^l$, respectively. Further for subsets S, T of Q, $\{S \cup T\}^l$ is denoted by $\{S, T\}^l$ and the set $\{S \cup \{a\}\}^l$ is denoted by $\{S, a\}^l$. Similar notations are used for the set of all upper bounds of subsets of a poset Q.

For any subsets S, T of a poset Q, we note that, $S \subseteq S^{ul}$ and $S \subseteq S^{lu}$ and if $S \subseteq T$ in Q then $S^u \supseteq T^u$ and $S^l \supseteq T^l$. In addition,$\{a^u\}^l = a^l$ and $\{a^l\}^u = a^u$. An element x_0 in Q is called the least upper bound S or supremum of S, denoted by $supS$, if $x_0 \in S^u$ and $x_0 \leq x \; \forall x \in S^u$. Dually we have the concept of the greatest lower bound of S or infimum of S which is denoted by $infS$. For $x, y \in Q$, we write $x \wedge y$ (read as 'x meet y') in place of $\inf\{x, y\}$ if it exists and $x \vee y$ (read as "x join y") in place of $\sup\{x, y\}$ if it exists. A poset Q is said to be a join-semi-lattice (respectively, a meet-semi-lattice) if $x \vee y$ (respectively, $x \wedge y$) exists for all $x, y \in Q$ and is said to be a lattice if it is both a join-semi-lattice and a meet-semi-lattice.

An element x_0 in Q is called the smallest (respectively, the largest) element of a poset Q if $x_0 \leq x$ (respectively, $x \leq x_0$) for all $x \in Q$. The smallest (respectively, the largest) element if it exists in Q is denoted by 0 (respectively, by 1). A poset $(Q \leq)$ is called bounded if it has 0 and 1.

A subset S of a poset Q is said to be a down-set if it is decreasing, in the sense that $s \in S$ and $t \leq s$ imply that $t \in S$.

Definition 2.1. [23] A subset I of a poset $(Q, \leq)$ is called an ideal in Q if $(a, b)^{ul} \subseteq I$ whenever $a, b \in I$.

We consider the following sets that are studied in [22]. For any ideals I and J of a poset Q, define subsets of Q by:

$C_1(I, J) = \bigcup\{(a, b)^{ul} : a, b \in I \cup J\}$ and $C_{n+1}(I, J) = \bigcup\{(a, b)^{ul} : a, b \in C_n(I, J)\}$ for each $n \in \mathbb{N}$, inductively.

It is easy to observe that the set $\{C_n(I, J) : n \in \mathbb{N}\}$ forms a chain and each $C_n(I, J)$ is a down set of Q.

We state the following concepts that are essentially introduced for lattices by Rav [33].

Let Q be a given poset and $(\mathcal{I}(Q), \subseteq)$ be the lattice of all ideals of a poset. Define an extension of an ideal I of Q, denoted by I^e, as

$$I^e = \{J \in \mathcal{I}(Q) : J \subseteq I\}$$

and for an ideal λ of the lattice $(\mathcal{I}(Q), \subseteq)$ of all ideals of a poset Q, define the contraction of λ, denoted by λ^c, as

$$\lambda^c = \bigcup\{J : J \in \lambda\}.$$

Definition 2.2. [24] A proper ideal P of a poset Q is called prime, if for all $a, b \in Q$, $(a, b)^l \subseteq P$ implies either $a \in P$ or $b \in P$.

Now, we consider the concept of a semi-prime ideal introduced by V. S. Khart and K. A. Mokbel in a poset and by Y. Rav in a lattice, as given in the following.

Definition 2.3. [27] A proper ideal I of a poset Q is called a semi-prime ideal of Q if for all $x, y, z \in Q$,

$$(x, y)^l \subseteq I \text{ and } (x, z)^l \subseteq I \text{ imply } \{x, (y, z)^u\}^l \subseteq I.$$

Dually we have the concept semi-prime filter of a poset Q.

Definition 2.4. [33] A proper ideal I of a lattice X is called a semi-prime ideal of X if for all $x, y, z \in X$,

$$x \wedge y \in I \text{ and } x \wedge z \in I \text{ together imply } x \wedge (y \vee z) \in I.$$

Dually we have the concept semi-prime filter of a lattice X.

Throughout this paper L stands for a complete lattice satisfying the infinite meet distributive law, Q stands for a poset with 0 unless otherwise stated and $\mathcal{I}(Q)$ stands for the lattice of all ideals of Q.

By an L-fuzzy subset μ of a poset Q, we mean a mapping from Q into L. We denote the set of L- fuzzy subsets of Q by L^Q. For each $\alpha \in L$, the α-level subset of μ, which is denoted by μ_α, is a subset of Q given by: $\mu_\alpha = \{x : \mu(x) \geq \alpha\}$.

For fuzzy subsets μ and σ of Q, we write $\mu \subseteq \sigma$ to mean $\mu(x) \leq \sigma(x)$ for all $x \in Q$ in the ordering of L. It can be easily verified that " $\subseteq$ " is a partial order on the set L^Q and is called the *point wise ordering*. We write $\mu \subset \sigma$ if $\mu \subseteq \sigma$ and $\mu \neq \sigma$.

An L- fuzzy subset μ of Q is said to have the sup property if for every non-empty subset A of Q, the supremum of $\{\mu(x) : x \in A\}$ is attained at a point of A.

Definition 2.5. [36] An L-fuzzy subset μ of a lattice X with 0 is said to be an L -fuzzy ideal of X, if $\mu(0) = 1$ and $\mu(a \vee b) = \mu(a) \wedge \mu(b)$ for all $a, b \in X$.

Dually, an L-fuzzy subset μ of a lattice X with 1 is said to be an L -fuzzy filter of X, if $\mu(1) = 1$ and $\mu(a \wedge b) = \mu(a) \wedge \mu(b)$ for all $a, b \in X$.

Definition 2.6. ([8], [9]) $\mu \in L^Q$ is called an L- fuzzy ideal of Q if it satisfies the following conditions:

1. $\mu(0) = 1$
2. for any $a, b \in Q$, $\mu(x) \geq \mu(a) \wedge \mu(b)$ for all $x \in (a, b)^{ul}$.

Dually $\mu \in L^Q$, where Q is a poset with 1, is called an L- fuzzy filter of Q if it satisfies the following conditions:

1. $\mu(1) = 1$
2. for any $a, b \in Q$, $\mu(x) \geq \mu(a) \wedge \mu(b)$ for all $x \in (a, b)^{lu}$.

Definition 2.7. [9] An L-fuzzy filter μ is called an l-L-fuzzy filter if for any $a, b \in Q$, there exists $x \in (a, b)^l$ such that $\mu(x) = \mu(a) \wedge \mu(b)$.

Lemma 2.8. [8] *$\mu \in L^Q$ is an L- fuzzy ideal of Q if and only if μ_α is an ideal of Q, for all $\alpha \in L$.*

Lemma 2.9. [8] *If μ is an L- fuzzy ideal of Q, then μ is anti-tone. That is, $\mu(x) \geq \mu(y)$ whenever $x \leq y$.*

Note that for any α in L, the constant L-fuzzy subset of Q which maps all elements of Q onto α is denoted by $\overline{\alpha}$.

Definition 2.10. [10] An L-fuzzy ideal μ of a poset Q is called proper, if μ is not the constant map $\overline{1}$.

Definition 2.11. [10] A proper L-fuzzy ideal μ of a poset Q is called an L-fuzzy prime, if for any $a, b \in Q$,

$$inf\{\mu(x) : x \in (a,b)^l\} = \mu(a) \text{ or } \mu(b).$$

Definition 2.12. [28] A proper L-fuzzy ideal μ of a lattice X is called L-fuzzy prime, if for any $a, b \in Q$,

$$\mu(a \wedge b) = \mu(a) \text{ or } \mu(b).$$

Definition 2.13. [11] An L-fuzzy ideal μ of a poset Q is called an L-fuzzy semi-prime ideal if for all $a, b, c \in Q$,

$$\mu(z) \geq \inf\{\mu(x) \wedge \mu(y) : x \in (a,b)^l, y \in (a,c)^l\} \;\; \forall z \in \{a, (b,c)^u\}^l.$$

Dually we have the concept of L-fuzzy semi-prime filter of a poset Q.

Lemma 2.14. *[11] An L- fuzzy ideal μ of Q is an L- fuzzy semi-prime ideal of Q if and only if μ_α is a semi-prime ideal of Q for all $\alpha \in L$.*

Definition 2.15. [11] An L-fuzzy ideal μ of a lattice X is called an L-fuzzy semi-prime ideal, if for all $a, b, c \in Q$,

$$\mu(a \wedge (b \vee c)) = \mu(a \wedge b) \wedge \mu(a \wedge c).$$

Dually we have the concept of L-fuzzy semi-prime filter of a lattice X.

Lemma 2.16. *Let μ be an L- fuzzy ideal of Q. Then for any $a, b \in Q$,*

$$inf\{\mu(x) : x \in (a,b)^l\} = \mu(a \wedge b),$$

whenever $a \wedge b$ exists in Q.

3. On L-Fuzzy Semi-prime Ideals of a Poset

In this section we study the relations between L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of a poset and L-fuzzy semi-prime (respectively, L-fuzzy prime) ideals of the lattice of all ideals of a poset are established. Some counter examples are also given.

We begin by introducing the notion of an extension of an L-fuzzy ideal of a poset and a contraction of an L-fuzzy ideal of a lattice of all ideals of a poset.

Definition 3.1. Let μ is an L-fuzzy ideal of Q and Φ is an L-fuzzy ideal of $\mathcal{I}(Q)$. Then

1. an extension of μ of Q, denoted by μ^e, is an L -fuzzy subset of $\mathcal{I}(Q)$ given by: for all I $\in \mathcal{I}(Q)$,
$$\mu^e(I) = \inf\{\mu(x) : x \in I\}$$
2. a contraction of Φ of $\mathcal{I}(Q)$, denoted by Φ^c, is an L-fuzzy subset of Q given by: for all $x \in Q$,
$$\Phi^c(x) = \sup\{\Phi(I) : x \in I\}.$$

Lemma 3.2. *Let μ be an L-fuzzy ideal of Q. Then*

$$(\mu^e)_\alpha = (\mu_\alpha)^e$$

for all $\alpha \in L$.

Lemma 3.3. *Let Φ be an L-fuzzy ideal of $\mathcal{I}(Q)$ with sup property and $\alpha \in L$. Then $(\Phi^c)_\alpha = (\Phi_\alpha)^c$*

LEMMA 3.4. *Let μ be an L-fuzzy ideal Q. Then its extension μ^e is an L-fuzzy ideal of $\mathcal{I}(Q)$.*

Proof. Now $\mu^e((0]) = \inf\{\mu(x) : x \in (0]\} = \mu(0) = 1$. Let $I, J \in \mathcal{I}(Q)$. Then

$$\begin{aligned} \mu^e(I) &= \inf\{\mu(x) : x \in I\} \\ &\geq \inf\{\mu(x) : x \in I \vee J\} \\ &= \mu^e(I \vee J) \end{aligned}$$

and similarly we have $\mu^e(J) \geq \mu^e(I \vee J)$. Thus $\mu^e(I) \wedge \mu^e(J) \geq \mu^e(I \vee J)$. Again to show the other inequality put $\alpha = \mu^e(I) \wedge \mu^e(J)$. Now

$$\begin{aligned} \alpha = \mu^e(I) \wedge \mu^e(J) &\Rightarrow \alpha \leq \mu^e(I) = \inf\{\mu(x) : x \in I\} \text{ and} \\ &\quad\ \alpha \leq \mu^e(J) = \inf\{\mu(y) : y \in I\} \\ &\Rightarrow \alpha \leq \mu(x) \text{ for all } x \in I \text{ and } \alpha \leq \mu(y) \text{ for all } y \in J \\ &\Rightarrow I \subseteq \mu_\alpha \text{ and `} J \subseteq \mu_\alpha \\ &\Rightarrow I \cup J \subseteq \mu_\alpha \\ &\Rightarrow I \vee J \subseteq \mu_\alpha \\ &\Rightarrow I \vee J \in (\mu_\alpha)^e = (\mu^e)_\alpha \\ &\Rightarrow \mu^e(I \vee J) \geq \alpha = \mu^e(I) \wedge \mu^e(J) \end{aligned}$$

Therefore $\mu^e(I \vee J) = \mu^e(I) \wedge \mu^e(J)$. Hence μ^e is an L-fuzzy ideal $\mathcal{I}(Q)$. □

LEMMA 3.5. *Let Φ be an L-fuzzy ideal of $\mathcal{I}(Q)$. Then Φ^c is an L-fuzzy ideal of Q.*

Proof. Now since

$$\begin{aligned} \Phi^c(0) &= sup\{\Phi(I) : 0 \in I\} \\ &\geq \Phi((0]) \\ &= 1 \end{aligned}$$

we have $\Phi^c(0) = 1$. Again let $a, b \in Q$ and $x \in (a, b)^{ul}$. Now

$$\begin{aligned} \Phi^c(a) \wedge \Phi^c(b) &= \sup\{\Phi(I) : a \in I\} \wedge \sup\{\Phi(J) : b \in J\} \\ &= \sup\{\Phi(I) \wedge \Phi(J) : a \in I, b \in J\} \\ &= \sup\{\Phi(I \vee J) : a \in I, b \in J\} \\ &\leq \sup\{\Phi(I \vee J) : x \in (a, b)^{ul} \subseteq I \vee J\} \\ &\leq \sup\{\Phi(K) : x \in K\} \\ &= \Phi^c(x) \end{aligned}$$

Therefore Φ^c is an L-fuzzy ideal of Q. □

LEMMA 3.6. *Let μ be an L-fuzzy ideal of a poset Q. Then $\mu^{ec} = \mu$.*

Proof. Let μ be an L-fuzzy ideal of a poset Q. Now we claim that $\mu^{ec} = \mu$. Now for any $x \in Q$, we have

$$\begin{aligned} (\mu^{ec})(x) &= (\mu^e)^c(x) \\ &= \sup\{\mu^e(I) : x \in I, I \in \mathcal{I}(Q)\} \\ &\geq \mu^e((x]) \\ &= \inf\{\mu(y) : y \in (x]\} = \mu(x). \end{aligned}$$

Thus we have $\mu^{ec} \supseteq \mu$. Again for any $x \in Q$, put $\mathcal{S}_x = \{I \in \mathcal{I}(Q) : x \in I\}$. Clearly $\mathcal{S}_x$ is non empty. Now for any $I \in \mathcal{S}_x$, we have $\mu(x) \geq inf\{\mu(y) : y \in I\} = \mu^e(I)$. This implies that

$$\mu(x) \geq \sup\{\mu^e(I) : I \in \mathcal{S}_x\} = \sup\{\mu^e(I) : x \in I\} = (\mu^e)^c(x).$$

Therefore $\mu \supseteq \mu^{ec}$. Hence the claim is true. □

We use the following Lemma in the result followed by it which is a relation between L-fuzzy semi-prime ideals of Q and L-fuzzy semi-prime ideals of a poset.

Lemma 3.7. *[22] Let Q be a poset and $I, J \in \mathcal{I}(Q)$. Then the supremum $I \vee J$ of I and J in $\mathcal{I}(Q)$ is given by:*

$$I \vee J = \bigcup\{\{C_n(I, J) : n \in \mathbb{N}\}$$

Theorem 3.8. *Let μ be an L-fuzzy semi-prime ideal of Q. Then μ^e is an L-fuzzy semi-prime ideal of $\mathcal{I}(Q)$.*

Proof. Let $I, J, K \in \mathcal{I}(Q)$. Now we prove that

$$\mu^e(I \cap J) \wedge \mu^e(I \cap K) = \mu^e(I \cap (J \vee K)).$$

Since μ^e is an L-fuzzy ideal of $\mathcal{I}(Q)$ and $I \cap J \subseteq I \cap (J \vee K)$ and $I \cap K \subseteq I \cap (J \vee K)$ we clearly have

$$\mu^e(I \cap J) \wedge \mu^e(I \cap K) \geq \mu^e(I \cap (J \vee K)).$$

Again to show the other inequality it is enough to show that for each $n \in \mathbb{N}$

$$\mu^e(I \cap J) \wedge \mu^e(I \cap K) \leq \mu(x) \text{ for all } x \in I \cap C_n(J, K),$$

in view of Lemma 3.7. We use induction on n.

1. Let $n = 1$ and $x \in I \cap C_1(J, K)$. Then $x \in I$ and $x \in (a, b)^{ul}$ for some $a, b \in J \cup K$. If $a, b \in J$ or K, then obviously $\mu^e(I \cap J) \wedge \mu^e(I \cap K) \leq \mu(x)$. So, let us suppose without loss of generality, that $a \in J$ and $b \in K$. Then $(x, a)^l \subseteq I \cap J$ and $(x, b)^l \subseteq I \cap K$. By L-fuzzy semi-primness of μ, we have

$$\begin{aligned}\mu^e(I \cap J) \wedge \mu^e(I \cap K) &= \inf\{\mu(y) : y \in I \cap J\} \wedge \inf\{\mu(y) : y \in I \cap K\} \\ &\leq \inf\{\mu(y) : y \in (x, a)^l\} \wedge \inf\{\mu(z) : z \in (x, b)^l\} \\ &\leq \inf\{\mu(y) \wedge \mu(z) : y \in (x, a)^l, z \in (x, b)^l\} \\ &\leq \mu(x)\end{aligned}$$

   Thus the statement is true for $n = 1$.
2. Suppose that $\mu^e(I \cap J) \wedge \mu^e(I \cap K) \leq \mu(x)$ for all $x \in I \cap C_n(J, K)$ holds for some $n \in \mathbb{N}$. We will prove that it also holds for n+1. Now $x \in I \cap C_{n+1}(J, K)$ implies that $x \in I$ and $x \in (a, b)^{ul}$ for some $a, b \in C_n(J, K)$. This implies that $(x, a)^l, \subseteq I \cap C_n(J, K)$ and $(x, b)^l \subseteq I \cap C_n(J, K)$. Thus, by induction hypothesis, we have

$$\mu^e(I \cap J) \wedge \mu^e(I \cap K) \leq \mu(y) \text{ for all } y \in (x, a)^l$$

   and

$$\mu^e(I) \wedge \mu^e(J) \leq \mu(z) \text{ for all } z \in (x, b)^l$$

and since μ is L-fuzzy semi-prime and $x \in \{x, (a,b)^u\}^l$ we have

$$\begin{aligned} \mu^e(I \cap J) \wedge \mu^e(I \cap K) &\leq \inf\{\mu(y) \wedge \mu(z) : y \in (x,a)^l, z \in (x,b)^l \\ &\leq \mu(x). \end{aligned}$$

Therefore $\mu^e(I \cap J) \wedge \mu^e(I \cap K) \leq \mu(x)$ for all $x \in I \cap C_n(J,K)$ for each $n \in \mathbb{N}$. Thus we have

$$\begin{aligned} \mu^e(I \cap J) \wedge \mu^e(I \cap K) &\leq \inf\{\mu(x) : x \in \bigcup\{I \cap C_n(J,K) : n \in \mathbb{N}\}\} \\ &\leq \inf\{\mu(x) : x \in I \cap \bigcup\{C_n(J,K) : n \in \mathbb{N}\} \\ &= \inf\{\mu(x) : x \in I \cap (J \vee K)\} \\ &= \mu^e(I \cap (J \vee K)) \end{aligned}$$

Therefore $\mu^e(I\cap J)\wedge\mu^e(I\cap K) = \mu^e(I\cap(J\vee K))$ and hence μ^e is an L-fuzzy semi-prime ideal of the lattice $\mathcal{I}(Q)$. □

Theorem 3.9. *Let Q be a finite poset and let Φ be an L-fuzzy semi-prime ideal of $\mathcal{I}(Q)$ with sup property. Then Φ^c is an L-fuzzy semi-prime ideal of Q.*

Proof. Clearly Φ^c is an L-fuzzy ideal of Q ,by Lemma 3.5. Now we show that Φ^c is an L-fuzzy semi-prime ideal Q. Let $a, b, c \in Q$ and $z \in \{a, (b,c)^u\}^l$. Now put

$$\alpha = \inf\{\Phi^c(x) \wedge \Phi^c(y) : x \in (a,b)^l, y \in (a,c)^l\}.$$

Then it is clear that

$$\Phi^c(x) \geq \alpha \text{ for all } x \in (a,b)^l \text{ and } \Phi^c(y) \geq \alpha \text{ for all } y \in (a,c)^l.$$

This implies that $(a,b)^l \subseteq (\Phi^c)_\alpha = (\Phi_\alpha)^c$ and $(a,c)^l \subseteq (\Phi^c)_\alpha = (\Phi_\alpha)^c$. Since Q is finite and $(\Phi_\alpha)^c = \bigcup\{I : I \in \Phi_\alpha\}$, there exist $I_1, I_2, \cdots, I_n$ and $J_1, J_2, \cdots, J_m$ in Φ_α such that

$$(a] \cap (b] = (a,b)^l \subseteq \bigcup_{i=1}^{n} I_i \subseteq \bigvee_{i=1}^{n} I_i \in \Phi_\alpha$$

and

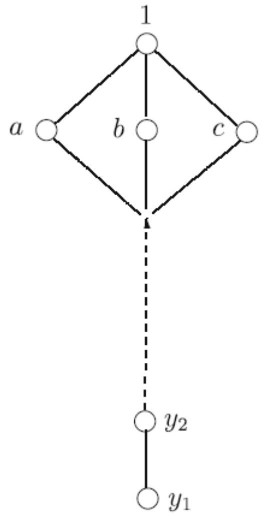
$$(a] \cap (c] = (a,c)^l \subseteq \bigcup_{j=1}^{m} J_j \subseteq \bigvee_{j=1}^{m} J_j \in \Phi_\alpha.$$

Since Φ_α a semi-prime ideal of $\mathcal{I}(Q)$, we have $(a] \cap ((b] \vee (c]) \in \Phi_\alpha$. Now

$$\begin{aligned} z \in \{a, (b,c)^u\}^l &\Rightarrow z \in (a] \cap ((b] \vee (c]) \in \Phi_\alpha \\ &\Rightarrow z \in (\Phi_\alpha)^c = (\Phi^c)_\alpha \\ &\Rightarrow \Phi^c(z) \geq \alpha \\ &\Rightarrow \Phi^c(z) \geq \inf\{\Phi^c(x) \wedge \Phi^c(y) : x \in (a,b)^l, y \in (a,c)^l\} \end{aligned}$$

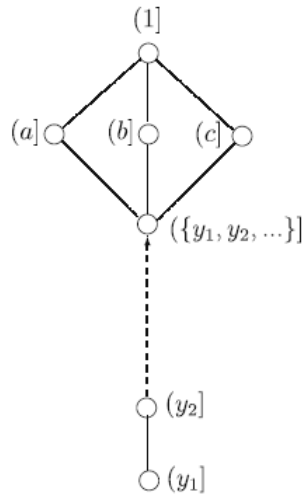
Hence Φ^c is an L-fuzzy semi-prime ideal of a poset Q. □

Remark 3.10. The finiteness conditions in the statement of the Theorem 3.9 is necessary. For example consider the infinite poset depicted in the Fig. 3.1 and its ideal lattice $\mathcal{I}(Q)$ in Fig. 3.2 given below.



Q

Fig. 3.1



$Id(Q)$

Fig. 3.2

Let $L = [0, 1]$. Then consider the L- fuzzy subset Φ of $\mathcal{I}(Q)$ given by:

$$\Phi(I) = \begin{cases} 1 & if\ I = (y_1] \\ 1 - \dfrac{\frac{1}{2}i}{1+i} & if\ I = (y_i]\ i = 2, 3, \cdots, \\ 0 & if \text{ otherwise} \end{cases}$$

for all $I \in \mathcal{I}(Q)$, which is an L-fuzzy semi-prime ideal of $\mathcal{I}(Q)$. Then its contraction, Φ^c, is given by:

$$\Phi^c(x) = \begin{cases} 1 & if\ x = y_1 \\ 1 - \dfrac{\frac{1}{2}i}{1+i} & if\ x = y_i\ i = 2, 3, \cdots \\ 0 & if \text{ otherwise} \end{cases}$$

for all $x \in Q$. But Φ^c is not an L-fuzzy semi-prime ideal of a poset Q as $a \in a^l = \{a, (b,c)^u\}^l$ but $\Phi^c(a) = 0 \ngeq \frac{1}{2} = \inf\{\Phi^c(x) \wedge \Phi^c(y) : x \in (a,b)^l, y \in (a,c)^l\}$.

However, if Q is a meet semi-lattice, we have the following theorem.

THEOREM 3.11. *Let Q be a meet semi-lattice and Φ be an L-fuzzy semi-prime ideal of $\mathcal{I}(Q)$ with sup property. Then Φ^c is an L-fuzzy semi-prime ideal of Q.*

Proof. Let Φ be an L-fuzzy semi-prime ideal of $\mathcal{I}(Q)$. Let $a, b, c \in Q$ and $z \in \{a, (b,c)^u\}^l$. Since Q is a meet semi-lattice it is clear that

$$\inf\{\Phi^c(x) \wedge \Phi^c(y) : x \in (a,b)^l, y \in (a,c)^l\} = \Phi^c(a \wedge b) \wedge \Phi^c(a \wedge c).$$

Now put $\alpha = \Phi^c(a \wedge b) \wedge \Phi^c(a \wedge c)$. Then we have

$$\Phi^c(a \wedge b) \geq \alpha \text{ and } \Phi^c(a \wedge c) \geq \alpha\ .$$

This implies that

$$(a] \cap (b] = (a \wedge b] \subseteq (\Phi^c)_\alpha = (\Phi_\alpha)^c \text{ and } (a] \cap (c] = (a \wedge c] \subseteq (\Phi^c)_\alpha = (\Phi_\alpha)^c.$$

Thus there exist I, J in Φ_α such that $(a] \cap (b] \subseteq I$ and $(a] \cap (c] \subseteq J$. This implies that $(a] \cap (b], (a] \cap (c] \in \Phi_\alpha$. Since Φ_α a semi-prime ideal of $\mathcal{I}(Q)$, $(a] \cap ((b] \vee (c]) \in \Phi_\alpha$. Now

$$\begin{aligned} z \in \{a, (b,c)^u\}^l &\Rightarrow z \in (a] \cap ((b] \vee (c]) \in \Phi_\alpha \\ &= z \in (\Phi_\alpha)^c = (\Phi^c)_\alpha \\ &= \Phi^c(z) \geq \alpha = \Phi^c(a \wedge b) \wedge \Phi^c(a \wedge c) \end{aligned}$$

Therefore Φ^c is an L-fuzzy semi-prime ideal of a meet semi-lattice Q. □

We also investigate the relationships between L-fuzzy prime ideal of a poset Q and L-fuzzy prime ideal of the lattice $\mathcal{I}(Q)$ in the next two theorems.

THEOREM 3.12. *Let μ be an L-fuzzy prime ideal of a poset Q. Then μ^e is an L-fuzzy prime ideal of the lattice $\mathcal{I}(Q)$.*

Proof. Let μ be an L-fuzzy prime ideal of a poset Q. Let $I, J \in \mathcal{I}(Q)$.
We need to show that

$$\mu^e(I \cap J) = \mu^e(I) \text{ or } \mu^e(J).$$

Indeed, on the contrary if both $\mu^e(I \cap J) \neq \mu^e(I)$ and $\mu^e(I \cap J) \neq \mu^e(J)$, then there exist $a, b \in Q$ such that $a \in I, b \in J$ and $\mu^e(I \cap J) \nleq \mu(a)$ and $\mu^e(I \cap J) \nleq \mu(b)$.
This implies that

$$\mu(x) \nleq \mu(a) \text{ and } \mu(x) \nleq \mu(b) \text{ for all } x \in I \cap J.$$

Since $(a,b)^l \subseteq I \cap J$, we have $\inf\{\mu(x) : x \in (a,b)^l\} \nleq \mu(a)$ and $\inf\{\mu(x) : x \in (a,b)^l\} \nleq \mu(b)$, which contradicts the fact that μ is an L-fuzzy prime ideal of a poset Q. Therefore μ^e is an L-fuzzy prime ideal of the lattice $\mathcal{I}(Q)$. □

THEOREM 3.13. *Let Q be a finite poset and Φ be an L-fuzzy prime ideal of $\mathcal{I}(Q)$ with sup property. Then Φ^c is an L-fuzzy prime ideal of Q.*

Proof. Suppose that Φ is an L-fuzzy prime ideal of $\mathcal{I}(Q)$ where Q is a finite poset. Then, by Lemma 3.5, Φ^c is an L-fuzzy ideal of Q. Let $a, b \in Q$ and put $\alpha = \inf\{\Phi^c(x) : x \in (a,b)^l\}$. This implies that

$$\Phi^c(x) \geq \alpha \text{ for all } x \in (a,b)^l.$$

Thus we have $(a,b)^l \subseteq (\Phi^c)_\alpha = (\Phi_\alpha)^c$. Since Q is finite and $(\Phi_\alpha)^c = \bigcup\{I : I \in \Phi_\alpha\}$, there exist $I_1, I_2, \cdots, I_n$ such that

$$(a] \cap (b] = (a,b)^l \subseteq \bigcup_{i=1}^n I_i \subseteq \bigvee_{i=1}^n I_i \in \Phi_\alpha$$

Since Φ_α a prime ideal of $\mathcal{I}(Q)$, we have either $(a] \in \Phi_\alpha$ or $(b] \in \Phi_\alpha$. Consequently $(a] \subseteq (\Phi_\alpha)^c = (\Phi^c)_\alpha$ or $(b] \subseteq (\Phi_\alpha)^c = (\Phi^c)_\alpha$ and therefore

$$\Phi^c(a) \geq \alpha = \inf\{\Phi^c(x) : x \in (a,b)^l\} \geq \Phi^c(a)$$

or

$$\Phi^c(b) \geq \alpha = \inf\{\Phi^c(x) : x \in (a,b)^l\} \geq \Phi^c(b),$$

that is, $\inf\{\Phi^c(x) : x \in (a,b)^l\} = \Phi^c(a)$ or $\Phi^c(b)$. Hence Φ^c is an L-fuzzy prime ideal of a poset Q. □

Remark 3.14. The statement of Theorem 3.13 is not necessarily true if the poset Q is not finite. Consider the infinite poset Q depicted in Fig.3.1 and its ideal lattice $\mathcal{I}(Q)$ in Fig. 3.2 on pages 10 and 11. Observe that the L- fuzzy subset Φ of $\mathcal{I}(Q)$ into $L = [0,1]$ defined by:

$$\Phi(I) = \begin{cases} 1 & if\ I = (y_1] \\ 1 - \dfrac{\frac{1}{3}i}{1+i} & if\ I = (y_i],\ i = 2,3,\cdots \\ 0 & if \text{ otherwise} \end{cases}$$

for all $I \in \mathcal{I}(Q)$ is an L-fuzzy prime ideal of $\mathcal{I}(Q)$.

Also see that Φ^c is given by:

$$\Phi^c(x) = \begin{cases} 1 & if\ x = y_1 \\ 1 - \dfrac{\frac{1}{3}i}{1+i} & if\ x = y_i,\ i = 2,3,\cdots \\ 0 & if \text{ otherwise} \end{cases}$$

for all $x \in Q$ is an L-fuzzy ideal of Q but not L-fuzzy prime ideal, as $\inf\{\Phi^c(x) : x \in (a,b)^l\} = \frac{2}{3}$ and neither equal to $\Phi^c(a)$ nor $\Phi^c(b)$.

However, if the poset is a meet semilattice, then we have the following theorem.

Theorem 3.15. *Let Q be a meet semi-lattice and Φ be an L-fuzzy prime ideal of $\mathcal{I}(Q)$ with sup property. Then Φ^c is an L-fuzzy prime ideal of Q.*

Proof. Suppose that Φ is an L-fuzzy prime ideal of $\mathcal{I}(Q)$ with sup property Now we claim that $\Phi^c(a \wedge b) = \Phi^c(a)$ or $\Phi^c(b)$, for all $a, b \in Q$. Now put $\alpha = \Phi^c(a \wedge b)$. This implies that $a \wedge b \in (\Phi^c)_\alpha = (\Phi_\alpha)^c$. Thus there exists $I \in \Phi_\alpha$ such that $a \wedge b \in I$. Therefore $(a] \cap (b] = (a \wedge b] \subseteq I \in \Phi_\alpha$ and hence $(a] \cap (b] \in \Phi_\alpha$.
Now, by primeness of Φ_α, we must have $(a] \in \Phi_\alpha$ or $(b] \in \Phi_\alpha$ and so $a \in (\Phi_\alpha)^c = (\Phi^c)_\alpha$ or $b \in (\Phi_\alpha)^c = (\Phi^c)_\alpha$. This implies that

$$\Phi^c(a) \geq \alpha = \Phi^c(a \wedge b) \geq \Phi^c(a) \text{ or } \Phi^c(b) \geq \alpha = \Phi^c(a \wedge b) \geq \Phi^c(b),$$

i.e. $\Phi^c(a \wedge b) = \Phi^c(a)$ or $\Phi^c(b)$. Hence Φ^c is an L-fuzzy prime ideal of Q. □

4. Separation Theorems

In this section, we extend and prove an analogue of Stone's Theorem for finite posets which has been studied by V. S. Kharat and K. A. Mokbel [26] using L-fuzzy semi-prime ideals. Some counter examples are also given.

Now we obtain an L-fuzzy filter in a poset Q with the help of an L-fuzzy filter in the lattice $\mathcal{I}(Q)$ of all L-fuzzy ideals of Q and study the L-fuzzy semi-primeness connection between them.

Definition 4.1. Let Q be a poset with 1 and Φ_F be an L-fuzzy filter of $\mathcal{I}(Q)$. Define an L-fuzzy subset μ_F of Q by:

$$\mu_F(x) = \Phi_F((x]) \text{ for all } x \in Q.$$

We have the following Lemma.

Lemma 4.2. *Let Q be a poset with 1. Then μ_F is an L-fuzzy filter of Q, where μ_F is an L-fuzzy subset of Q given in Definition 4.1 above.*

Proof. Clearly $\mu_F(1) = 1$. Let $a, b \in Q$ and $x \in (a,b)^{lu}$. This implies that $(a] \cap (b] = (a,b)^l \subseteq x^l = (x]$. Thus

$$\begin{aligned} \mu_F(a) \wedge \mu_F(b) &= \Phi_F((a]) \wedge \Phi_F((b]) \\ &= \Phi_F((a] \cap (b]) \\ &\leq \Phi_F((x]) \\ &= \mu_F(x) \end{aligned}$$

Therefore μ_F is an L-fuzzy filter of Q. □

In the case of finite posets we have the following.

Lemma 4.3. *Let Q be a finite poset and Φ_F be an L-fuzzy filter of $\mathcal{I}(Q)$ and μ_F be an L-fuzzy filter given as in Definition 4.1 above. Then the following statements hold.*

1. $\Phi_F((a] \vee (b]) = \inf\{\mu_F(x) : x \in (a,b)^u\}$ *for any $a, b \in Q$.*
2. *if Φ_F is an L-fuzzy semi-prime filter, then μ_F is an L-fuzzy semi-prime filter.*

Proof. 1. Let $a, b \in Q$. Since Q is finite, $(a,b)^u$ is finite and hence we clearly have $\bigcap_{x \in (a,b)^u} (x] \subseteq (a,b)^{ul} \subseteq (a] \vee (b]$. Thus

$$\begin{aligned} \inf\{\mu_F(x) : x \in (a,b)^u\} &= \inf\{\Phi_F((x]) : x \in (a,b)^u\} \\ &= \Phi_F(\bigcap_{x \in (a,b)^u} (x]) \\ &\leq \Phi_F((a] \vee (b]) \end{aligned}$$

Again let $x \in (a,b)^u$. Then $a \leq x$ and $b \leq x$ and hence $(a] \subseteq (x]$ and $(b] \subseteq (x]$. This implies that $(a] \vee (b] \subseteq (x]$ for all $x \in (a,b)^u$ and thus $(a] \vee (b] \subseteq \bigcap_{x \in (a,b)^u} (x]$. Since Φ_F is an L-fuzzy filter of $\mathcal{I}(Q)$ and hence isotone, we have

$$\begin{aligned} \Phi_F((a] \vee (b]) &\leq \Phi_F(\bigcap_{x \in (a,b)^u} (x]) \\ &= \inf\{\Phi_F((x]) : x \in (a,b)^u\} \\ &= \inf\{\mu_F(x) : x \in (a,b)^u\} \end{aligned}$$

Hence (1) holds.

2. Let $a, b, c \in Q$ and $z \in \{a, (b, c)^l\}^u$. Then it is clear that $(a] \subseteq (z]$ and $(b] \cap (c] \subseteq (z]$. Thus we have $(a] \vee ((b] \cap (c]) \subseteq (z]$.

$$\begin{aligned} & \inf\{\mu_F(x) \wedge \mu_F(y) : x \in (a, b)^u, y \in (a, c)^u\} \\ = & \inf\{\mu_F(x) : x \in (a, b)^u\} \wedge \inf\{\mu_F(y) : y \in (a, c)^u\} \\ = & \Phi_F((a] \vee (b]) \wedge \Phi_F((a] \vee (c]) \cdots \text{ (by 1)} \\ = & \Phi_F((a] \vee ((b] \cap (c])) \\ \leq & \Phi_F((z]) = \mu_F(z) \end{aligned}$$

Hence (2) holds.

□

Remark 4.4. We give an example to show that the assertion of Lemma 4.3 is not necessarily true if we drop the finiteness condition. Consider the dual of the infinite poset Q that is depicted in Fig 3.1, say Q^d and its ideal lattice $\mathcal{I}(Q^d)$ which is the dual of the ideal lattice $\mathcal{I}(Q)$ depicted in Fig 3.2. Consider the L- fuzzy filter Φ_F of $\mathcal{I}(Q^d)$ into $L = [0, 1]$ which is given by:

$$\Phi_F(I) = \begin{cases} 1 & if\ I = (y_1] \\ 1 - \dfrac{\frac{1}{3}i}{1+i} & if\ I = (y_i]\ for\ i = 2, 3, \cdots \\ 0 & if \text{ otherwise} \end{cases}$$

for all $I \in \mathcal{I}(Q^d)$. Observe that the L-fuzzy subset μ_F of Q^d into $L = [0, 1]$ which is given by:

$$\mu_F(x) = \begin{cases} 1 & if\ x = y_1 \\ 1 - \dfrac{\frac{1}{3}i}{1+i} & if\ x = y_i\ for\ i = 2, 3, \cdots \\ 0 & if \text{ otherwise} \end{cases}$$

for all $x \in Q$ is an L-fuzzy filter of Q^d. But

$$\Phi_F((a] \vee (b]) = \Phi_F((\{y_1, y_2, \cdots\}]) = 0 \neq \tfrac{2}{3} = \inf\{\mu_F(z) : z \in (a, b)^u\}.$$

Moreover, Φ_F is an L-fuzzy semi-prime filter of $\mathcal{I}(Q^d)$. But μ_F is not an L-fuzzy semi-prime filter, as $a \in a^u = \{a, (b, c)^l\}^u$ and

$$\mu_F(a) = 0 \ngeq \tfrac{2}{3} = \inf\{\mu_F(x) \wedge \mu_F(y) : x \in (a, b)^u, y \in (a, c)^u.$$

However, in the case of join semi-lattices we have the following corollary.

Corollary 4.5. *Let Q be a join semi-lattice with 1, Φ_F be an L-fuzzy filter of $\mathcal{I}(Q)$ and μ_F be an L-fuzzy filter defined as in in Definition 4.1. Then the following statements hold.*

1. *$\Phi_F((a] \vee (b]) = \mu_F(a \vee b)$ for any $a, b \in Q$.*
2. *if Φ_F is an L-fuzzy semi-prime filter, then μ_F is an L-fuzzy semi-prime filter.*

Now we obtain an L-fuzzy filter in the lattice $\mathcal{I}(Q)$ of all L-fuzzy ideals of Q using an l-L-fuzzy filter of a poset Q with 1.

Definition 4.6. Let σ be an l-L-fuzzy filter of a poset Q with 1, Define an L-fuzzy subset Ω of $\mathcal{I}(Q)$ as follows:

$$\Omega(I) = \sup\{\sigma(x) : x \in I\} \text{ for all‘} I \in \mathcal{I}(Q).$$

We establish the following result.

LEMMA 4.7. *Let Q be a poset with 1. Then Ω is an L-fuzzy filter of $\mathcal{I}(Q)$, where Ω is an L-fuzzy subset of $\mathcal{I}(Q)$ as given in Definition 4.6 given above.*

Proof. Let σ be an l-L-fuzzy filter of a poset Q. Then clearly $\Omega((1]) = 1$.
Let $I, J \in \mathcal{I}(Q)$. Then

$$\begin{aligned} \Omega(I) \wedge \Omega(J) &= \sup\{\sigma(x) : x \in I\} \wedge \sup\{\sigma(y) : y \in J\} \\ &= \sup\{\sigma(x) \wedge \sigma(y) : x \in I, y \in J\} \\ &\leq \sup\{\sigma(x) \wedge \sigma(y) : (x, y)^l \subseteq I \cap J\} \end{aligned}$$

Since σ is an l-L-fuzzy filter of Q and $x, y \in Q$, there there exists $z \in (x, y)^l$ such that $\sigma(z) = \sigma(x) \wedge \sigma(y)$. Therefore

$$\Omega(I) \wedge \Omega(J) \leq \sup\{\sigma(z) : z \in I \cap J\} = \Omega(I \cap J)$$

Again

$$\begin{aligned} \Omega(I \cap J) &= \sup\{\sigma(x) : x \in I \cap J\} \\ &\leq \sup\{\sigma(x) : x \in I\} \\ &= \Omega(I) \end{aligned}$$

Therefore $\Omega(I \cap J) \subseteq \Omega(I)$. Similarly we can show that $\Omega(I \cap J) \subseteq \Omega(J)$ and hence $\Omega(I \cap J) \subseteq \Omega(I) \wedge \Omega(J)$. Therefore

$$\Omega(I \cap J) = \Omega(I) \wedge \Omega(J)$$

and hence Ω is an L-fuzzy filter of $\mathcal{I}(Q)$. □

We prove the following lemma, which is analogous to Rav's Separation Theorem for semi-prime ideals in Lattice Theory. [33]

LEMMA 4.8. *Let α be a prime element in L, μ be an L-fuzzy semi-prime ideal and σ be an L-fuzzy filter of a lattice X such that $\mu \cap \sigma \subseteq \overline{\alpha}$. Then there exists an L-fuzzy semi-prime filter σ^F such that $\sigma \subseteq \sigma^F$ and $\mu \cap \sigma^F \subseteq \overline{\alpha}$.*

Proof. Let μ be an L-fuzzy semi-prime ideal and σ be an L-fuzzy filter of the lattice X such that $\mu \cap \sigma \subseteq \overline{\alpha}$. Now put

$$I = \{x \in X : \mu(x) \nleq \alpha\} \text{ and } K = \{x \in X : \sigma(x) \nleq \alpha\}.$$

Then, clearly I is a semi-prime ideal and K is a filter of X such that $I \cap K = \emptyset$. Therefore by Rav's Separation Theorem for semi-prime ideals in Lattice, there exists a semi-prime filter F such that $K \subseteq F$ and $I \cap F = \emptyset$. Then, note that the L-fuzzy subset σ^F of X defined by:

$$(\sigma^F)(x) = \begin{cases} 1 & if\ x \in F \\ \alpha & if\ x \notin F \end{cases}$$

for all $x \in X$ is an l-fuzzy semi-prime filter. Now we claim that $\sigma \subseteq \sigma^F$ and $\mu \cap \sigma^F \subseteq \overline{\alpha}$. Let $x \in X$. Now if $x \in F$, then $\sigma(x) \leq 1 = \sigma^F(x)$ and, if $x \notin F$, then $x \notin K$, so that $\sigma(x) \leq \alpha = \alpha^F(x)$. Hence $\sigma \subseteq \alpha^F$. Again if $x \in F$, then $x \notin I$, so that $\mu(x) \leq \alpha$. Thus

$$(\mu \cap \sigma^F)(x) = \mu(x) \wedge \sigma^F(x) = \mu(x) \wedge 1 = \mu(x) \leq \alpha = \overline{\alpha}(x)$$

and if $x \notin F$, then

$$(\mu \cap \sigma^F)(x) \leq \mu(x) \wedge \alpha \leq \alpha = \overline{\alpha}(x).$$

Hence $\mu \cap \sigma^F \subseteq \overline{\alpha}$. Therefore the claim is true. □

Now we extend an analogue of Stone's Theorem for finite posets which has been studied by V. S. Kharat and K. A. Mokbel [26] using L-fuzzy semi-prime ideals as given in Theorem 4.9 below.

THEOREM 4.9. *Let Q be a finite poset and α be a prime element in L. If μ be an L-fuzzy semi-prime ideal and σ be an l-L-fuzzy filter of Q for which $\mu \cap \sigma \subseteq \overline{\alpha}$, then there exists an L-fuzzy semi-prime filter σ' of Q such that $\sigma \subseteq \sigma'$ and $\mu \cap \sigma' \subseteq \overline{\alpha}$.*

Proof. Suppose that μ is an L-fuzzy semi-prime ideal and σ is an l-L-fuzzy filter of a finite poset Q such that $\mu \cap \sigma \subseteq \overline{\alpha}$, where α is a prime element in L. By Theorem 3.8, μ^e is an L-fuzzy semi-prime ideal of $\mathcal{I}(Q)$. Since σ is an l-L-fuzzy filter, the L-fuzzy subset Ω of $\mathcal{I}(Q)$ given in Definition 4.6 is an L-fuzzy filter of $\mathcal{I}(Q)$. (See Lemma 4.7). Now we claim that $\mu^e \cap \Omega \subseteq \overline{\alpha}$. Suppose not. Then there exists $I \in \mathcal{I}(Q)$ such that $\mu^e(I) \nleq \alpha$ and $\Omega(I) \nleq \alpha$. This implies that

$$\mu(x) \nleq \alpha \text{ for all } x \in I \text{ and } \sigma(x) \nleq \alpha \text{ for some } \ x \in I.$$

This contradicts the hypothesis $\mu \cap \sigma \subseteq \overline{\alpha}$. Hence the claim holds. Now, since $\mathcal{I}(Q)$ is a lattice, by Lemma 4.8, there exists an L-fuzzy semi-prime filter, say Φ_F of $\mathcal{I}(Q)$ such that $\Omega \subseteq \Phi_F$ and $\mu^e \cap \Phi_F \subseteq \overline{\alpha}$. Consider the L-fuzzy subset μ_F of Q given in definition 4.1 which is an L-fuzzy semi-filter of Q. (See Lemma 4.2). Put $\sigma' = \mu_F$ and observe that $\sigma \subseteq \sigma'$; for, if $x \in Q$, then

$$\sigma(x) \leq \sup\{\sigma(y) : y \in (x]\} = \Omega((x]) \leq \Phi_F((x]) = \mu_F(x) = \sigma'(x).$$

Further, we must have $\mu \cap \sigma' \subseteq \overline{\alpha}$. Otherwise if $\mu \cap \sigma' \nsubseteq \overline{\alpha}$, there exists $x \in Q$ such that $\mu(x) \nleq \alpha$ and $\sigma'(x) = \mu_F(x) \nleq \alpha$. This implies that $\mu^e((x]) \nleq \alpha$ and $\Phi_F((x]) \nleq \alpha$, which is a contradiction to the fact that $\mu^e \cap \Phi_F \subseteq \overline{\alpha}$. □

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Derso Abeje Engidaw
Department of Mathematics, University of Gondar, Gondar, Ethiopia
E-mail: deab02@yahoo.com

Tilahun Bimerew Alemu
Department of Mathematics, University of Gondar, Gondar, Ethiopia
E-mail: tilahunb88@gmail.com